Lee Model with Two V Particles*†

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For a relativistic field theory including two particles V_i with the same quantum numbers described by Heisenberg fields $\psi_i(x)$, the values of the matrix elements $(0|\psi_i(x)|\mathbf{V}_j)$ are obtained by the use of the asymptotic conditions. For a Lee model including two V particles these matrix-elements are used to obtain expressions for the renormalization constants. The consistency between the dispersion theoretic and Hamiltonian methods for this model is verified by discussions of several particular situations.

I. INTRODUCTION

In this paper, we seek to present a nontrivial example of a field theory, describing in its framework two particles having the same quantum numbers, but different masses. The major complication in such a theory is in the definition of Heisenberg field operators for the two particles. The reason being that the Heisenberg fields require an extension off the mass shell, and, if we have more than one field having the same quantum numbers differing only through their masses, then the two fields naturally get mixed. We seek to remove this arbitrariness by the requirement that the asymptotic fields, defined by the Yang-Feldman equations,[1] describe independent particles. That is to say, that they have the usual commutation rules required for noninteracting independent fields. It is found that such conditions can be postulated in general.

The model used for illustration is a simple generalization of the original Lee model. Just as in the original Lee model,[2] our model is essentially nonrelativistic, as our fields have no antiparticles. The model describes a very restricted set of interactions of three fermions V_1, V_2, and N with one boson θ; ($V_1 \rightleftharpoons N+\theta$ and $V_2 \rightleftharpoons N+\theta$). The fermions are assumed to be infinitely heavy, while the boson obeys a relativistic energy-momentum relation. The form of interaction allows us to define two quantum numbers B and S which are conserved in the interaction. The quantum numbers of the V particles are $B=1$, $S=1$; those of the N particle are $B=1$, $S=0$, and those of the θ particle are $B=0$, $S=1$. It is clear that there are no other states with the same quantum numbers as N or θ, and, hence, these do not require any renormalization. The V particles, on the other hand, can dissociate in $N\theta$ pairs, and, hence, the fields associated with them need normalization. The vacuum state needs no renormalization.

In Sec. II we shall formulate the conditions which we shall have to impose to insure a proper description of the asymptotic fields. In Secs. III and IV the calculation of the renormalization constants and the $N-\theta$ scattering amplitude will be carried out in the usual Hamiltonian formalism (solution of Schrödinger equation). In Sec. V a connection between the dispersion theoretic and the Hamiltonian methods will be made clear by illustration of the use of asymptotic fields, asymptotic conditions, defined in II and by showing that they are consistent with relations obtained in Secs. III and IV. This will complete the foundation of using this hybrid approach in this model, and will be applied to $V-\theta$ sector in a later communication.

II. ASYMPTOTIC FIELDS AND THEIR COMMUTATION RELATIONS

Let us consider two fields $\phi_1(x)$ and $\phi_2(x)$ which correspond to particles whose masses are μ_1 and μ_2, and whose other quantum numbers are all identical. We wish to discover sufficient conditions, such that the asymptotic fields in the remote past and the distant future describe independent particles; that is to say, they obey free-field equations and commute with each other. For simplicity we assume the two particles to be spinless; hence, the fields obey

$$(\Box+\mu_1^2)\phi_1(x)=j_1(x) \tag{2.1}$$

and

$$(\Box+\mu_2^2)\phi_2(x)=j_2(x). \tag{2.2}$$

Here $j_1(x)$ and $j_2(x)$ depend on $\phi_1(x)$, $\phi_2(x)$, and also on all the other fields which may be relevant in a particular problem under consideration. Indeed, these equations may be taken as a definition of the current operators $j_i(x)$. We can integrate these equations formally to give the Yang-Feldman equations, defining the asymptotic fields in remote past, in terms of the Heisenberg fields

$$\phi_1^{\text{in}}(x)=\phi_1(x)-\int_{-\infty}^{\infty}\Delta_R(\mu_1, x-y)j_1(y)d^4y, \tag{2.3}$$

and

$$\phi_2^{\text{in}}(x)=\phi_2(x)-\int_{-\infty}^{\infty}\Delta_R(\mu_2, x-y)j_2(y)d^4y, \tag{2.4}$$

where $\Delta_R(\mu, x-y)$ is the retarded Green's function of a Klein-Gordon equation with mass μ. The asymptotic

