a proof of the Bloch-Wangness equations that is free of the random phase inconsistency pointed out by Van Hove.[5]

We conclude with some remarks about the diagonal singularity condition of Van Hove.[6] In the Wigner-Weisskopf theory of line broadening the method consists of an exact solution of a problem where the only states considered are those in which only one photon at a time appears in any intermediate state and the photons in successive intermediate states are independent of each other. A consequence of our derivation is that we consider only those intermediate states of the system+reservoir interaction in which only one reservoir particle appears at a time and the reservoir particles in successive intermediate states are independent of each other. This allows us to effectively discard the spent reservoir particle and the system "sees" a canonically distributed reservoir before each collision.

We can see the connection between the above discussion and the diagonal singularity condition by considering the jth term in the expansion of the time evolution operator of the system, $e^{t\mathbf{K}}$, which is

$$(1/j!)t^j\mathbf{K}^j = (1/j!)(\lambda^2 t)^j[\textstyle\sum_n\langle\mathbf{n}|V|\mathbf{m}\rangle P_m\langle\mathbf{m}|V|\mathbf{n}\rangle]^j. \quad (6.2)$$

The intermediate states of the system in Eq. (6.2) are arbitrary. However, in the reservoir Hilbert space where the system operators are c numbers, Eq. (6.2) is of the form $[\sum(VAV)_d]^j$, where $A=P$ is diagonal in the reservoir coordinates and where d indicates "take the diagonal matrix element of $(\cdots)$." This is just the term that the diagonal singularity condition for internal relaxation gives for the coefficient of $(\lambda^2 t)^j$. Thus, our derivation which starts with the assumption that the off-diagonal matrix elements of the system +reservoir density matrix are determined by $Pe^{t\mathbf{K}}$ has, as a consequence, that in the reservoir Hilbert space the interaction between system and reservoir satisfies the diagonal singularity condition of Van Hove.

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Lorentz-Invariant Equations of Motion of Point Masses in the General Theory of Relativity

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After a general discussion of the problem of motion in the general theory of relativity a simple derivation of the law of motion is given for single poles of the gravitational field, which is based on a method originally developed by Mathisson. This law follows from the covariant conservation law for the matter energy-momentum tensor alone, without reference to any field equations, and takes the form of a geodesic of the (unknown) metric. Expanding this metric in terms of a power series in a parameter λ and using the Minkowski proper time to parametrize the world lines of the particles, the (Lorentz-invariant) form of the approximate laws of motion follows. A method is developed to obtain the equations of motion (including the explicit form of the metric in terms of the particle variables) from Einstein's field equations. A systematic linearization procedure leads to a series of second-order differential equations for the metric; the nth order approximation of the equations of motion, as well as the explicit form of the matter tensor in $(n+1)$st order, is obtained as an integrability condition on the $(n+1)$st order opproximation for the metric. No coordinate conditions are required to obtain the general form of the equations of motion; they are needed only to reduce the approximation equations to wave equations and thus to allow their explicit integration in terms of retarded or symmetric potentials. In developing the approximation method it is shown that consistency requires that any set of approximate equations is solved "up to" rather than "in" nth order; this implies that the form of the lower-order metric be maintained, but with the motion corresponding to the nth order solutions rather than to lower order ones. In particular, the equations for the first-order metric imply zero-order equations of motion which restrict the particles to zero acceleration; the equations for the second-order metric imply first-order equation of motion involving the first-order metric, but without the previous restriction. In the retarded case the equations of motion contain retarded interactions and radiation reaction terms of the form familiar from electrodynamics; no such terms appear in the symmetric case. The equations of the symmetric case are derivable from a Fokker-type variational principle. The relation of the results obtained to work on Lorentz-invariant equations by other authors is discussed. In Appendix I a discussion of alternative derivations is presented; Appendix II contains remarks on Wheeler-Feynman type considerations for general relativistic equations of motion.

I. INTRODUCTION

WITHIN the conceptual framework of Newtonian particle mechanics there is a sharp division between laws of motion and force laws. The laws of motion ($F=ma$) are assumed to be the same for all matter; the force laws (Newton's law of gravitation, Coulomb's law etc.) are different for different types of particles, their specific form to be determined by experiment.[1]

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[1] In the following we shall call "laws of motion" the expressions relating the variation of some particle variables to the (unspeci-

After a fruitless search for simple force laws for rapidly moving electric charges it was realized that the laws of electrodynamics take a simpler form if expressed in terms of field laws than in terms of force laws. At first these field laws (Maxwell's equations) were considered to be independent of the equations of motion (the combination of Newton's or Einstein's law of motion with the expression for the Lorentz force). However, it was soon realized by Lorentz that if one assumed that the fields due to the electric charges were retarded rather than half-retarded, half-advanced, as seemed to be necessary for the description of electromagnetic radiation, then the equations of motion had to be modified if one wished to maintain the laws of conservation of energy and momentum for a closed system. Lorentz's work[2] led to serious difficulties both if the electrons were treated as points (infinite self-terms) and if they were assumed to be of finite extension (structure-dependent terms, differential equations of infinite order) and showed that the equations of motion were not a simple consequence of the field equations, but established clearly that these two sets of equations could not be postulated independently without leading to inconsistencies. Later work by Dirac[3] and others[4] showed that for a very general class of special relativistic field equations it was always possible to obtain finite equations of motion for point particles by requiring the detailed conservation of energy-momentum and of angular momentum.

In the general theory of relativity it was at first again assumed that the field equations and the equations of motion were independent and it was postulated that a test particle was moving along a geodesic of the background metric.[5] However, it was soon realized by Weyl[6] and by Einstein and Grommer[7] that the field equations imposed limitations on the motion of particles. Indeed, several authors were able to show that Einstein's field equations implied that a test particle had to move along a geodesic.[6,8] For bodies with comparable masses, a method of successive approximations for obtaining the equations of motion was devised in a fundamental paper by Einstein, Infeld, and Hoffmann.[9] This method is based on the assumption that the time derivative of any field quantity is much smaller than the spatial derivatives. Such an unsymmetric treatment of the four coordinates not only does violence to the spirit of general covariance, but does not even satisfy the requirements of special relativity. The resulting approximate equations of motion are thus not applicable to motions of bodies with relative velocities comparable to the velocity of light. Similarly they are not well suited for an investigation of the problem of gravitational radiation. Einstein, Infeld, and Hoffmann obtain the Newtonian equations of motion, including the Newtonian gravitational interaction, in the fourth order of approximation and a new ("post-Newtonian") set of equations in the sixth order[10,11]; an investigation of possible radiation damping terms requires a study of the equations in the tenth order.[12]

In this paper we shall present an approximation procedure in which time and space coordinates are on the same footing and where all the calculations and results are Lorentz invariant.[13] The equations obtained are thus applicable to the study of the motion of bodies of high relative velocity and allow the investigation of the problem of radiation damping with the methods familiar from the special relativistic equations.

While gaining the help of these special relativistic methods, we also acquire all the mathematical difficulties inherent in special relativistic equations of motion. In the equations of motion of Newtonian mechanics the interactions are given explicitly as functions of the simultaneous positions of the particles. In special relativity they are usually given as functions of the separations on or even within the light cones of the particles, which are known explicitly only if the motion is known. No mathematical methods are known for handling such equations (other than approximation methods using the nonrelativistic motion as a first approximation), and it is not even known how to formulate the initial value problem correctly.[14,15]

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fied) force, and "equations of motion" these same expressions but with the force specified either in terms of fields or in terms of the variables describing the particles exerting the force. Except in the introductory discussion we shall consider only gravitational interactions. Other interactions will be considered elsewhere.

[2] H. A. Lorentz, *Collected Papers* (M. Nijhoff, The Hague, 1936), Vol. II, pp. 281 and 343; and *The Theory of Electrons* (B. G. Teubner, Leipzig, 1909), pp. 49 and 253.

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).

[4] For a review of this work see P. Havas, in "Argonne National Laboratory Summer Lectures on Theoretical Physics, 1958," ANL-5982 (unpublished), p. 124.

[5] A. Einstein, Ann. Physik **49**, 769 (1916).

[6] H. Weyl, *Raum, Zeit, Materie* (Springer-Verlag, Berlin, 1921), 4th ed., Sec. 36; in more detail in the 5th ed. (1923).

[7] A. Einstein and J. Grommer, Sitzber. preuss. Akad. Wiss., 2 and 235 (1927).

[8] M. von Laue, *Die Relativitätstheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1921), 1st ed., Vol. 2, Sec. 15; A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, New York, 1923), Sec. 56; C. Lanczos, Z. Physik **59**, 514 (1930); M. Mathisson, *ibid.* **67**, 270 (1931); H. P. Robertson, Proc. Edinburgh Math. Soc. **5**, 63 (1936); L. Infeld and A. Schild, Revs. Modern Phys. **21**, 408 (1949).

[9] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 66 (1938). These authors treated the particles as singularities which are simple poles of the gravitational field. Whenever we refer to "mass points," "singularities," or "test particles" in this paper, we imply such simple poles unless otherwise stated.

[10] In the following we shall refer to these post-Newtonian equations as the EIH equations. There have been many modifications of the method of derivation of these equations, all of which have in common the assumption of different orders of magnitude for the spatial and temporal derivatives ("slow motion approximation"). For the early literature see A. E. Scheidegger, Revs. Modern Phys. **25**, 451 (1953); for later references, e.g., L. Infeld, *ibid.* **29**, 398 (1957) and reference 11.

[11] L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press, New York, 1960).

[12] J. N. Goldberg, Phys. Rev. **99**, 1873 (1955).

[13] The main results of this paper were stated in P. Havas, Phys. Rev. **108**, 1351 (1957), which also contains additional results the details of which will be published in subsequent papers.

[14] For an example of a solution of a two-body problem showing some peculiar features due to the non-instantaneous interaction law, see P. Havas, Acta Phys. Austriaca **3**, 342 (1949).

[15] For a discussion of the relativistic initial value problem, see P. Havas and J. Plebański, Bull. Am. Phys. Soc. **5**, 433 (1960). A detailed account is in preparation.

In general relativity the situation is even worse. It has been realized by several authors that even in the case of interacting particles the laws of motion are given by the differential equations for the geodesics of the metric at the position of each particle[16] (a simple derivation is given in the next section). However, the metric is unknown; not even its functional dependence on the particle coordinates (including time) can be given, and its possible dependence on the self-actions of the particles also is not known. Unfortunately, the metric always has to be determined by some approximation method. In most previous work such methods were devised with the aim of obtaining equations of motion patterned after those of Newtonian mechanics, i.e. containing explicit instantaneous interactions. Although this may be justified for most astronomical applications, this way of proceeding is not well suited for studying those features of the problem of motion which are characteristic of the general theory of relativity, and which are qualitatively different from nonrelativistic theory, such as noninstantaneous interactions and possible radiation effects. Therefore, we have used a method of approximation which allows a better treatment of these features by at least leading to equations of motion of special relativistic form.

In this method of approximation the Minkowski metric has a special standing. However, this special role is a purely formal, mathematical one; it is not implied that the Minkowski "metric" actually represents the physical metric of space.[17] Similarly, the manifest Lorentz invariance of our equations does not imply as in special relativity that there exists an infinity of inertial frames of reference, but only assures us that the equations are valid for all velocities $\leq c$. On the other hand, our results as well as those of the EIH method at any given stage of approximation are not generally covariant.[18] Such a covariance can only be expected (for either method) for the final, exact result obtained by carrying the approximation method to infinite order (the possibility of which has not been proved for either method). The advantage of our method consists in the fact that there always exist suitable coordinate systems in which all particle velocities are less than c,[19] but not always such that they are all small compared to c, and that therefore we can expect our low-order results to be better approximations in the case of high velocities than those of the EIH method.

[16] M. von Laue, reference 8; A. S. Eddington, reference 8; L. Infeld and J. Plebański, Bull. acad. polon. sci., Classe III, **4**, 757 (1956). As an assumption the geodesic law was already used by W. de Sitter, Monthly Notices Roy. Astron. Soc. **77**, 155 (1916), to obtain the equations of motions of the N-body problem.

[17] Such an interpretation is, however, adopted by other authors, e.g., N. Rosen, Phys. Rev. **57**, 150 (1940); A. Papapetrou, Proc. Roy. Irish Acad. **52A**, 11 (1948); S. N. Gupta, Revs. Modern Phys. **29**, 334 (1957) and references given there.

[18] Except in the sense that any set of equations whatever can be written in covariant form; see E. Kretschmann, Ann. Physik **53**, 575 (1917) and A. Einstein, *ibid.* **55**, 241 (1918).

[19] Excepting cosmological modifications, which are excluded by the assumption of asymptotic flatness of the metric (see Sec. II).

The method used here has also been applied to the case of dipoles and of the presence of nongravitational fields. The results[13] will be described in subsequent papers.

II. THE LAW OF MOTION

We shall first derive the exact form of the law of motion. As mentioned in Sec. I, this has been shown to be a geodesic of a certain (unknown) metric by several authors before; we shall present a simple derivation, making use of a method which we shall use extensively later. This method is based on one due to Mathisson,[20] and has been described previously.[21,22]

We consider a Riemannian four space with coordinates x^ρ and metric tensor $g_{\mu\nu}$ (Greek indices running from 0 to 3); furthermore we introduce the Minkowski metric $\eta_{\mu\nu}$ (with signature -2). The velocity of light is taken as unity. We shall use the notation

$$\partial_\rho \equiv \partial/\partial x^\rho, \quad \partial_{\rho\sigma\ldots} \equiv \partial_\rho \partial_\sigma \cdots, \quad \Box \equiv \eta^{\rho\sigma}\partial_\rho\partial_\sigma. \tag{1}$$

We shall be concerned with the motion of N singularities. The coordinates of the ith particle are denoted by z_i^ρ, and we shall use the abbreviation

$$s_i^\rho = x^\rho - z_i^\rho. \tag{2}$$

In this paper we do not consider any nongravitational fields; matter is taken to consist of simple poles of the gravitational field and to be described by an energy-momentum tensor $P^{\mu\nu}$ of the form

$$P^{\mu\nu} = \sum_{i=1}^{N} \int_{-\infty}^{\infty} p_i^{\mu\nu}(\tau_i)\delta^4(s_i^\rho)d\tau_i, \tag{3}$$

where δ^4 is a fourfold product of Dirac δ functions and the $p_i^{\mu\nu}$, whose exact form remains to be determined, are functions of some scalars τ_i parametrizing the world lines.[23] The representation of singular quantities by integrals of the type (3) is well known from special relativity.[3]

The requirement of the existence of a covariant law of conservation of energy and momentum of matter implies that the covariant divergence of $P^{\mu\nu}$ must vanish. One form of writing this law is[24]

$$\partial_\sigma(g_{\mu\rho}\mathfrak{P}^{\rho\sigma}) - \tfrac{1}{2}\mathfrak{P}^{\rho\sigma}\partial_\mu g_{\rho\sigma} = 0, \quad \mathfrak{P}^{\rho\sigma} \equiv (-g)^{\frac{1}{2}} P^{\rho\sigma}, \tag{4}$$

where $g \equiv |g_{\mu\nu}|$. Because of the form (3) of $P^{\mu\nu}$ this equation gives a nontrivial relation only along the world lines of the particles. Now we multiply Eq. (4) by a

[20] M. Mathisson, Acta Phys. Polon. **6**, 163 (1937).

[21] P. Havas, in *Recent Developments in General Relativity* (Pergamon Press, New York, 1962), p. 259.

[22] For a similar development of Mathisson's method, see W. Tulczyjew, Acta Phys. Polon. **18**, 393 (1959).

[23] Generally, Roman letters denote tensors and German ones tensor densities. However, for convenience no such rule is implied for $p_i^{\mu\nu}$ introduced by Eq. (3) and the related quantity $\mathfrak{p}_i^{\mu\nu}$ and its constituent parts, defined by Eqs. (6) and (9), respectively; the structure of these is to be determined in the course of the calculation.

[24] Compare, e.g., reference 6, Sec. 28.

vector function $\xi^\mu(x^\rho)$ and integrate over all x^ρ to obtain

$$\int [\partial_\sigma(g_{\mu\rho}\mathfrak{P}^{\rho\sigma})-\tfrac{1}{2}\mathfrak{P}^{\rho\sigma}\partial_\mu g_{\rho\sigma}]\xi^\mu(x^\rho)d^4x=0. \tag{5}$$

ξ^μ is assumed to be completely arbitrary except for vanishing at the limits of the τ_i integrations together with all its derivatives.[25]

We can transfer the derivative in the first term of (5) by an integration by parts to ξ^μ. Taking account of the form (3) of $P^{\rho\sigma}$, we can then carry out the x integration; then g, $g_{\mu\nu}$, ξ^μ and their derivatives, being evaluated at the positions $z_i^\rho(\tau_i)$, become functions of the τ_i's. Thus we get

$$\int \sum_i [g_{\mu\rho}\mathfrak{p}_i^{\rho\sigma}\partial_{i\sigma}\xi^\mu+\tfrac{1}{2}\mathfrak{p}_i^{\rho\sigma}\partial_{i\mu}g_{\rho\sigma}\xi^\mu]d\tau_i=0,$$

$$\mathfrak{p}_i^{\rho\sigma}\equiv(-g)^{\frac{1}{2}}p_i^{\rho\sigma}, \tag{6}$$

where

$$\partial_{i\sigma}\equiv\partial/\partial z_i^\sigma. \tag{7}$$

Now we break up $\mathfrak{p}_i^{\rho\sigma}$ in components parallel and perpendicular to the four-velocity v_i^ρ defined by

$$v_i^\rho\equiv dz_i^\rho/d\tau_i, \tag{8}$$

such that

$$\mathfrak{p}_i^{\rho\sigma}=M_i(\tau_i)v_i^\rho v_i^\sigma+n_i^\rho(\tau_i)v_i^\sigma+n_i^\sigma v_i^\rho+{}^*\mathfrak{p}_i^{\rho\sigma}(\tau_i),$$

$${}^*\mathfrak{p}_i^{\rho\sigma}={}^*\mathfrak{p}_i^{\sigma\rho},\quad {}^*\mathfrak{p}_i^{\rho\sigma}g_{\sigma\alpha}v_i^\alpha=0,\quad n_i^\rho g_{\rho\sigma}v_i^\sigma=0. \tag{9}$$

We substitute these expressions into Eq. (6) and note that

$$v_i^\sigma\partial_{i\sigma}=d/d\tau_i \tag{10}$$

along the ith world line. Thus, carrying out an integration by parts, we obtain

$$\int \sum_i \left\{g_{\mu\rho}({}^*\mathfrak{p}_i^{\rho\sigma}+v_i^\rho n_i^\sigma)\partial_{i\sigma}\xi^\mu-\frac{d}{d\tau_i}[g_{\mu\rho}(M_iv_i^\rho+n_i^\rho)]\xi^\mu +\tfrac{1}{2}\mathfrak{p}_i^{\rho\sigma}\partial_{i\mu}g_{\rho\sigma}\xi^\mu\right\}d\tau_i=0. \tag{11}$$

Because of the arbitrariness of ξ^μ and its derivatives, the terms involving $\partial_{i\sigma}\xi^\mu$ as well as those involving ξ^μ must vanish separately for all i. Thus we must have

$$g_{\mu\rho}({}^*\mathfrak{p}_i^{\rho\sigma}+v_i^\rho n_i^\sigma)=0, \tag{12}$$

from which we get by contraction with v_i^μ, using the definition (9), that n_i^σ and ${}^*\mathfrak{p}_i^{\rho\sigma}$ must vanish separately and thus

$$\mathfrak{p}_i^{\rho\sigma}=M_iv_i^\rho v_i^\sigma. \tag{13}$$

Using this result, we get from the vanishing of the coefficients of ξ^μ in Eq. (11)

$$\frac{d}{d\tau_i}(M_ig_{\mu\rho}v_i^\rho)=\tfrac{1}{2}M_iv_i^\rho v_i^\sigma\partial_{i\mu}g_{\rho\sigma}. \tag{14}$$

This is the law of motion of the ith particle.

To determine M_i we contract Eq. (14) with v_i^μ and use Eq. (10). Denoting derivatives with respect to τ_i by a dot, we obtain

$$\dot{M}_ig_{\mu\rho}v_i^\mu v_i^\rho+M_i\dot{g}_{\mu\rho}v_i^\mu v_i^\rho +M_ig_{\mu\rho}v_i^\mu\dot{v}_i^\rho-\tfrac{1}{2}M_i\dot{g}_{\rho\sigma}v_i^\rho v_i^\sigma=0$$

or

$$\dot{M}_ig_{\alpha\beta}v_i^\alpha v_i^\beta=-\tfrac{1}{2}M_i\frac{d}{d\tau_i}(g_{\alpha\beta}v_i^\alpha v_i^\beta), \tag{15}$$

and thus

$$M_i=m_i(g_{\alpha\beta}v^\alpha v^\beta)^{-\frac{1}{2}}, \tag{16}$$

where m_i is a constant (characterizing the mass of the ith particle).[26] Inserting this into Eq. (14) we obtain as our final form of the law of motion

$$m_i\frac{d}{d\tau_i}\frac{g_{\mu\rho}v_i^\rho}{(g_{\alpha\beta}v_i^\alpha v_i^\beta)^{\frac{1}{2}}}=\frac{1}{2}\frac{m_iv_i^\rho v_i^\sigma}{(g_{\alpha\beta}v_i^\alpha v_i^\beta)^{\frac{1}{2}}}\partial_{i\mu}g_{\rho\sigma}. \tag{17}$$

Equation (17) is the well-known equation of a geodesic of the metric $g_{\mu\nu}$.[27] It should be noted, however, that it is not restricted to test particles, as we did not have to assume our $g_{\mu\nu}$ to be independent of m_i in the course of our derivation.[28] Indeed, the law of motion (17) is a direct consequence of the existence of a covariant conservation law (4) for the singular energy-momentum tensor (3) alone, without any reference whatever to the equations (if any) satisfied by the metric. As the field equations of general relativity (both with and without a cosmological term) were constructed so as to entail such a conservation law, Eq. (17) is the law of motion for both forms of this theory; but any other theory with such a conservation law (even one which is not built on field-theoretical concepts) must include the same law of motion.

Although the law of motion is thus known, $g_{\mu\nu}$ by the very nature of the derivation remains undetermined. Furthermore, we have tacitly implied that $g_{\mu\nu}$ is finite. From our derivation it is clear, however, that the problem of constructing a theory with finite $g_{\mu\nu}$ (and determining its dependence on the masses m_i) is separate from the problem of the form of the law of motion.

In the following we shall only be concerned with the determination of $g_{\mu\nu}$ in the case of Einstein's field equations without cosmological term:

$$G_{\mu\nu}\equiv R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi GP_{\mu\nu}. \tag{18}$$

[25] Mathisson (reference 20) did not make use of δ functions and thus his original method is much more cumbersome than the form used here; he therefore also used the Einstein equations (18) as a starting point rather than the conservation law (4). A form of his method closer to the one used here, but only developed for special relativity, was given by him in Proc. Cambridge Phil. Soc. **36**, 331 (1940).

[26] A result equivalent to our Eqs. (3), (13), and (16) has been established by W. Tulczyjew, Bull. acad. polon. sci., Classe III, **5**, 279 (1957).

[27] Compare, e.g., C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952), p. 230.

[28] In this respect our result goes beyond that established by Mathisson (reference 20).

No entirely rigorous proof has been given thus far that it is indeed possible to formulate an exact theory with singularities for which $g_{\mu\nu}$ is finite on the world lines.[29] In this paper we shall only show that this is possible at each stage of the approxlmation method we are considering.

Equation (17) holds for any parametrization of the world lines. It might appear natural to use the general relativistic line element $(g_{\alpha\beta}dz_i^{\alpha}dz_i^{\beta})^{\frac{1}{2}}$ for $d\tau_i$. This is indeed most convenient for those cases in which a direct application of Eq. (17) is possible, in particular for test particles. However, it is very inconvenient for any calculation involving the simultaneous determination of the metric and the motion, because the line element itself involves the unknown metric. The same objection applies to the use of the "tweedled" line element of Infeld and Plebański.[16] In the following it is most convenient to use the *special* relativistic line element given by

$$d\tau_i=(\eta_{\alpha\beta}dz_i^{\alpha}dz_i^{\beta})^{\frac{1}{2}}, \tag{19}$$

for the parametrization, since it does not involve any unknown function, but does allow us to make use of the tools of special relativity.

Our approximation method is a consistent development of Einstein's original linear approximation method.[30] We expand the metric tensor in a power series in some parameter λ, with the Minkowski metric as the zero-order approximation:

$$g_{\mu\nu}=\eta_{\mu\nu}+\sum_{n=1}^{\infty}\lambda^n\ {}_ng_{\mu\nu}. \tag{20}$$

This parameter will be used only to keep track of the different orders in our approximation, and will be absorbed into the definitions of the ${}_ng_{\mu\nu}$ whenever convenient, without explicitly noting this procedure.