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[1] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[2] T. D. Lee, Phys. Rev. **95**, 1329 (1954); see also M. L. Goldberger and S. B. Trieman, *ibid.* **113**, 1663 (1959), for a dispersion-theoretic treatment of the Lee model.

fields $\phi_1^{\rm in}(x)$ and $\phi_2^{\rm in}(x)$ satisfy free-field equations with mass μ_1 and μ_2, respectively,

$$(\Box+\mu_1^2)\phi_1^{\rm in}(x)=0,\quad (\Box+\mu_2^2)\phi_2^{\rm in}(x)=0. \quad (2.5)$$

Equations (2.3) and (2.4) imply, in a certain sense, a convergence of the operator $\phi(x)$ to $\phi^{\rm in}(x)$ as the time $t\rightarrow -\infty$. Indeed it is required to be a weak-operator convergence defined by Lehmann, Symanzik, and Zimmermann[3]:

$$\lim_{t\rightarrow-\infty}|\langle\Psi|(\phi_i^{f_i}(t)-[\phi_i^{\rm in}]^{f_i})|\Phi\rangle|=0 \quad (2.6)$$

for all pairs of states $|\Psi\rangle$ and $|\Phi\rangle$, where

$$\phi_i^f(t)=i\int d^3x\,\phi_i(x)\overleftrightarrow{\partial}_0 f^*(x). \quad (2.7)$$

Here $f_i(x)$ is a normalized solution of the Klein-Gordon equation with mass μ_i. As both $\phi_i^{\rm in}(x)$ and $f_i(x)$ satisfy the Klein-Gordon equation with the same mass, $[\phi_i^{\rm in}]^{f_i}$ is actually time-independent.

In order that the in-fields describe noninteracting particles, we not only want them to satisfy Eqs. (2.5) but also proper commutation relations. These are

$$[\phi_1^{\rm in}(x),\phi_1^{\rm in}(x')]=i\Delta(\mu_1,x-x'),$$
$$[\phi_2^{\rm in}(x),\phi_2^{\rm in}(x')]=i\Delta(\mu_2,x-x'),$$

and

$$[\phi_1^{\rm in}(x),\phi_2^{\rm in}(x)]=0. \quad (2.8)$$

These lead to the following matrix elements for the in-fields,

$$\begin{aligned}&\langle 0|\phi_1^{\rm in}(x)|p_2\rangle=0,\quad \langle 0|\phi_2^{\rm in}(x)|p_1\rangle=0,\\ &\langle 0|\phi_1^{\rm in}(x)|p_1\rangle=e^{ip_{1\mu}x_\mu}/(2\pi)^{3/2},\\ &\langle 0|\phi_2^{\rm in}(x)|p_2\rangle=e^{ip_{2\mu}x_\mu}/(2\pi)^{3/2},\end{aligned} \quad (2.9)$$

where $|p_1\rangle$ and $|p_2\rangle$ are eigenstates of the total Hamiltonian with masses μ_1 and μ_2, respectively. All other matrix elements are zero.

Let us now investigate what Eqs. (2.9) imply in terms of the Heisenberg field. From Lorentz invariance, we have

$$\langle 0|\phi_i(x)|p_j\rangle=\langle 0|\phi_i(0)|p_j\rangle e^{ip_{j\mu}x_\mu}/(2\pi)^{3/2}. \quad (2.10)$$

Thus, the space time-dependence is explicitly factored out. By relativistic invariance $\langle 0|\phi_i(0)|p_j\rangle$ is a function of $p_j^2=-\mu_j^2$; hence, is a constant. We can thus easily substitute for $|p_j\rangle$ a state with wave function $g_j(x)$; thus

$$\langle 0|\phi_i^{f_i}(t)|g_j\rangle=iC_{ij}\int g_j(x)\overleftrightarrow{\partial}_0 f_i(x)d^3x, \quad (2.11)$$

where the C_{ij}'s are constant, equal to $\langle 0|\phi_i(0)|p_j\rangle$. Application of the asymptotic conditions (2.6) and comparison with (2.9) thus yields

$$C_{ij}=\langle 0|\phi_i(0)|p_j\rangle=\delta_{ij}. \quad (2.12)$$

Using (2.10), (2.1), and (2.2), we obtain

$$\langle 0|j_i(0)|p_j\rangle=0. \quad (2.12)$$

These thus give the general conditions which the fields describing two particles having identical quantum numbers must obey.