We now insert the series (20) into the law of motion (17). The lowest order of approximation is obtained by considering the terms of order λ^0. Using the relation

$$\eta_{\alpha\beta}v_i^{\alpha}v_i^{\beta}=1, \tag{21}$$

which follows from the definitions (8) and (19), we get for the zero-order law of motion $m_i\eta_{\mu\rho}\dot{v}_i^{\rho}=0$, or

$$m_i\dot{v}_i^{\mu}=0. \tag{22}$$

The first-order law is obtained by collecting the terms of orders λ^0 and λ^1. We obtain

$$m_i\frac{d}{d\tau_i}[(\eta_{\mu\rho}+{}_1g_{\mu\rho})v_i^{\rho}-\tfrac{1}{2}\eta_{\mu\rho}v_i^{\rho}\ {}_1g_{\alpha\beta}v_i^{\alpha}v_i^{\beta}]=\tfrac{1}{2}m_iv_i^{\rho}v_i^{\sigma}\partial_{i\mu}\ {}_1g_{\rho\sigma}. \tag{23}$$

Whereas Eq. (22) was fully determined (and thus, in our terminology,[1] is an equation, as well as a law, of motion), Eq. (23) is not, as ${}_1g_{\mu\nu}$ is not known; in the next section we shall describe the method used in determining its form. It should be noted that it is not sufficient to have any approximation method whatever for determining the metric (20); it has to be established that the procedure is consistent with the approximate laws of motion (22), (23), and similar higher order approximations.

Thus, our interest here is concentrated on the equations of motion—the general form of which is established by Eq. (17) once and for all—and on the metric. For the arguments presented here we do not have to consider the energy-momentum pseudotensor, which until now has been almost exclusively used to investigate the problem of radiation, following the original work by Einstein,[30] and thus do not have to be concerned with the recent controversy on the definition and interpretation of this quantity.[31,32]

Of course the metric and the law of motion (17) could have been expanded in a manner different from the one just discussed. In particular one could also use an expansion suitable for slow motion, but clearly the development (20) is best suited to exploit the symmetry provided by the covariance of the theory.

III. THE APPROXIMATION METHOD

Up to this time we have proceeded in a purely formal manner without asking whether the metric tensor satisfies field equations or any other conditions. In this section we shall formulate a method of successive approximations for the solution of the Einstein field equations (18). We shall find that the nth order equations of motion corresponding to (17) will be the consistency condition for the existence of a solution for the metric in the $(n+1)$st order, when the solution is known for all lower orders.

For comparison with the results in the previous section it is convenient to write the gravitational field equations in the form[31,33]

$$\begin{aligned}-2\mathfrak{G}_\mu{}^\nu&\equiv\partial_\sigma U_\mu{}^{[\nu\sigma]}-t_\mu{}^\nu=16\pi G\mathfrak{P}_\mu{}^\nu,\\ \mathfrak{G}_\mu{}^\nu&\equiv(-g)^{\frac{1}{2}}G_\mu{}^\nu,\\ U_\mu{}^{[\nu\sigma]}&=(-g)^{\frac{1}{2}}g_{\mu\lambda}\partial_\rho[g(g^{\lambda\sigma}g^{\nu\rho}-g^{\rho\sigma}g^{\nu\lambda})]\end{aligned} \tag{24}$$

rather than in the covariant tensor form given in (18). An alternative derivation, starting from Eq. (18), convenient for other purposes, is given in Appendix I. $t_\mu{}^\nu$ is the Einstein energy-momentum pseudotensor; however,

[29] In previous work (see reference 16) it was either tacitly assumed that the contributions of the self-action terms are finite or, as in the work of Infeld and Plebański, certain mathematical properties of the metric had to be assumed to be able to establish the finiteness.

[30] A. Einstein, Sitzber. preuss. Akad. Wiss. 688 (1916).

[31] J. N. Goldberg, Phys. Rev. **111**, 315 (1958) and references given there.

[32] A. Komar, Phys. Rev. **113**, 934 (1959); C. Møller, Ann. Phys. (New York) 4, 347 (1958); and **12**, 118 (1961); *Max-Planck-Festschrift 1958* (VEB Deutscher Verl. Wiss., Berlin, 1958), p. 139; and Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 14 (1959); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961).

[33] V. Bargmann, Revs. Modern Phys. **29**, 161 (1957).

the development presented here is not associated with any particular physical interpretation of this quantity. As $t_\mu{}^\nu$ is homogeneous quadratic in the first derivatives of the metric tensor, all the second derivatives in the field equations originate in $U_\mu{}^{[\nu\sigma]}$. In order to obtain the approximate field equations we shall expand the metric with respect to the parameter λ discussed in the previous section.

The quantities ${}_n g^{\mu\nu}$ from the expansion

$$g^{\mu\nu}=\eta^{\mu\nu}+\sum_{n=1}^{\infty}\lambda^n\, {}_n g^{\mu\nu} \tag{25}$$

are determined in terms of the ${}_n g_{\mu\nu}$ of Eq. (20) from the relation $g_{\mu\rho}g^{\rho\nu}=\delta_\mu{}^\nu$ as

$${}_n g^{\mu\nu}=-\eta^{\mu\rho}\eta^{\nu\sigma}\, {}_n g_{\rho\sigma}+\cdots, \tag{26}$$

where the dots indicate additional nonlinear terms involving ${}_m g_{\mu\nu}$, $m<n$; we shall not need these additional terms here. It is convenient to introduce the abbreviation

$${}_n\gamma_{\mu\nu}={}_n g_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\, {}_n g_{\alpha\beta}. \tag{27}$$

Again we represent the matter tensor by Eq. (3) and further assume an expansion of the form

$$\mathfrak{P}^{\mu\nu}=\sum_{n=1}^{\infty}\lambda^n\, {}_n\mathfrak{P}^{\mu\nu}, \tag{28}$$

with

$${}_n\mathfrak{P}^{\mu}=\sum_{i=1}^{N}\int {}_n\mathfrak{p}_i{}^{\mu\nu}(\tau_i)\delta^4(s_i{}^\rho)d\tau_i. \tag{29}$$

Substituting the above expansions into the field equations (24), we obtain[34]

$$\sum_{n=1}^{\infty}\lambda^n(\eta_{\mu\lambda}\partial_{\rho\sigma}\, {}_nK^{[\lambda\rho][\nu\sigma]}-2\, {}_n\Lambda_\mu{}^\nu)=16\pi G\sum_{n=1}^{\infty}\lambda^n\, {}_n\mathfrak{P}_\mu{}^\nu, \tag{30}$$

$$\begin{aligned}{}_nK^{[\lambda\rho][\nu\sigma]}=[\eta^{\nu\alpha}(\eta^{\lambda\sigma}\eta^{\rho\beta}-\eta^{\lambda\beta}\eta^{\rho\sigma})\\ -\eta^{\sigma\alpha}(\eta^{\lambda\nu}\eta^{\rho\beta}-\eta^{\lambda\beta}\eta^{\rho\nu})]\, {}_n\gamma_{\alpha\beta}.\end{aligned} \tag{31}$$

Here ${}_nK^{[\lambda\rho][\nu\sigma]}$ contains only terms linear in ${}_n\gamma_{\mu\nu}$, whereas ${}_n\Lambda_\mu{}^\nu$ contains only terms nonlinear in the ${}_m\gamma_{\mu\nu}$ of orders $m\leq n-1$. For reasons which will be discussed in detail later in this section, one can not equate the coefficients of equal power of λ in Eq. (30). Rather, one must solve the equations by successive approximations. That is, in the first approximation one solves

$$\eta_{\mu\lambda}\partial_{\rho\sigma}\, {}_1K^{[\lambda\rho][\nu\sigma]}=16\pi G\, {}_1\mathfrak{P}_\mu{}^\nu \tag{32}$$

and in the $(n+1)$st approximation

$$\sum_{l=1}^{n+1}\lambda^l(\eta_{\mu\lambda}\partial_{\rho\sigma}\, {}_lK^{[\lambda\rho][\nu\sigma]}-2\, {}_l\Lambda_\mu{}^\nu)=16\pi G\sum_{l=1}^{n+1}\lambda^l\, {}_l\mathfrak{P}_\mu{}^\nu. \tag{33}$$

We assume that as we proceed from one approximation to the next, the explicit form of the lower order solutions and the explicit form of the lower order matter-tensor are unchanged. Only the implicit dependence on λ is changed; that is, only the world lines are changed. Thus, in the nth approximation the unknowns are ${}_n\gamma_{\mu\nu}$ and ${}_n\mathfrak{p}_i{}^{\rho\sigma}$ and these quantities occur in Eq. (33) linearly. This procedure will be discussed again later.

Equation (32) was first derived by Einstein[30]; it has since been noted by many authors[35] that formally a systematic linearization of Eqs. (18) or (24) can be achieved. It should be noted that the linearization procedure does not require the use of any coordinate condition.

We now assume that the field equations have been solved for all approximations $m\leq n$, i.e. we know ${}_l\gamma_{\mu\nu}$ and ${}_l\mathfrak{p}_i{}^{\rho\sigma}$ for all $l\leq m\leq n$, and wish to solve Eq. (33). This equation implies

$$2\sum_{l=1}^{n+1}\lambda^l\partial_\nu\, {}_l\Lambda_\mu{}^\nu=-16\pi G\sum_{l=1}^{n+1}\lambda^l\partial_\nu\, {}_l\mathfrak{P}_\mu{}^\nu, \tag{34}$$

because of the symmetry properties of ${}_lK^{[\lambda\rho][\nu\sigma]}$. We can consider this equation as an integrability condition for Eq. (33).

On account of the Bianchi identities[24]

$$\partial_\nu\mathfrak{G}_\mu{}^\nu-\tfrac{1}{2}\partial_\mu g_{\rho\sigma}\mathfrak{G}^{\rho\sigma}\equiv 0, \tag{35}$$

we have

$$\sum_{l=1}^{n+1}\lambda^l\partial_\nu\, {}_l\Lambda_\mu{}^\nu=\tfrac{1}{2}\sum_{m=1}^{n}\lambda^m\partial_\mu\, {}_m g_{\rho\sigma}\sum_{l=1}^{n+1-m}\lambda^l\, {}_l\mathfrak{G}^{\rho\sigma}. \tag{36}$$

If Eq. (33) has been solved for all $m\leq n$, it follows from Eq. (24) that to the same approximation

$$\sum_{l=1}^{m}\lambda^l\, {}_l\mathfrak{G}^{\rho\sigma}=-8\pi G\sum_{l=1}^{m}\lambda^l\, {}_l\mathfrak{P}^{\rho\sigma},\quad m\leq n.$$

Substituting this result into Eq. (36) we have in the $(n+1)$st order

$$\sum_{l=1}^{n+1}\lambda^l\partial_\nu\, {}_l\Lambda_\mu{}^\nu=-4\pi G\sum_{m=1}^{n}\lambda^m\partial_\mu\, {}_m g_{\rho\sigma}\sum_{l=1}^{n+1-m}\lambda^l\, {}_l\mathfrak{P}^{\rho\sigma}$$

and thus we can eliminate the ${}_l\Lambda_\mu{}^\nu$'s from the integrability condition (34) and obtain

$$\sum_{l=1}^{n+1}\lambda^l\partial_\nu\, {}_l\mathfrak{P}_\mu{}^\nu-\tfrac{1}{2}\sum_{m=1}^{n}\sum_{l=1}^{n+1-m}\lambda^{l+m}\partial_\mu\, {}_m g_{\rho\sigma}\, {}_l\mathfrak{P}^{\rho\sigma}=0. \tag{37}$$

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[34] The notation used here is suggested by that used in L. Infeld and J. Plebański, reference 11, p. 79.

[35] A probably incomplete list includes: K. Lancius (Lanczos), Z. Physik **13**, 7 (1923); M. Mathisson, *ibid.* **67**, 826 (1931); A. Papapetrou, Proc. Roy. Irish Acad. **52A**, 11 (1948); S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 608 (1952); B. Bertotti, Nuovo cimento **4**, 898 (1956).

This is a condition on the matter tensor in the $(n+1)$st order and the metric in orders n and lower; thus we can satisfy the consistency conditions, i.e. determine the matter tensor and the equations of motion, before the field equations of the $(n+1)$st order have been solved.

In Sec. II we expanded the law of motion and called the equation obtained by keeping the terms up to ${}_n g_{\mu\nu}$ the nth order law of motion. In agreement with this terminology we call Eq. (37) the nth order equation of motion. From the point of view taken in Sec. II it was to be expected that this equation can be obtained without the need for an explicit solution of the $(n+1)$st order equation for the metric.

In order to understand how the matter tensor and the equations of motion are to be obtained in the general case, we shall consider the first and second approximations explicitly. For $n=0$, Eq. (37) becomes simply

$$\partial_\nu \, {}_1\mathfrak{P}_\mu{}^\nu = 0. \tag{38}$$

Following the procedure described in the previous Section, we contract with an arbitrary function ξ^μ and integrate over all space. Substituting from Eq. (29) we obtain

$$\int \eta_{\mu\rho} \, {}_1\mathfrak{p}_i{}^{\rho\nu} \, \partial_\nu \xi^\mu(\tau_i) d\tau_i$$

if ξ^μ and its derivatives are nonvanishing only in the neighborhood of the ith world line. We break up ${}_i\mathfrak{p}^{\rho\sigma}$ into components parallel and perpendicular to $v_i{}^\rho$ as in Eq. (9), except for lowering indices with $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$, and using m_i in place of ${}_1M_i$. Then we get

$${}_1\mathfrak{p}_i{}^{\rho\sigma} = m_i v_i{}^\rho v_i{}^\sigma \tag{39}$$

and

$$\frac{d}{d\tau_i}(m_i v_i{}^\mu) = 0,$$

which implies

$$\dot{m}_i = 0, \quad \dot{v}_i{}^\mu = 0. \tag{40}$$

Thus our zero-order equation of motion agrees with Eq. (22) as it should.

The result (40) has first been derived by Lubański,[36] as the equations of motion of a test particle moving in a Minkowskian background metric. However, it was apparently not realized that these equations were also valid as the lowest order approximation for the general relativistic equations of motion for any number of particles. Furthermore, the general validity of the result was obscured by the use of a coordinate condition, whereas our derivation is independent of such a condition to the order considered. The first-order approximation Eq. (32) implies Eq. (39) and thus (40) as discussed above; Lubański, however, following Einstein,[30] introduced the coordinate condition

$$\eta^{\nu\lambda} \partial_\lambda \, {}_1\gamma_{\mu\nu} = 0 \tag{41}$$

into Eq. (32) to obtain the wave equation[37]

$$\Box \, {}_1\gamma_{\mu\nu} = -16\pi G \eta_{\mu\rho} \eta_{\nu\sigma} \, {}_1\mathfrak{P}^{\rho\sigma}, \tag{42}$$

which he integrated explicitly. He then derived Eq. (40) from the fact that the solution obtained had to satisfy the condition (41).

The implications of Lubański's result appear to have been completely ignored in the literature in spite of their importance for the problem of gravitational radiation. Following Einstein's fundamental paper,[30] most investigations of this problem (both classical and quantum) have been based on Eq. (42). This equation clearly allows wave-like solutions; however, having been obtained from Eq. (32) which implies (38), it also requires the equation of motion (40) for consistency, which state that these waves do not affect the motion of the particles at all. Thus on the basis of the first order approximation to the metric the mathematically permissible gravitational waves imply no physical consequences whatever.[38] This does not necessarily mean that all the conclusions drawn in Einstein's paper and subsequent investigations are wrong; however, they can not be justified without going beyond the first order approximation. We shall return to this question in Sec. V and elsewhere.[39]

Now we consider the second approximation. In passing from the first to the second order we are confronted with a difficulty of the same kind as was encountered in the EIH method.[40] The problem is the following: The equations of motion (40) imply that the singularities move along straight lines and that the first-order metric becomes infinite along these lines. But the first-order metric enters the terms of order λ^2 in Eq. (33) through ${}_2\Lambda_\mu{}^\nu$, and thus these terms also become infinite along the straight lines rather than the lines corresponding to the actual motion of the N particles in the second order (and similarly for higher orders), which does not allow a consistent treatment of the next approximation, as will be shown later.

To avoid this obviously meaningless and inconsistent result, we must devise a method by which we can relax the restrictions imposed by the nth order equations before proceeding to determine the solution of the $(n+1)$st order.

Such a method is suggested by a closer analysis of the meaning of an expansion such as (20) or (30). If the quantity λ were a variable capable of taking arbitrary values, we could rigorously conclude that in any equation the coefficients of all powers of λ had to vanish

[36] J. Lubański, Acta Phys. Polon. **6**, 163 (1937). For a nonsingular matter tensor an equivalent result was already established by C. Lanczos, reference 8.

[37] Lubański did not make use of δ functions and thus wrote zero on the right-hand side of Eq. (52) and similar singular expressions; however, his calculations are in complete accord with the δ formalism.

[38] This has also been noted (independently of Lubański's work) by H. Weyl, Am. J. Math. **66**, 591 (1944), who describes the linear theory as one in which "the gravitational field remains a powerless shadow."

[39] S. F. Smith and P. Havas (to be published).

[40] See especially A. Einstein and L. Infeld, Can. J. of Math. **1**, 209 (1949). The technique used there to overcome the difficulty is not suitable for our purposes.

separately. However, λ is simply a parameter used to group terms of the same order of magnitude, and thus no such conclusion is warranted. "nth approximation" then does not mean that we solve n separate equations involving terms of different orders in λ, but that we solve a single equation containing all the terms up to nth order as well as possible.

This point of view has already been made use of in Sec. II as well as earlier in this section; in considering what information we could obtain from the exact law of motion (17) about the form of the approximate law, we obtain the first-order law (23) by collecting the terms of orders λ^0 and λ^1. From this point of view it is clear that no other procedure is possible. A set of equations of motion (together with suitable initial conditions) determines a motion, i.e., a set of world lines, completely; as there is no meaning in adding world lines of different orders, one can not obtain a new motion by adding results obtained from two different equations of motion, but only by replacing one set of equations of motion by another, such as Eq. (22) by Eq. (23).

Thus, we propose to look for solutions and integrability conditions of successive sets of equations of the form (33), summed up to successive orders. No difficulty is encountered with the integrability conditions, which will be given by Eqs. (34) or (37) summed up to successive orders. However, in attempting to solve Eq. (33) we are now faced with the difficulty that from our present point of view it is no longer linear, because we should now treat all the γ's up to order $n+1$ as unknowns rather than consider those up to order n as determined by the equations of lower order.

We propose to overcome this difficulty as follows. From the first-order equation (32) we concluded that ${}_1\mathfrak{p}_i{}^{\rho\sigma}$ must be of the form (39) with the further restrictions (40) on the quantities entering (39). From our previous considerations we must clearly renounce these restrictions, as these are restrictions placed on the motion; however, we do not necessarily have to renounce the form (39) of the energy-momentum density. Of course, we expect this density to be modified by the second-order equations; but these modifications are by definition included in ${}_2\mathfrak{p}_i{}^{\rho\sigma}$. We proceed similarly in higher orders. At each step we maintain the form of the energy-momentum tensor and thus of the metric as determined by the equations up to a given order, and use these in the equations (37) summed up to the next order to obtain a correction to the energy-momentum tensor and a new set of equations of motion. But while we maintain the *form* of these quantities in terms of the coordinates and their derivatives, these variables will now correspond to the *new* motion rather than the motions following in lower approximations.

Thus for the second approximation, $n=1$, Eq. (37) is

$$\partial_\nu[(\eta_{\mu\rho}+{}_1g_{\mu\rho})\,{}_1\mathfrak{P}^{\rho\nu}+\eta_{\mu\rho}\,{}_2\mathfrak{P}^{\rho\nu}]-\tfrac{1}{2}\partial_\mu\,{}_1g_{\rho\sigma}\,{}_1\mathfrak{P}^{\rho\sigma}=0. \quad (43)$$

We assume that ${}_1g_{\mu\nu}$ is known except for the motion of the particles. That is, we have obtained a solution for Eq. (32) and then we have relaxed the equations of motion (40); however, we maintain (39). Therefore, in Eq. (43) only ${}_2\mathfrak{p}_i{}^{\rho\sigma}$ and the motion are unknown. Again following the procedure of contraction with an appropriate function ξ^μ, integrating over all space, and breaking up ${}_2\mathfrak{p}_i{}^{\rho\sigma}$ as before, we obtain

$${}_2\mathfrak{p}_i{}^{\rho\sigma}={}_2M_i\,v_i{}^\rho v_i{}^\sigma, \quad (44)$$

$$\frac{d}{d\tau_i}[{}_1M_i\,v_i{}^\rho(\eta_{\mu\rho}+{}_1g_{\mu\rho})+{}_2M_i\,v_i{}^\rho\eta_{\rho\mu}]-\tfrac{1}{2}\,{}_1M_i\,v_i{}^\rho v_i{}^\sigma\partial_{i\mu}\,{}_1g_{\rho\sigma}=0. \quad (45)$$

Contracting this with $v_i{}^\mu$ we get

$${}_2\dot{M}_i=-\tfrac{1}{2}\frac{d}{d\tau_i}(m_i\,{}_1g_{\alpha\beta}\,v_i{}^\alpha v_i{}^\beta)$$

provided that we still have ${}_1\dot{M}_i\equiv\dot{m}_i=0$; thus the constancy of m_i is necessary to allow the determination of the form of ${}_2M_i$ from Eq. (45). We get

$${}_2M_i=-\tfrac{1}{2}m_i\,{}_1g_{\alpha\beta}\,v_i{}^\alpha v_i{}^\beta+{}_2C_i, \quad (46)$$

where ${}_2C_i$ is a constant of integration. If there are no difficulties with singularities arising from the self-field, ${}_2C_i$ should be set equal to zero, for it merely would introduce an arbitrary constant without connection with the gravitational interaction. However, we shall see later that by an appropriate choice of ${}_2C_i$ the infinite self-field can be removed so that finite equations of motion result. With (46) the equation of motion (45) becomes

$$m_i\frac{d}{d\tau_i}\{[v_i{}^\rho-(\tfrac{1}{2}\,{}_1g_{\alpha\beta}\,v_i{}^\alpha v_i{}^\beta-{}_2C_i)v_i{}^\rho]\eta_{\mu\rho}+{}_1g_{\mu\rho}\,v_i{}^\rho\}=\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1g_{\alpha\beta}\,v_i{}^\alpha v_i{}^\beta. \quad (47)$$

With ${}_2C_i=0$, this has the form of Eq. (23), obtained by expansion of the law of motion.

We note that our interpretation of the meaning of an expansion in λ was necessary for the derivation of this equation. Had we insisted in taking as our first-order potentials the solutions of Eq. (32) for straight line motion, then all the singularities appearing in Eq. (43) except that due to ${}_2\mathfrak{P}^{\rho\sigma}$ would have been placed on these straight lines. But then we would have had to conclude from Eq. (43) that the terms due to ${}_2\mathfrak{p}^{\rho\sigma}$ lead to an equation of the same form as Eq. (38) and then we would again have deduced Eq. (40). We could then still have derived Eqs. (47), but with all the singularities moving with constant velocity. Thus, they would no longer be differential equations determining the accelerations (which from the derivation would have to vanish), but rather algebraic relations between the positions, which in general would not allow any motion at all.

We now assume that the solution through the nth approximation has been obtained and consider the integrability conditions (37). The same procedure as used previously yields

$$\int\{-\sum_{l=1}^{n+1}{}_l\mathfrak{p}_i{}^{\rho\sigma}(\eta_{\mu\rho}+\sum_{m=1}^{n+1-l}{}_mg_{\mu\rho})\partial_\nu\xi^\mu -\tfrac{1}{2}\sum_{m=1}^{n}\sum_{l=1}^{n+1-m}\partial_{i\mu}\,{}_mg_{\rho\sigma}\,{}_l\mathfrak{p}_i{}^{\rho\sigma}\xi^\mu\}d\tau_i=0.$$

For $l\leq n$, ${}_l\mathfrak{p}_i{}^{\rho\sigma}$ is known. Thus this equation serves to determine ${}_{n+1}\mathfrak{p}_i{}^{\rho\sigma}$ and the nth order equations of motion. By induction one finds that

$$ {}_{n+1}\mathfrak{p}_i{}^{\rho\sigma}={}_{n+1}M_i\,v_i{}^\rho v_i{}^\sigma \tag{48}$$

and

$$\frac{d}{d\tau_i}\{\sum_{l=1}^{n+1}{}_lM_i\,v_i{}^\rho(\eta_{\mu\rho}+\sum_{m=1}^{n+1-l}{}_mg_{\mu\rho})\} =\tfrac{1}{2}\sum_{m=1}^{n}\sum_{l=1}^{n+1-m}{}_lM_i\partial_{i\mu}\,{}_mg_{\rho\sigma}\,v_i{}^\rho v_i{}^\sigma, \tag{49}$$

which are of the same form as the series expansions of Eqs. (13) and (14). The mass corrections ${}_{n+1}M_i$, $n>1$, may be obtained at each step as from Eq. (45), provided we maintain $\dot{m}_i=0$. Then it is not difficult to convince oneself that the resulting expression is the series expansion of the law of motion (16) and (17) except for the intrusion of certain constants of integration similar to ${}_2C_i$ of Eq. (43) which are needed in order to obtain finite equations of motion.