III. DESCRIPTION OF THE MODEL

The model used here describes the interactions of three fermions, V_1, V_2, and N with a boson θ. The allowed interactions are $V_1\rightleftharpoons N\theta$ and $V_2\rightleftharpoons N\theta$. These interactions, along with the restrictions that there are no antiparticles in the model, allow only a very restricted class of reactions, and thus make the model especially simple. The model is described by a Hermitian Hamiltonian,

$$\begin{aligned}H=&(m_1+\delta m_1)Z_1\psi_1^\dagger\psi_1+(m_2+\delta m_2)Z_2\psi_2^\dagger\psi_2\\ &+B(\psi_1^\dagger\psi_2+\text{H.c.})+\textstyle\sum_{\mathbf{k}}\omega\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}\\ &+\{(g_1\psi_1^\dagger+g_2\psi_2^\dagger)\psi_N A+\text{H.c.}\},\end{aligned} \quad (3.1)$$

where

$$A=\textstyle\sum_{\mathbf{k}}[u(\omega)/(2\omega\Omega)^{1/2}]\alpha_{\mathbf{k}},\quad \omega=(\mu^2+k^2)^{1/2}. \quad (3.2)$$

Here ψ_1 and ψ_2 are renormalized field operators describing V_1 and V_2 in a certain asymptotic sense which shall be made explicit later. ψ_N is the field operator for the N particle, and α_k is the annihilation operator for the θ particle in the state of momentum k. m_1 and m_2 are the renormalized masses of the particles V_1 and V_2, respectively, and μ is the mass of the θ particle. The mass of the N particle is taken as the zero for energy. It is clear from the Hamiltonian that V and N particles are taken to be fixed in position; hence, their energy is the same as their mass, whereas the θ particle obeys a relativistic energy-momentum relation. We have quantized in a box of volume Ω; later the limit $\Omega\rightarrow\infty$ will be taken in all integrals. The cutoff function $u(\omega)$ is so chosen that all the relevant integrals exist and that there are no ghost V-particle states.

The equal-time commutation relations obeyed by the field operators are

$$\begin{aligned}&[\alpha_{\mathbf{k}},\alpha_{\mathbf{k}'}^\dagger]=\delta_{\mathbf{k},\mathbf{k}'},\quad \{\psi_N^\dagger,\psi_N\}=1,\\ &\{\psi_1^\dagger,\psi_1\}=1/Z_1,\quad \{\psi_2^\dagger,\psi_2\}=1/Z_2,\end{aligned} \quad (3.3)$$

and

$$\{\psi_1^\dagger,\psi_2\}=1/Z_3.$$

All other anticommutators between fermion operators vanish. It shall be noticed that the anticommutator of the field operators $\psi_1^\dagger$ and ψ_2 does not vanish, as is usual for field operators of particles having different quantum numbers. In that case it follows[4] from the

[3] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

[4] As the anticommutator is a c-number, we have

$$\begin{aligned}\{\psi_1^\dagger,\psi_2\}&=\langle 0|\{\psi_1^\dagger,\psi_2\}|0\rangle\\ &=\textstyle\sum_s[\langle 0|\psi_1^\dagger|s\rangle\langle s|\psi_2|0\rangle+\langle 0|\psi_2|s\rangle\langle s|\psi_1^\dagger|0\rangle],\end{aligned}$$

fact that the equal time anticommutators of the Heisenberg fields are c-numbers in any proper Hamiltonian theory quantized by the canonical commutation rules. In the present case, since V_1 and V_2 have the same quantum numbers, we cannot put $\{\psi_1^\dagger,\psi_2\}=0$; however, even here we shall assume that it is a c-number.

IV. PHYSICAL V-PARTICLE STATES

We define the physical V-particle states as the eigenstates of the total Hamiltonian with eigenvalues m_1 and m_2:

$$H|\mathbf{V}_1\rangle=m_1|\mathbf{V}_1\rangle, \tag{4.1}$$

$$H|\mathbf{V}_2\rangle=m_2|\mathbf{V}_2\rangle. \tag{4.2}$$

These equations are however not sufficient to determine all the renormalization constants in (3.1) and (3.3). We thus require further restrictions which are provided by the discussion in Sec. II. We require in analogy with (2.11)

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=1,\quad \langle 0|\psi_2|\mathbf{V}_1\rangle=0, \tag{4.3}$$

and

$$\langle 0|\psi_1|\mathbf{V}_2\rangle=0,\quad \langle 0|\psi_2|\mathbf{V}_2\rangle=1. \tag{4.4}$$

We had shown the consequences of such conditions for a relativistic theory involving bosons; however, it is also valid for the nonrelativistic model under consideration.