Equation (47), just like Eq. (23), was obtained without the use of any coordinate condition [as was Eq. (49)]. But now it was derived as the integrability condition of Eq. (33), making use of the form (39) of the first-order quantities ${}_1\mathfrak{p}_i{}^{\rho\sigma}$, which in turn determine the first-order metric ${}_1g_{\mu\nu}$ (which enters the equations of motion explicitly) through Eq. (32). It is only the mathematical problem of integrating Eq. (32) which forces us to use a coordinate condition, in exactly the same way in which the problem of integrating a similar system of second order differential equations in electrodynamics forces us to choose a gauge condition for the electromagnetic potentials. In both cases it is natural to make this choice so as to reduce the unmanageable system of equations for several unknown functions to a set of wave equations for a single unknown function, and the natural choice in our case is Einstein's condition (41) which leads to the wave equation (42). More generally, we can reduce Eq. (30) to the inhomogeneous wave equation

$$\sum_{m=1}^{\infty}\lambda^m\{\Box\,{}_m\gamma_{\mu\nu}+(16\pi G\,{}_m\mathfrak{P}_\mu{}^\alpha+2\,{}_m\Lambda_\mu{}^\alpha)\eta_{\nu\alpha}\}=0 \tag{50}$$

by the coordinate condition

$$\sum_{m=1}^{\infty}\lambda^m\eta^{\nu\lambda}\partial_\lambda\,{}_m\gamma_{\mu\nu}=0 \tag{51}$$

because of the form (31) of ${}_mK^{[\lambda\rho][\nu\sigma]}$. This condition has to be understood in the same way as the other expansions discussed before, i.e., in passing from one order of the approximation to the next we relax the condition previously required to order n by requiring it for the sum of all terms up to order $n+1$ instead.[41] In particular, in passing beyond the first-order set (32) we no longer require condition (41) to be satisfied rigorously, which would imply the equations of motion (40) to hold, but relax it to allow solutions of Eq. (42) which do not correspond to straight line motion. These solutions are substituted in ${}_2\Lambda_\mu{}^\nu$ and we can proceed to integrate Eq. (50) for ${}_2\gamma_{\mu\nu}$, the motion (as an integrability condition) being determined by Eq. (47), and similarly for higher orders.

IV. THE FIRST-ORDER METRIC AND THE SELF-ACTION TERMS

Now we turn to the explicit determination of the metric from the wave equation (42). This equation can be integrated with the help of the well known Green's functions. The retarded Green's function is given by

$$\begin{aligned} D_r&=\frac{1}{2\pi}\delta(u^2) \quad \text{for} \quad u^0\geqslant(\sum_{k=1}^{3}u^ku^k)^{\frac{1}{2}} \\ &=\quad 0 \qquad\qquad \text{otherwise,}\end{aligned} \tag{52}$$

where

$$u^\rho\equiv x^\rho-x'^\rho,\quad u^2\equiv\eta_{\rho\sigma}u^\rho u^\sigma. \tag{53}$$

Similarly we have for the advanced Green's function

$$\begin{aligned} D_a&=\frac{1}{2\pi}\delta(u^2) \quad \text{for} \quad u^0>(\sum_{k=1}^{3}u^ku^k)^{\frac{1}{2}} \\ &=\quad 0 \qquad\qquad \text{otherwise,}\end{aligned} \tag{54}$$

and for the symmetric Green's function

$$D_s=\tfrac{1}{2}(D_r+D_a). \tag{55}$$

Just as in the case of electrodynamics we can obtain different solutions of the wave equation by using any one of these functions (or any linear combination of D_r and D_a the sum of whose coefficients is one). In the following we shall carry out all calculations for the retarded and for the symmetric case explicitly and note the modifications necessary for the advanced case.

Performing the integration of Eq. (42) as in electrodynamics,[3] we obtain in the retarded case, taking into

[41] A similar treatment of a coordinate condition has also been suggested by A. Papapetrou, Ann. Physik 20, 399 (1957).

account the form (24) with (39) of the energy-momentum tensor,

$$_1\gamma_{\mu\nu}{}^{\text{ret}}(x^\rho) = -4G\sum_j\left(\frac{m_j\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta}{\kappa_j}\right)_r, \tag{56}$$

where[42]

$$\kappa_j = \eta_{\rho\sigma}s_j{}^\rho v_j{}^\sigma \tag{57}$$

with $s_j{}^\rho$ given by Eq. (2), and where the subscript r indicates that the expression is to be evaluated at the "retarded point" defined by

$$\eta_{\rho\sigma}s_j{}^\rho s_j{}^\sigma = 0, \quad s_j{}^0 > 0. \tag{58}$$

Similarly we get

$$_1\gamma_{\mu\nu}{}^{\text{adv}}(x^\rho) = +4G\sum_j\left(\frac{m_j\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta}{\kappa_j}\right)_a, \tag{59}$$

where the subscript a indicates evaluation at the "advanced point"

$$\eta_{\rho\sigma}s_j{}^\rho s_j{}^\sigma = 0, \quad s_j{}^0 < 0, \tag{60}$$

and thus

$$_1\gamma_{\mu\nu}{}^{\text{sym}}(x^\rho) = -2G\sum_j\left[\left(\frac{m_j\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta}{\kappa_j}\right)_r - \left(\frac{m_j\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta}{\kappa_j}\right)_a\right]. \tag{61}$$

From Eqs. (50) and (61) and the definition (27) we get for the metric

$$_1g_{\mu\nu}{}^{\text{ret}}(x^\rho) = -4G\sum_j\left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta - \frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_r \tag{62}$$

in the retarded case, and

$$_1g_{\mu\nu}{}^{\text{sym}}(x^\rho) = -2G\sum_j\left[\left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta - \frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_r - \left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta - \frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_a\right] \tag{63}$$

in the symmetric case. Equations (56) and (62), or (61) and (63), are now to be substituted into Eq. (47) to obtain the explicit form of the equations of motion in terms of the particle variables in the retarded or symmetric case.

However, all of these quantities become infinite at the positions of the particles. Thus, a simple substitution would lead to meaningless expressions, and we have to investigate the behavior of the field quantities on approaching the particles more closely and remove the infinities in some manner.

In the approximation we are considering we are only dealing with the linear field equation (42) and the mathematical problem of obtaining finite equations is essentially the same as for the special relativistic equations of motion of point particles interacting through nongravitational fields,[3,4,21] and several methods of handling this problem are available. In the general case we are dealing with nonlinear equations and, as noted in Sec. II, no satisfactory method of handling this case exists.

The mathematically most rigorous way of handling the problem in our case of successive approximations of the Eqs. (50) appears to be the method of analytical continuation due to Riesz.[43] This method assures the existence of solutions of any inhomogeneous wave equation which are finite at the positions of the singularities and satisfy all other required conditions.[44] In applying this method to Eqs. (50) we first calculate the first-order "Riesz potential" corresponding to $_1\gamma_{\mu\nu}$, then substitute this into $_2\Lambda_\mu{}^\nu$ and use this to calculate the second order Riesz potential, and similarly for higher orders.[45] This procedure guarantees the existence of finite equations of motion in all orders, but it remains to be proved that the successive approximations obtained by substituting Riesz potentials of lower order to obtain those of higher order indeed converge to the finite potential of the full nonlinear equations, for which no rigorous mathematical treatment exists.

To obtain finite first-order equations of motion, however, we do not need the heavy guns of the still rather unfamiliar Riesz method, but can make use of previous work on the wave equation by Bhabha and Harish-Chandra[46,47] (based on Dirac's method[3]) which provides explicit formulas for the evaluation of the needed terms.

We first note that for any field quantity f

$$f^{\text{ret}} = f^{\text{sym}} + \tfrac{1}{2}(f^{\text{ret}} - f^{\text{adv}}), \quad f^{\text{sym}} \equiv \tfrac{1}{2}(f^{\text{ret}} + f^{\text{adv}}), \tag{64}$$

and similarly for f^{adv}.

All the self-fields of the ith particle are of the form

$$f_i{}^{\text{ret}} = (S_i/\kappa_i)_r, \quad f_i{}^{\text{adv}} = -(S_i/\kappa_i)_a, \tag{65}$$

or derivatives of such terms, and we have to evaluate them for $\kappa_i \to 0$, which we shall indicate by a subscript 0. For these limits we have [reference 47, Eqs. (3.8)

[42] In reference 13 the definition for κ_j was given erroneously with the wrong sign.

[43] M. Riesz, Acta Math. **81**, 1 (1949) and references given there.

[44] For applications of this method to the electromagnetic and the meson case see N. E. Fremberg, Proc. Roy. Soc. (London) **A188**, 18 (1946).

[45] A similar procedure has been used in an analogous problem of equations of motion in the case of interacting electromagnetic and meson fields by Wm. C. Schieve, Lehigh University thesis, 1959 (unpublished); Wm. C. Schieve and P. Havas (to be published).

[46] H. J. Bhabha and Harish Chandra, Proc. Roy. Soc. (London) **A185**, 250 (1946).

[47] Harish Chandra, Proc. Roy. Soc. (London) **A185**, 269 (1946).

and (3.11)]

$$\left[\left(\frac{S_i}{\kappa_i}\right)_r+\left(\frac{S_i}{\kappa_i}\right)_a\right]_0=-2S_{i0},$$

$$\left\{\partial_{i\sigma}\left[\left(\frac{S_i}{\kappa_i}\right)_r+\left(\frac{S_i}{\kappa_i}\right)_a\right]\right\}_0 \tag{66}$$

$$=-\eta_{\rho\sigma}\{\tfrac{2}{3}S_i(\ddot{v}_i{}^\rho-v_i{}^\rho\eta_{\alpha\beta}v_i{}^\alpha\ddot{v}_i{}^\alpha)+2\dot{v}_i{}^\rho\dot{S}_i+2v_i{}^\rho\ddot{S}_i\}_0.$$

Applying these formulas and using the notation

$$\tfrac{1}{2}(f_i{}^{\text{ret}}-f_i{}^{\text{adv}})_0\equiv f_i{}^{\text{rad}}, \tag{67}$$

we obtain

$$\begin{aligned}{}_1g_{i\alpha\beta}{}^{\text{rad}}v_i{}^\alpha v_i{}^\beta=0,\quad {}_1g_{i\mu\rho}{}^{\text{rad}}v_i{}^\rho=4Gm_i\eta_{\mu\rho}\dot{v}_i{}^\rho,\\ \tfrac{1}{2}\partial_{i\mu}\,{}_1g_{i\alpha\beta}{}^{\text{rad}}v_i{}^\alpha v_i{}^\beta=Gm_i\eta_{\mu\rho}[\tfrac{1}{3}(\ddot{v}_i{}^\rho-v_i{}^\rho\eta_{\alpha\beta}v_i{}^\alpha\ddot{v}_i{}^\beta)\\ +4v_i{}^\rho\eta_{\alpha\beta}v_i{}^\alpha\ddot{v}_i{}^\beta],\end{aligned} \tag{68}$$

and thus for the retarded case the radiation terms of Eq. (47) add up to

$$\begin{aligned}&\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1g_{i\alpha\beta}{}^{\text{rad}}v_i{}^\alpha v_i{}^\beta\\ &\quad-m_i\frac{d}{d\tau_i}({}_1g_{i\mu\rho}{}^{\text{rad}}v_i{}^\rho-\tfrac{1}{2}\eta_{\mu\rho}v_i{}^\rho\,{}_1g_{i\alpha\beta}{}^{\text{rad}}v_i{}^\alpha v_i{}^\beta)\\ &\qquad=(11/3)Gm_i{}^2\eta_{\mu\rho}(\ddot{v}_i{}^\rho-v_i{}^\rho\eta_{\alpha\beta}v_i{}^\alpha\ddot{v}_i{}^\beta).\end{aligned} \tag{69}$$

Now we turn to the evaluation of the limit of the first term in Eq. (64),

$$\tfrac{1}{2}(f_i{}^{\text{ret}}+f_i{}^{\text{adv}})\equiv f_i{}^{\text{self}}, \tag{70}$$

which, in contrast to the second term (67), becomes infinite on the ith world line. Using the series (26) of reference 46 we get then, by using the techniques of reference 47 which led to Eq. (66),[48]

$$\begin{aligned}\left[\left(\frac{S_i}{\kappa_i}\right)_r-\left(\frac{S_i}{\kappa_i}\right)_a\right]_0&=\lim_{\epsilon\to0}2\frac{S_i}{\epsilon},\\ \left\{\partial_{i\sigma}\left[\left(\frac{S_i}{\kappa_i}\right)_r-\left(\frac{S_i}{\kappa_i}\right)_a\right]\right\}_0&=\lim_{\epsilon\to0}\frac{S_i\eta_{\rho\sigma}v_i{}^\rho}{\epsilon}.\end{aligned} \tag{70}$$

Thus the divergent self-action terms of Eq. (47) add up to

$$\begin{aligned}&\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1g_{i\alpha\beta}{}^{\text{self}}v_i{}^\alpha v_i{}^\beta\\ &\quad-m_i\frac{d}{d\tau_i}({}_1g_{i\mu\rho}{}^{\text{self}}v_i{}^\rho-\tfrac{1}{2}\eta_{\mu\rho}v_i{}^\rho\,{}_1g_{i\alpha}{}^{\text{self}}v_i{}^\alpha v_i{}^\beta)\\ &\qquad=\lim_{\epsilon\to0}\tfrac{1}{2}Gm_i{}^2\frac{\eta_{\mu\rho}\dot{v}_i{}^\rho}{\epsilon}.\end{aligned} \tag{71}$$

Thus these terms only enter the equations of motion in the form of a constant times the acceleration. But since in our derivation of the equations of motion the quantity ${}_2M_i$, which enters only in the form $d({}_2M_i\eta_{\mu\rho}v_i{}^\rho)/d\tau_i$ [see Eq. (45)], is only determined up to the constant of integration ${}_2C_i$ by Eq. (46), we can compensate for the term (71) by a suitable choice of this constant. We put

$${}_2C_i=\lim_{\epsilon\to0}\tfrac{1}{2}\frac{Gm_i{}^2}{\epsilon}, \tag{72}$$

and then no infinite quantities will be contained in Eq. (47).