Besides (4.3) and (4.4), there are the usual normalization and orthogonality requirements on these eigenstates,

$$\langle \mathbf{V}_1|\mathbf{V}_1\rangle=1,\quad \langle \mathbf{V}_2|\mathbf{V}_2\rangle=1, \tag{4.5}$$

$$\langle \mathbf{V}_1|\mathbf{V}_2\rangle=0. \tag{4.6}$$

Let us define, for convenience in writing,

$$|1\rangle=\psi_1^\dagger|0\rangle,\quad |2\rangle=\psi_2^\dagger|0\rangle,$$

and

$$|N\theta_{\mathbf{k}}\rangle=\alpha_{\mathbf{k}}^\dagger\psi_N^\dagger|0\rangle. \tag{4.7}$$

It should be noted that we carefully avoid calling states $|1\rangle$ and $|2\rangle$ V-particle states, as they do not represent bare V-particle states. They are not even orthogonal to each other.

We now expand the physical V-particle states in terms of $|1\rangle$, $|2\rangle$, and $|N\theta_{\mathbf{k}}\rangle$.

$$|\mathbf{V}_1\rangle=\alpha_1|1\rangle+\alpha_2|2\rangle+\sum_{\mathbf{k}} f_1(\mathbf{k})|N\theta_{\mathbf{k}}\rangle. \tag{4.8}$$

Imposing the requirements (4.3), we can deduce the following relations between α_1 and α_2.

$$\alpha_2=-(Z_2/Z_3)\alpha_1, \tag{4.9}$$

and

$$(\alpha_1/Z_1)[1-(Z_1Z_2/|Z_3|^2)]=1. \tag{4.10}$$

Using the Hamiltonian (3.1), the commutation rules (3.3) and Eqs. (4.9) and (4.10), we have

$$\begin{aligned}H|\mathbf{V}_1\rangle&=Z_1(m_1+\delta m_1)|1\rangle+B^*|2\rangle+g_1\sum_k\frac{u(\omega)}{(2\omega\Omega)^{1/2}}|N\theta_{\mathbf{k}}\rangle\\&\quad+\sum_k\frac{u(\omega)f_1(\mathbf{k})}{(2\omega\Omega)^{1/2}}(g_1|1\rangle+g_2|2\rangle)\\&\quad+\sum_k\omega f_1(\mathbf{k})|N\theta_{\mathbf{k}}\rangle\\&=m_1|\mathbf{V}_1\rangle\\&=m_1\alpha_1|1\rangle-m_1(Z_2/Z_3)\alpha_1|2\rangle\\&\quad+m_1\sum_k f_1(\mathbf{k})|N\theta_{\mathbf{k}}\rangle.\end{aligned} \tag{4.11}$$

Comparing the coefficients of the states $|1\rangle$, $|2\rangle$, and $|N\theta_{\mathbf{k}}\rangle$, we obtain the equations

$$m_1+\delta m_1-(m_1\alpha_1/Z_1)+\frac{g_1}{Z_1}\sum_{\mathbf{k}}[u(\omega)/(2\omega\Omega)^{1/2}]f_1(\mathbf{k})=0, \tag{4.12}$$

$$B^*+g_2\sum_{\mathbf{k}}[u(\omega)/(2\omega\Omega)^{1/2}]f_1(\mathbf{k})+m_1\alpha_1(Z_2/Z_3)=0, \tag{4.13}$$

and

$$(\omega-m_1)f_1(\mathbf{k})+g_1[u(\omega)/(2\omega\Omega)^{1/2}]=0. \tag{4.14}$$

Substituting $f_1(k)$ from Eq. (4.14) in (4.12) and (4.13), and taking the limit of infinite volume, we have

$$m_1+\delta m_1-(m_1\alpha_1/Z_1)=(g_1^2/Z_1)\phi(m_1) \tag{4.15}$$

and

$$B^*+(m_1\alpha_1/Z_1)(Z_1Z_2/Z_3)=g_1g_2\phi(m_1), \tag{4.16}$$

where

$$\phi(\omega)=\frac{1}{4\pi^2}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{\omega'-\omega}=\lim_{t\to\infty}\sum_{k'}\frac{u^2(\omega')}{(2\omega'\Omega)^{1/2}}\frac{1}{\omega'-\omega}. \tag{4.17}$$