Thus both in the retarded and in the symmetric case the finite equations of motion of the ith particle are obtained by omitting the symmetric part of the field of the particle itself in the sums appearing in Eqs. (56), (61), (62), and (63); in the retarded case the remaining radiation part of the field is given by Eq. (69). Therefore we finally have

$$\begin{aligned}&m_i\frac{d}{d\tau_i}[(\eta_{\mu\rho}+{}_1{}^ig_{\mu\rho}{}^{\text{sym}})v_i{}^\rho-\tfrac{1}{2}\eta_{\mu\rho}v_i{}^\rho\,{}_1{}^ig_{\alpha\beta}{}^{\text{sym}}v_i{}^\alpha v_i{}^\beta]=\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1{}^ig_{\alpha\beta}{}^{\text{sym}}v_i{}^\alpha v_i{}^\beta,\\ &{}_1{}^ig_{\mu\nu}{}^{\text{sym}}=-2G\sum_{j\neq i}\left[\left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta-\frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_r-\left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta-\frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_a\right]\end{aligned} \tag{73}$$

as the equations of motion in the symmetric case, and

$$\begin{aligned}&m_i\frac{d}{d\tau_i}[(\eta_{\mu\rho}+{}_1{}^ig_{\mu\rho}{}^{\text{ret}})v_i{}^\rho-\tfrac{1}{2}\eta_{\mu\rho}v_i{}^\rho\,{}_1{}^ig_{\alpha\beta}{}^{\text{ret}}v_i{}^\alpha v_i{}^\beta]=\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1{}^ig_{\alpha\beta}{}^{\text{ret}}v_i{}^\alpha v_i{}^\beta-\frac{11}{3}Gm_i{}^2\eta_{\mu\rho}(\ddot{v}_i{}^\rho-v_i{}^\rho\eta_{\alpha\beta}v_i{}^\alpha\ddot{v}_i{}^\beta),\\ &{}_1{}^ig_{\mu\nu}{}^{\text{ret}}=-4G\sum_{j\neq i}\left(\frac{m_j(\eta_{\mu\alpha}\eta_{\nu\beta}v_j{}^\alpha v_j{}^\beta-\frac{1}{2}\eta_{\mu\nu})}{\kappa_j}\right)_r\end{aligned} \tag{74}$$

in the retarded case.[49] In the advanced case we similarly get an equation of the form (74), with ${}_1{}^ig_{\mu\nu}{}^{\text{ret}}$ replaced by

[48] These expressions were not given explicitly in references 46 or 47 because it had already been established in general by the authors that all singular terms can be eliminated. The elimination is carried out here explicitly for the benefit of readers unfamiliar with the special relativistic techniques involved. A use of the Riesz potentials discussed earlier would also make unnecessary the explicit consideration of infinite self-action terms, or of nonzero ${}_nC_i$'s.

[49] In the equivalent Eq. (10) of reference 13 the radiation term was erroneously given with the wrong sign.

${}_1{}^i g_{\mu\nu}{}^{\mathrm{adv}}$, and with the opposite sign of the radiation reaction term. The superscripts i in Eqs. (73) and (74) indicate that the field of the ith particle is excluded.

Just as for the special relativistic equations of motion of point charges originally considered by Wheeler and Feynman[50] and of point singularities interacting through meson fields[51] it is possible to pass from the equations of the symmetric case to those of the retarded (or advanced) case. The details of this demonstration, as well as some considerations on the significance and limitations of the result, will be found in Appendix II.

Just as we have shown above that from the point of view of field theory the field equations imply the equations of motion, we shall now show that it is possible to formulate the theory in terms of particle variables only, but nevertheless to imply a set of field equations in addition to the equations of motion. For this purpose we start from a Fokker-type variational principle and proceed in complete analogy to electro-[52,53] and mesodynamics[51]; as in those theories we are generally limited to the symmetric case. We consider

$$J \equiv \sum_{j=1}^{N} m_j \int_{-\infty}^{\infty} (\eta_{\alpha\beta} v_j{}^\alpha v_j{}^\beta)^{\frac{1}{2}} d\tau_j - \sum_{j<k}^{N} G m_j m_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_s[\eta_{\kappa\lambda}(z_j{}^\kappa - z_k{}^\kappa)(z_j{}^\lambda - z_k{}^\lambda)] \times \frac{\eta_{\mu\alpha}\eta_{\nu\beta} v_j{}^\alpha v_j{}^\beta v_k{}^\mu v_k{}^\nu - \frac{1}{2}\eta_{\alpha\beta} v_j{}^\alpha v_j{}^\beta - \frac{1}{2}\eta_{\alpha\beta} v_k{}^\alpha v_k{}^\beta + \frac{1}{2}}{(\eta_{\gamma\delta} v_j{}^\gamma v_j{}^\delta)^{\frac{1}{2}} (\eta_{\rho\sigma} v_k{}^\rho v_k{}^\sigma)^{\frac{1}{2}}} d\tau_j d\tau_k = \text{extremum}. \tag{75}$$

Variation of J with respect to the coordinates $z_i{}^\mu$ of the ith particle leads us by well-known procedures[53,51] to Eq. (73). To obtain "field equations" we define the "adjunct"[50] field quantities

$${}_1 g_{j\mu\nu}{}^{\mathrm{sym}}(x^\rho) = -4Gm_j \int_{-\infty}^{\infty} (\eta_{\mu\alpha}\eta_{\nu\beta} v_j{}^\alpha v_j{}^\beta - \tfrac{1}{2}\eta_{\mu\nu}) D_s(\eta_{\kappa\lambda} s_j{}^\kappa s_j{}^\lambda) d\tau_j \tag{76}$$

with $s_j{}^\rho$ defined by Eq. (2), and note that[3]

$$\Box D_s(\eta_{\kappa\lambda} s_j{}^\kappa s_j{}^\lambda) = 4\pi \delta^4(s_j{}^\rho). \tag{77}$$

Multiplying both sides by $Gm_j(\eta_{\mu\alpha}\eta_{\nu\beta} v_j{}^\alpha v_j{}^\beta - \frac{1}{2}\eta_{\mu\nu}) d\tau_j$ and integrating from $-\infty$ to ∞ we obtain

$$\Box\, {}_1 g_{j\mu\nu}{}^{\mathrm{sym}} = -16\pi G m_j \int_{-\infty}^{\infty} (\eta_{\mu\alpha}\eta_{\nu\beta} v_j{}^\alpha v_j{}^\beta - \tfrac{1}{2}\eta_{\mu\nu}) \delta^4(s_j{}^\rho) d\tau_j, \tag{78}$$

which from the definitions (27) and (29) and Eq. (39) is equivalent to Eq. (42), with the source due to the jth particle only. Our previous ${}_1{}^i g_{\mu\nu}{}^{\mathrm{sym}}$ equals $\sum_{j\neq i} {}_1 g_{j\mu\nu}{}^{\mathrm{sym}}$ on the ith world line.

Just as in electro- and mesodynamics, we could also define an energy-momentum tensor of the "field" in terms of particle variables alone with the help of the quantities (76), and obtain detailed conservation of energy-momentum as in field theory.[53,51] Possible explicit forms will be discussed elsewhere.[39]

V. DISCUSSION

The equations of motion (73) of the symmetric, (74) of the retarded, and the analogous one of the advanced case have the features familiar from special relativity,[3,4] especially from electrodynamics. The fields acting on the particles are determined not by the simultaneous positions of the other particles, but by the retarded (or advanced) ones; these latter positions are not known initially, but have to be determined in the process of solving the equations for the actual motion, and the retarded and advanced fields of the particles which depend on their velocities and accelerations are those corresponding to this motion. Furthermore, the equations (except in the symmetric case) contain radiation reaction terms which differ from those of electrodynamics only by a constant factor.

This structure of the equations follows necessarily from our derivation, in particular from our handling of the expansion in λ as implying that the field equations had to be solved up to a given order rather than in each order independently. However, they differ from the results of other authors[54–57] who attempted to obtain Lorentz-invariant equations of motion, in one or both of the features discussed above.

In particular, the equations of Bertotti[55] and Geissler[56] are of the general form (47), but using expressions ${}_1 g_{\alpha\beta}$ corresponding to the first-order motion with constant velocity, and similarly putting all accelerations appearing in the derivative on the left-hand side of Eq. (47) equal to zero except the one appearing with the factor m_i only. But such a procedure amounts to allow-

[50] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).
[51] P. Havas, Phys. Rev. **87**, 309 (1952); **91**, 997 (1953).
[52] A. D. Fokker, Z. Physik **58**, 386 (1929).
[53] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **21**, 425 (1949).
[54] F. J. Belinfante, Phys. Rev. **89**, 914 (1953).
[55] B. Bertotti, Nuovo cimento **4**, 898 (1956).
[56] D. Geissler, Z. Naturforsch. **14a**, 689 (1959).
[57] R. P. Kerr, Nuovo cimento **13**, 469, 492, and 693 (1959).

ing the metric to become infinite on two different sets of world lines (of the first and second approximation, respectively), and is mathematically inconsistent.

Belinfante's purpose in considering certain equations of motion[54] was to obtain a linear special relativistic theory of gravitation which is able to reproduce the three Einstein effects of general relativity. He started from a Lagrangian containing a large number of adjustable constants. For a certain choice of these, the equations of motion of the form (47) with the field equations (42) follow. However, Belinfante explicitly omits self-action terms in his considerations. For a consideration of the Einstein effects this is entirely justified, but for an application to higher order problems this would lead to inconsistencies. Such terms can easily be included within the framework of his considerations, however, with results entirely consistent with those of this paper.[58]

Radiation damping terms are also omitted by Kerr[57] as being of higher order. On describing the approximation method in Sec. III we discussed why consistency requires that one solve the field equations up to a given order and then determine the equations of motion which follow from the integrability conditions of the next order solution. In this procedure one no longer has the privilege of omitting terms "of higher order" although it is true that one must be careful in their physical interpretation. To argue that additional terms of a similar nature will arise from higher order solutions does not alter the fact that radiation damping terms appear already in the second order.

It should be noted particularly that terms of the same order as the explicit radiation reaction terms appear implicitly in the equations of motion (74) through the retardation effects in the metric; they appear with opposite sign and cancel the main contribution of the explicit terms (thereby assuring that in low-velocity approximation the main radiation term is of quadrupole rather than dipole character). Thus omission of the explicitly appearing terms introduces an error of lower order than their retention could possibly have.[59]

In addition to this point, Kerr's paper differs from ours by his attempt to avoid the use of δ functions, which he considers to be mathematically and physically unsatisfactory, in his derivation. The relation between the methods of obtaining equations of motion using extended sources and singularities, respectively, has been extensively discussed in the literature[60] and we shall not take up this question here again.

In a series of papers whose arguments are summarized in his book with Plebański,[10] Infeld has argued that gravitational radiation does not occur. However, in his argument the choice of coordinate conditions, and hence boundary conditions, plays an important role. Instead of Eq. (51) Infeld takes

$$\sum \lambda^n \partial_\mu \, {}_n\gamma^{o\mu}=0, \quad \sum \lambda^n \partial_l \, {}_n\gamma^{kl}=0, \quad k, l=1, 2, 3$$

as his coordinate condition. The requirement that the field fall off as $1/r$ for large spatial distances leads to an inconsistency unless the radiation term vanishes, as otherwise logarithmic terms appear in higher orders. This situation does not occur with the coordinate conditions used in this paper, as Infeld remarks. It is easy to show that in the presence of radiation the transformation leading from the conditions (51) to those used by Infeld involves logarithmic terms. Although there may be a wide choice of coordinate conditions, one must be careful that the choice does not throw out physically interesting and important solutions. In the absence of experimental evidence to fix the boundary conditions it is necessary to examine under what conditions gravitational radiation effects may be expected. If experiment tells us that these conditions can not be satisfied, then certainly Infeld's argument is correct; it is of course compatible with the choice of the symmetric equations of motion (73).

Other mathematical arguments concerning the restrictions imposed on the solutions of the field equations in higher order by requiring the boundary condition $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ have been raised by several authors.[41,61] Recent work[62] on the asymptotic metric in the presence of a radiation field refutes the argument that the field necessarily diverges.

As discussed in Sec. IV, the method of Riesz potentials guarantees that the metric will be finite at the location of the point particles in all orders. Thus it is possible to write down the ${}_mg_{\mu\nu}$'s appearing in our nth order equations of motion (49) in the form of integrals which are known to be finite, and in a sense we have thus solved the problem of obtaining finite Lorentz-invariant equations of motion in all orders. However, it is desirable to exhibit the radiation reaction terms explicitly as functions of the particle variables. Unfortunately, the integrals are so complicated[45] that an explicit evaluation of the radiation reaction terms has so far not been possible beyond the second order. A different approach to the integration of the higher order inhomogeneous wave equations have been taken by Bertotti and Plebański,[63] who studied in detail the structure of a set of generalized Green's functions. However, they were not able to eliminate the infinite self-action terms arising in their method.

The method used in this paper to obtain Lorentz-

[58] P. Havas, Bull. Am. Phys. Soc. 6, 346 (1961). A detailed account is in preparation.

[59] This point will be discussed in detail in reference 39.

[60] For recent discussions see, e.g., L. Infeld and J. Plebański, reference 10, or T. H. Pham, Nuovo cimento 9, 647 (1958), and references given there.

[61] A. Papapetrou, Ann. Physik 1, 186 (1958); 2, 87 (1958); A. Peres and N. Rosen, Phys. Rev. 115, 1085 (1959).

[62] H. Bondi, Nature 186, 535 (1960); H. Bondi, A. W. K. Metzner, and M. G. J. Van der Burg, Proc. Roy. Soc. (London) (to be published); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 121, 1556 (1961).

[63] B. Bertotti and J. Plebański, Ann. Phys. (New York) 11, 169 (1960).

invariant equations of motion of simple poles of the gravitational field can easily be generalized to include particles with intrinsic angular momentum, and nongravitational fields. Some results were mentioned in reference 13, and detailed papers on this subject are in preparation.

The presentation of the application of the equations to specific problems has also been left to other papers. We shall, however, briefly discuss some simple applications and the result of a calculation of the two-body problem.[13,39,64] First of all it is clear that our equations for very low velocities and accelerations reduce to Newton's equations (regardless of the choice of retarded, symmetric, or any other Green's functions). If, using the same equations, we carry our calculations to the order v^2/c^2 (again regardless of the Green's function chosen), we obtain equations which for the two-body problem give an advance of the periastron of 7/6 of the Einstein value both in the test particle limit[12,54] and for comparable masses.[64,39] Although it may be disappointing that we do not obtain the exact value in the test particle limit (the exact value for comparable masses is not known and thus the correctness of the EIH result[65] is not established), it should be noted that we do obtain an advance of almost the right magnitude in our *first* nontrivial order of approximation rather than in the *second* nontrivial one with the EIH method (and that, from the meaning of our expansion parameter, our method is better suited for the case of comparable masses than for the test particle limit). In comparing the relative merits of the method presented here and that of EIH in the calculation of specific results this should be kept in mind. Similarly it should be kept in mind that the EIH method was specifically designed for the treatment of problems involving low relative velocities, and thus can be expected to give more accurate answers than the Lorentz-invariant one in this range; the latter method, however, having been designed specifically for the case of high relative velocities, can be expected to be better suited for that range. In this sense we believe the two methods to be fully compatible, and can not agree with some points raised by Infeld.[66]

Carrying the calculation of the two-body problem beyond v^2/c^2, we have to distinguish between the results of different Green's functions. There are no radiation effects in the symmetric case. In the retarded case, a detailed investigation of the two-body system[64,39] shows that there is an apparent energy gain of the system, rather than a loss as expected from electrodynamics.[67,68] The significance of this result is not clear, but it appears that this means that we have pushed the limits of applicability of our second-order equations too far. This is the more likely because the energy loss is not, as may be thought at first sight, a simple consequence of the sign of the radiation reaction term in Eq. (74), which is opposite to that of the electrodynamic one. Rather it follows from the sign of the small difference between this self-action term and a similar term originating in the retarded field of the other particle (which we discussed above). Unfortunately it is impossible to separate out this difference in a Lorentz-invariant fashion, and we are thus forced to keep the full equation of the form (74), and to await the calculation of the results of the higher order equations.

To understand the significance of the apparent energy gain, similar calculations have been undertaken for a class of linear theories of gravitation.[58] It turned out that all linear theories derivable from a Lagrangian which give the correct advance of the perihelion in the test particle limit also give an energy gain. Thus it appears that there either is really an energy gain in the retarded case, or that it is not possible to get meaningful results about gravitational radiation on the basis of any equations resembling the linear equations familiar from electrodynamics and other special relativistic field theories.

We wish to acknowledge numerous discussions with colleagues on various aspects of the work in the course of the preparation of this paper, particularly in connection with the final section. We also wish to acknowledge the hospitality extended us at the Institutes of Theoretical Physics of Copenhagen and Stockholm in the final stages of the preparation of the manuscript.

APPENDIX I. ALTERNATIVE DERIVATIONS

The reason we chose to start from the form (24) of the Einstein equation rather than from the form (18) was that the former allowed a more convenient use of the Bianchi identities [to go from Eq. (34) to Eq. (37)] than the latter. We now wish to discuss some alternate derivations based on Eq. (18). We expand the metric (20) as before, but now expand the tensors $P_{\mu\nu}$ and $G_{\mu\nu}$ rather than $\mathfrak{P}^{\mu\nu}$ and $\mathfrak{G}^{\mu\nu}$. We put

$$P_{\mu\nu}=\sum_{n=1}^{\infty}\lambda^n\,{}_nP_{\mu\nu},\quad p_{i\mu\nu}=\sum_{n=1}^{\infty}\lambda^n\,{}_np_{i\mu\nu}(\tau_i),$$
$$G_{\mu\nu}=\sum_{n=1}^{\infty}\lambda^n\,{}_nG_{\mu\nu}\equiv\sum_{n=1}^{\infty}\lambda^n\tfrac{1}{2}({}_nL_{\mu\nu}-{}_nU_{\mu\nu}),\tag{79}$$

and Eq. (18) becomes

$$\sum_{n=1}^{\infty}\lambda^n({}_nL_{\mu\nu}+16\pi G\,{}_nP_{\mu\nu}-{}_nU_{\mu\nu})=0.\tag{80}$$

[64] S. F. Smith, Lehigh University thesis, 1960 (unpublished); S. F. Smith and P. Havas, Bull. Am. Phys. Soc. **5**, 53 (1960).

[65] H. P. Robertson, Ann. Math. **39**, 101 (1938); the same result is contained in a paper by A. S. Eddington and G. L. Clark, Proc. Roy. Soc. (London) **A166**, (1938), who had independently obtained the EIH equations by correcting an error in the calculations of de Sitter, reference 16.

[66] L. Infeld, Bull. acad. polon. sci., Classe III, **9**, 93 (1961).

[67] The relation of this result to the various conflicting slow motion calculations will be discussed in reference 39.

[68] Conversely, there is an energy loss in the advanced case, and the possibility that in the case of gravitational radiation we have to choose the advanced equations will have to be considered.

Here, the functions

$$ {}_nL_{\mu\nu}\equiv\tfrac{1}{2}\eta^{\rho\sigma}[\partial_\nu(\partial_\mu\,{}_ng_{\sigma\rho}-\partial_\sigma\,{}_ng_{\mu\rho})-\partial_\rho(\partial_\mu\,{}_ng_{\sigma\nu}-\partial_\sigma\,{}_ng_{\mu\nu})-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\eta^{\rho\sigma}[\partial_\beta(\partial_\alpha\,{}_ng_{\sigma\rho}-\partial_\sigma\,{}_ng_{\alpha\rho})-\partial_\rho(\partial_\alpha\,{}_ng_{\sigma\beta}-\partial_\sigma\,{}_ng_{\alpha\beta})] \quad (81) $$

are linear in the ${}_ng_{\alpha\beta}$'s. The nth order functions ${}_nU_{\mu\nu}$ only involve $g_{\alpha\beta}$'s of lower order. In particular we have ${}_1U_{\mu\nu}=0$ and

$$ \begin{aligned} {}_2U_{\mu\nu}=\ &\partial_\nu({}_1g_{\rho\sigma}\,\eta^{\rho\alpha}\eta^{\sigma\beta}\partial_\mu\,{}_1g_{\alpha\beta})-\partial_\rho[\eta^{\rho\alpha}\eta^{\sigma\beta}\,{}_1g_{\alpha\beta}(\partial_\mu\,{}_1g_{\sigma\nu}+\partial_\nu\,{}_1g_{\sigma\mu}-\partial_\sigma\,{}_1g_{\mu\nu})] \\ &-\tfrac{1}{2}\eta^{\rho\alpha}(\partial_\sigma\,{}_1g_{\rho\mu}+\partial_\mu\,{}_1g_{\rho\sigma}-\partial_\rho\,{}_1g_{\mu\sigma})[\eta^{\sigma\beta}(\partial_\alpha\,{}_1g_{\beta\nu}+\partial_\nu\,{}_1g_{\alpha\beta}-\partial_\beta\,{}_1g_{\alpha\nu})-\delta_\nu{}^\sigma\partial_\alpha\,{}_1g_{\kappa\lambda}\eta^{\kappa\lambda}] \\ &+\tfrac{1}{2}{}_1g_{\mu\nu}[\eta^{\rho\sigma}\Box\,{}_1g_{\rho\sigma}-2\eta^{\rho\alpha}\eta^{\beta\sigma}\partial_{\rho\sigma}({}_1g_{\alpha\beta}-\tfrac{1}{2}\eta_{\alpha\beta}\eta^{\kappa\lambda}\,{}_1g_{\kappa\lambda})] \\ &+\tfrac{1}{2}\eta_{\mu\nu}\{\eta^{\rho\alpha}\eta^{\sigma\beta}\,{}_1g_{\alpha\beta}[2\eta^{\tau\gamma}\partial_{\gamma\sigma}({}_1g_{\rho\tau}-\tfrac{1}{2}\eta_{\rho\tau}\eta^{\kappa\lambda}\,{}_1g_{\kappa\lambda})-\Box\,{}_1g_{\rho\sigma}] \\ &-\eta^{\kappa\lambda}\partial_\kappa({}_1g_{\rho\sigma}\eta^{\rho\alpha}\eta^{\sigma\beta}\partial_\lambda\,{}_1g_{\alpha\beta})+2\partial_\gamma[\eta^{\alpha\rho}\eta^{\gamma\sigma}\,{}_1g_{\rho\sigma}\eta^{\beta\tau}\partial_\tau({}_1g_{\alpha\beta}-\tfrac{1}{2}\eta_{\alpha\beta}\eta^{\kappa\lambda}\,{}_1g_{\kappa\lambda})] \\ &+\tfrac{1}{2}\eta^{\rho\beta}(\partial_\alpha\,{}_1g_{\beta\gamma}+\partial_\gamma\,{}_1g_{\alpha\beta}-\partial_\beta\,{}_1g_{\alpha\gamma})\eta^{\sigma\alpha}\eta^{\tau\gamma}(\partial_\rho\,{}_1g_{\sigma\tau}+\partial_\tau\,{}_1g_{\rho\sigma}-\partial_\sigma\,{}_1g_{\rho\tau}) \\ &-2\eta^{\gamma\delta}\partial_\rho\,{}_1g_{\gamma\delta}\eta^{\rho\alpha}\eta^{\sigma\beta}\partial_\sigma({}_1g_{\alpha\beta}-\tfrac{1}{2}\eta_{\alpha\beta}\eta^{\kappa\lambda}\,{}_1g_{\kappa\lambda})\}. \end{aligned} \quad (82) $$

We note that we have identically

$$ \eta^{\nu\lambda}\partial_\lambda\,{}_nL_{\mu\nu}=0 \quad (83) $$

and thus Eq. (80) implies

$$ \textstyle\sum_m \lambda^m\eta^{\nu\lambda}\partial_\lambda(16\pi G\,{}_mP_{\mu\nu}-{}_mU_{\mu\nu})=0. \quad (84) $$

In the original derivation of the equations of motion (73) and (74) (sketched, but not given in detail in reference 13) the method of Lubański[37] was used without essential modification; we introduced the coordinate condition (51) and thus obtained the wave equations

$$ \Box\,{}_1\gamma_{\mu\nu}=-16\pi G\,{}_1P_{\mu\nu} $$

and

$$ \sum_{m=1}^{2}(\Box\,{}_m\gamma_{\mu\nu}+16\pi G\,{}_mP_{\mu\nu}-{}_mU_{\mu\nu})=0 \quad (85) $$

in first and second order. Equations (85) were integrated by means of the different Green's functions of Sec. IV and then the solutions were subjected to the coordinate condition (51), which led by Lubański's techniques after somewhat cumbersome calculations to the second-order equations of motion as conditions on ${}_2P_{\mu\nu}$.