A similar procedure can be used for the $|V_2\rangle$ state; namely, we begin by defining

$$|\mathbf{V}_2\rangle=\beta_1|1\rangle+\beta_2|2\rangle+\sum_{\mathbf{k}} f_2(\mathbf{k})|N\theta_{\mathbf{k}}\rangle. \tag{4.18}$$

Following the procedure of Eqs. (4.9)–(4.17), we have

$$m_2+\delta m_2-m_2(\beta_2/Z_2)=(g_2^2/Z_2)\phi(m_2), \tag{4.19}$$

$$B^*+m_2(\beta_2/Z_2)(Z_1Z_2/Z_3)=g_1g_2\phi(m_2), \tag{4.20}$$

$$(\beta_2/Z_2)[1-(Z_1Z_2/|Z_3|^2)]=1, \tag{4.21}$$

and

$$f_2(k)=-g_2[u(\omega)/(2\omega\Omega)^{1/2}][1/(\omega-m_2)]. \tag{4.22}$$

A simple algebraic manipulation of Eqs. (4.10), (4.12), and (4.13), (4.14)–(4.16) yields the following

where $|s\rangle$ is a complete set of states. If the fields ψ_1 and ψ_2 have different quantum numbers, then for every member of the set $|s\rangle$ the summand in the above expression vanishes. Hence, $\{\psi_1^\dagger,\psi_2\}=0$.

results:

$$Z_1=K\alpha_1,\quad Z_2=K\alpha_2,$$
$$1/Z_3=(1/K)(g_1g_2/\alpha_1\beta_2)F(m_1,m_2), \tag{4.23a}$$

where

$$K=[1-(Z_1Z_2/|Z_3|^2)]$$
$$=1-(g_1^2g_2^2/\alpha_1\beta_2)F^2(m_1,m_2). \tag{4.23b}$$

$$\delta m_1=m_1[(1/K)-1]+(g_1^2/Z_1)\phi(m_1), \tag{4.23c}$$

$$\delta m_2=m_2[(1/K)-1]+(g_2^2/Z_2)\phi(m_2), \tag{4.23d}$$

and

$$B=g_1g_2[E(m_1,m_2)-(m_1+m_2)F(m_1,m_2)], \tag{4.23e}$$

where

$$F(m_1,m_2)=\frac{1}{4\pi^2}\int_\mu^\infty\frac{ku^2(\omega)d\omega}{(\omega-m_1)(\omega-m_2)} \tag{4.24}$$

and

$$E(m_1,m_2)=\frac{1}{4\pi^2}\int_\mu^\infty\frac{\omega ku^2(\omega)d\omega}{(\omega-m_1)(\omega-m_2)}. \tag{4.25}$$

The constants α_1 and β_2 can be easily determined by the use of the normalization condition (4.5):

$$1=\langle\mathbf{V}_1|\mathbf{V}_1\rangle$$
$$=|\alpha_1|^2(1/Z_1)[1-(Z_1Z_2/|Z_3|^2)]+\sum_{\mathbf{k}}|f_1(\mathbf{k})|^2,$$

i.e.,

$$\alpha_1{}^*=1-g_1^2F(m_1,m_1). \tag{4.26}$$

Similarly,

$$\beta_2{}^*=1-g_2^2F(m_2,m_2). \tag{4.27}$$

It is clear from the expression (4.24) that as m_1 and m_2 are less than μ the constants α_1 and β_2 are real and so are all the renormalization constants Z_1, Z_2, Z_3, B, δm_1, and δm_2. We shall thus ignore the asterisks from all our subsequent equations.