A much simpler derivation was obtained later by a modification of Lubański's method, sketched in references 4 and 64. This was actually a special case of Mathisson's method as used in this paper, use being made of a special function [the retarded Green's function D_r defined by Eq. (52)] instead of the arbitrary function ξ^μ of Eq. (5); this method no longer required an integration of Eq. (85), but made use of Eq. (84) only. In first order in λ this becomes

$$ \eta^{\nu\lambda}\partial_\lambda\,{}_1P_{\mu\nu}=0, \quad (86) $$

and up to λ^2 it equals

$$ \eta^{\nu\lambda}\partial_\lambda[16\pi G({}_1P_{\mu\nu}+{}_2P_{\mu\nu})-{}_2U_{\mu\nu}]=0; \quad (87) $$

carrying out the last differentiation in (87) using Eq. (82) one obtains after a rather lengthy calculation

$$ \eta^{\nu\lambda}\partial_\lambda({}_1P_{\mu\nu}+{}_2P_{\mu\nu}-\eta^{\rho\sigma}\,{}_1\gamma_{\sigma\nu}\,{}_1P_{\mu\rho})-\tfrac{1}{2}\eta^{\nu\lambda}\eta^{\rho\sigma}\partial_\mu\,{}_1g_{\sigma\lambda}\,{}_1P_{\nu\rho}=0. \quad (88) $$

It is this lengthy calculation which is avoided by the use of the Bianchi identities in going from Eq. (34) to (37), but while a corresponding treatment of Eq. (87) is also possible, it is more awkward.

We shall not carry out the modified Lubański method here, but proceed with Mathisson's method as in Sec. III. From Eq. (86) we obtain easily

$$ {}_1p_{i\mu\nu}=\eta_{\mu\alpha}\eta_{\nu\beta}m_iv_i{}^\alpha v_i{}^\beta. \quad (89) $$

From Eq. (88) we get

$$ \int\{-\eta^{\nu\lambda}({}_1p_{i\mu\nu}+{}_2p_{i\mu\nu}-\eta^{\rho\sigma}\,{}_1\gamma_{\sigma\nu}\,{}_1p_{i\mu\rho})\partial_{i\lambda}\xi^\mu-\tfrac{1}{2}\eta^{\nu\lambda}\eta^{\rho\sigma}\partial_{i\mu}\,{}_1g_{\sigma\lambda}\,{}_1p_{i\mu\rho}\,\xi^\mu\}d\tau_i=0. \quad (90) $$

Here we substitute (89) and break up ${}_2p_{i\mu\nu}$ as in Sec. III:

$$ \begin{aligned} {}_2p_{i\mu\nu}&={}_2M_i\,\eta_{\mu\alpha}\eta_{\nu\beta}v_i{}^\alpha v_i{}^\beta+{}_2n_{i\mu}\,\eta_{\nu\alpha}v_i{}^\alpha+{}_2n_{i\nu}\,\eta_{\mu\alpha}v_i{}^\alpha+{}_2{}^*p_{i\mu\nu}, \quad (91) \\ {}_2{}^*p_{i\mu\nu}&={}_2{}^*p_{i\nu\mu},\quad {}_2{}^*p_{i\mu\nu}\,v_i{}^\nu=0,\quad {}_2n_{i\mu}\,v_i{}^\mu=0. \end{aligned} $$

We must similarly break up the third term in Eq. (90):

$$ \begin{aligned} -m_i\eta^{\rho\sigma}\,{}_1\gamma_{\sigma\nu}\,\eta_{\mu\alpha}\eta_{\rho\beta}v_i{}^\alpha v_i{}^\beta&=\eta_{\nu\alpha}v_i{}^\alpha\,{}_2Q_{i\mu}+{}_2{}^*Q_{i\mu}, \\ {}_2{}^*Q_{i\mu\nu}\,v_i{}^\nu&=0. \end{aligned} \quad (92) $$

Contracting this with $v_i{}^\nu$ we obtain

$$ \begin{aligned} {}_2Q_{i\mu}&=-m_i\,{}_1\gamma_{\beta\nu}\,v_i{}^\nu\eta_{\mu\alpha}v_i{}^\alpha v_i{}^\beta, \\ {}_2{}^*Q_{i\mu\nu}&=m_i\eta_{\mu\alpha}v_i{}^\alpha v_i{}^\beta(\eta_{\nu\lambda}v_i{}^\lambda\,{}_1\gamma_{\beta\rho}\,v_i{}^\rho-{}_1\gamma_{\beta\nu}). \end{aligned} \quad (93) $$

Now we can transfer the terms involving $v_i{}^\lambda\partial_{i\lambda}$ by an integration by parts to obtain

$$ \sum_i\int\Big\{-\eta^{\nu\lambda}({}_2{}^*p_{i\mu\nu}+{}_2n_{i\nu}\,\eta_{\mu\alpha}v_i{}^\alpha+{}_2{}^*Q_{i\mu\nu})\partial_{i\lambda}\xi^\mu+\frac{d}{d\tau_i}[(m_i+{}_2M_i)\eta_{\mu\alpha}v_i{}^\alpha+{}_2n_{i\mu}+{}_2Q_{i\mu}]\xi^\mu-\tfrac{1}{2}m_i\partial_{i\mu}\,{}_1g_{\alpha\beta}\,v_i{}^\alpha v_i{}^\beta\xi^\mu\Big\}d\tau_i=0. \quad (94) $$

Proceeding as before, we conclude from this that

$$ {}_2n_i = {}_2{}^*Q_{i\mu\nu}\, v_i{}^{\mu} = m_i v_i{}^{\beta}({}_1\gamma_{\beta\nu} - \eta_{\nu\lambda} v_i{}^{\lambda}\, {}_1\gamma_{\beta\rho} v_i{}^{\rho}), \qquad {}_2{}^*p_{i\mu\nu} = 0 \tag{95} $$

and

$$ \frac{d}{d\tau_i}[(m_i + {}_2M_i)\eta_{\mu\alpha} v_i{}^{\alpha} + {}_2n_{i\mu} + {}_2Q_{i\mu}] - \tfrac{1}{2} m_i \partial_{i\mu}\, {}_1g_{\alpha\beta}\, v_i = 0. \quad {}^{\alpha}v_i{}^{\beta} \tag{96} $$

Substituting for ${}_2Q_{i\mu}$ and ${}_2n_{i\mu}$ from Eqs. (93) and (95) and then contracting with $v_i{}^{\mu}$ we get

$$ \frac{d}{d\tau_i}(m_i + {}_2M_i - m_i\, {}_1\gamma_{\alpha\beta}\, v_i{}^{\alpha} v_i{}^{\beta}) - m_i\, {}_1\gamma_{\alpha\beta}\, v_i{}^{\alpha} \dot{v}_i{}^{\beta} - \tfrac{1}{2} m_i \partial_{i\mu}\, {}_1g_{\beta}\, v_i{}^{\alpha} v_i{}^{\beta} = 0. \tag{97} $$

This equation for ${}_2M_i$ is again integrable provided $\dot{m}_i = 0$. Then it can be written

$$ {}_2\dot{M}_i = \frac{d}{d\tau_i}(\tfrac{3}{2} m_i {}_1\gamma_{\alpha\beta}\, v_i{}^{\alpha} v_i{}^{\beta} + \tfrac{1}{4} m_i \eta^{\alpha\beta}\, {}_1g_{\alpha\beta}), \tag{98} $$

and thus

$$ {}_2M_i = \tfrac{3}{2} m_i\, {}_1\gamma_{\alpha\beta}\, v_i{}^{\alpha} v_i{}^{\beta} + \tfrac{1}{4} m_i \eta^{\alpha\beta}\, {}_1g_{\alpha\beta} + {}_2C_i. \tag{99} $$

Substituting this back into Eq. (96) we again obtain Eq. (47). The rest of the derivation is identical with that of Sec. IV.

It should be noted that the successive terms obtained by expanding $P_{\mu\nu}$ are more complicated than those resulting from the expansion of $\mathfrak{P}^{\mu\nu}$ and do not reduce to the simple form (48); this is of course to be expected from the considerations of Sec. II. Nevertheless, again in accordance with Sec. II, we obtain the same equations of motion.

Whichever form of Einstein's equations is taken as the starting point, it is essential to expand the covariant $g_{\mu\nu}$ and break up the energy-momentum tensor with respect to the contravariant $v_i{}^{\rho}$. This procedure leads to expressions for the ${}_n\dot{M}_{i\mu\nu}$ which are immediately integrable. Other procedures do not lead to such perfect differentials directly. It should also be noted that regardless of our starting point, Eq. (51) suggests itself naturally as the appropriate coordinate condition.

APPENDIX II. REMARKS ON WHEELER-FEYNMAN TYPE CONSIDERATIONS

We first wish to show that, subject to considerations analogous to those of Wheeler and Feynman in electrodynamics, we can pass from the equations of motion of the symmetric case (73) to those of the retarded or advanced case. From Eq. (64) we have

$$ \sum_{j\neq i} f_j{}^{\mathrm{sym}} = \sum_{j\neq i} f_j{}^{\mathrm{ret}} - \tfrac{1}{2}\sum_j (f_j{}^{\mathrm{ret}} - f_j{}^{\mathrm{adv}}) + \tfrac{1}{2}(f_i{}^{\mathrm{ret}} - f_i{}^{\mathrm{adv}}). \tag{100} $$

The Wheeler-Feynman condition [reference 50, Eq. (37)] requires

$$ \textstyle\sum_j f_j{}^{\mathrm{ret}} = \sum_j f_j{}^{\mathrm{adv}} \tag{101} $$

and thus we have on the world line of the ith particle, using the notations of Eqs. (67), (73), and (74),

$$ {}^i g^{\mathrm{sym}} = {}^i g^{\mathrm{ret}} + g_i{}^{\mathrm{rad}}. \tag{102} $$

Therefore, Eq. (73) together with condition (101) is equivalent to Eq. (74) of the retarded case, again subject to condition (101)[69,70]; by a similar argument it can be shown to be equivalent to the equation of the advanced case together with Eq. (101). An argument of statistical nature is required to discriminate between these two alternatives.

The question naturally arises whether in higher orders of our approximation method we can still expect a multiplicity of solutions for the metric (corresponding to retarded and advanced fields and their combinations) and thus of equations of motion, and whether these equations can still be related by a Wheeler-Feynman type argument.

If we impose the coordinate condition (51), the metric satisfies the inhomogeneous wave equation (50) in all orders. This equation can be integrated with the help of the Green's functions (52) or (54) regardless of the order of approximation. Furthermore, as discussed in Sec. III, we are treating Eq. (50) as a single equation to be solved up to order n rather than as n separate equations, which, as noted before, automatically assures that the *same* Green's function is used in all successive approximations to the metric. Therefore we have indeed the same multiplicity of solutions and of equations of motion in all orders as in the order discussed in detail above.

As for the second part of the question raised above, we note that the condition (101) requires the equality of the *total* retarded field and the *total* advanced field, i.e. the fields calculated from Eq. (50) with all N particles included as sources. Thus it requires the equality of the solutions of this equation obtained by using the Green's functions (52) or (54), and therefore also of all linear combinations of these functions with appropriate coefficients. But as the equations of motion are a consequence of the field equations (in the sense discussed in Sec. III), the validity of the condition (101) implies the existence of a unique set of equations of motion subject to this condition.[71]

[69] See also J. Weber and J. A. Wheeler, Bull. Am. Phys. Soc. 2, 13 (1957).

[70] However, Eqs. (73) and (101) are not fully equivalent to Eq. (74) alone, exactly as for the corresponding equations in electrodynamics. For a discussion of this difficulty see P. Havas, Phys. Rev. **86**, 974 (1952).

[71] This implication of the Wheeler-Feynman condition is identical to the one in electrodynamics, discussed by P. Havas, Phys. Rev. **74**, 456 (1948); the arguments are also identical except for the last step, which must be based on the energy-momentum tensor of the electromagnetic field rather than on the field equations. It should be noted that since only the *total* field is involved, the argument is valid even if the equations obeyed by this field are nonlinear.

Thus the answer to our question hinges on the validity of Eq. (101). The most general argument for this condition was given in "Derivation IV" of reference 50; to the order of approximation considered above, all the steps of this electrodynamic derivation can be duplicated in the gravitational case. However, this derivation depends essentially on the linearity of the fundamental equation obeyed by the field—in our case Eq. (42)—and the possibility of its extension to the higher order equations of the form (50) has not been established. Furthermore, the Wheeler-Feynman derivation has been questioned on cosmological grounds; on the other hand, it was noted that a different derivation of Eq. (101) might be possible from cosmological considerations,[72] which clearly could not require linearity. Pending a satisfactory derivation of this or possibly another type, our question remains open.

The Wheeler-Feynman theory was originally formulated in terms of action-at-a-distance concepts. It was shown subsequently, however, that its main result, the derivation of the electromagnetic radiation reaction force for the case of time-symmetric interactions, can be obtained on the basis of field-theoretical concepts as well.[71] A similar remark holds in our case. From the point of view of action at a distance we would simply postulate the time-symmetric equations of motion (73) [or the variational principle (75) which implies these equations] and then derive Eq. (74) by the procedure indicated. From the point of view of field theory Eq. (73) is obtained from the field equations, which are considered to be more fundamental, but the way from Eq. (73) to Eq. (74) is the same; it can even be considered to be more appropriate from the point of view of field theory, because its basic condition (101) involves the total field at all points in space-time rather than only the interactions between pairs of particles. The possibility of extending this argument to higher orders depends on the considerations discussed above.

[72] Reference 71, footnote 13.