The orthogonality of $|\mathbf{V}_1\rangle$ and $|\mathbf{V}_2\rangle$ is a direct consequence of (4.14), (4.22), and (4.23a–e).

$$\langle\mathbf{V}_1|\mathbf{V}_2\rangle=\alpha_1\beta_2(-Z_1/Z_3\langle 1|+\langle 2|)(|1\rangle-Z_2/Z_3|2\rangle)$$
$$+\sum_{\mathbf{k}}f_1(\mathbf{k})f_2(\mathbf{k})$$
$$=-K\alpha_1\beta_2/Z_3+g_1g_2F(m_1,m_2)=0. \tag{4.28}$$

We are now in a position to prove by direct calculation the equations corresponding to Eq. (12) of Sec. II. Before we undertake this, we have to define the operators at arbitrary time in terms of the operators we have been working with until now (i.e., operators at time $t=0$).[5] This is easily done by the transformation,

$$O(t)=e^{iHt}Oe^{-iHt}. \tag{4.29}$$

The current operators can now be defined by the following equations:

$$[-i(d/dt)+m_1]\psi_1(t)=f_1(t), \tag{4.30}$$

and

$$[-i(d/dt)+m_2]\psi_2(t)=f_2(t). \tag{4.31}$$

[5] The time label of all the operators is ignored when referring to time $t=0$.

Using the Hamiltonian (3.1) and commutation rules (3.3), we obtain

$$f_1(t)=[H,\psi(t)]+m_1\psi_1(t)$$
$$=-\delta m_1\psi_1(t)-\frac{Z_2}{Z_3}(m_2+\delta m_2)\psi_2(t)-\frac{B}{Z_1}\psi_2(t)$$
$$-\frac{B}{Z_3}\psi_1(t)-\frac{g_1}{Z_1}\psi_N(t)A(t)-\frac{g_2}{Z_3}\psi_N(t)A(t), \tag{4.32}$$

and

$$f_2(t)=-\delta m_2\psi_2(t)-\frac{Z_1}{Z_3}(m_1+\delta m_1)\psi_1(t)-\frac{B}{Z_2}\psi_1(t)$$
$$-\frac{B}{Z_3}\psi_2(t)-\frac{g_2}{Z_2}\psi_N(t)A(t)-\frac{g_1}{Z_3}\psi_N(t)A(t). \tag{4.33}$$

Similarly we can define the θ-current operator $j(t)$,

$$j(t)=[(2\omega\Omega)^{1/2}/u(\omega)][-i(d/dt)+\omega]\alpha_{\mathbf{k}}(t). \tag{4.34}$$

Thus,

$$j(t)=-(g_1\psi_N{}^\dagger\psi_1+g_2\psi_N{}^\dagger\psi_2). \tag{4.35}$$

Taking the matrix element of (4.32) between vacuum and one-V-particle states, we get

$$\langle 0|f_1|\mathbf{V}_1\rangle=-\delta m_1-\frac{B}{Z_3}+\frac{g_1^2}{Z_1}\phi(m_1)+\frac{g_1g_2}{Z_3}\phi(m_1), \tag{4.36}$$

$$\langle 0|f_1|\mathbf{V}_2\rangle=-(m_2+\delta m_2)\frac{Z_2}{Z_3}-\frac{B}{Z_1}$$
$$+\frac{g_1g_2}{Z_1}\phi(m_2)+\frac{g_2^2}{Z_3}\phi(m_2). \tag{4.37}$$

The expressions on the right of these two equations can each be easily shown to vanish by use of the relations (4.23). Similarly, we verify directly that

$$\langle 0|f_2|\mathbf{V}_1\rangle=0\quad\text{and}\quad\langle 0|f_2|\mathbf{V}_2\rangle=0. \tag{4.38}$$

The constants g_1 and g_2 appearing in the Hamiltonian (3.1) can be easily seen to be the renormalized coupling constant, as taking the matrix elements of (4.35), we get

$$\langle N|j|\mathbf{V}_1\rangle=-g_1\quad\text{and}\quad\langle N|j|\mathbf{V}_2\rangle=-g_2. \tag{4.39}$$

V. THE N-θ SCATTERING STATES

Now we are interested in the scattering of the θ-particle by an N particle, and calculate the scattering amplitude. We shall see that the form of the scattering amplitude is as required by unitarity and the mass spectrum.

The scattering states can be got as the continuum solution of the eigenvalue equation

$$(H-\omega-i\epsilon)|N\theta_\omega{}^\dagger\rangle=0. \tag{5.1}$$

The positive imaginary part for ω is introduced in the equation to insure the proper boundary condition at time $t=-\infty$ where we assume that only incoming plane waves exist. We split the state as

$$|N\theta_\omega^+\rangle = |N\theta_{\mathbf{k}}\rangle + |\chi^+(\omega)\rangle. \qquad (5.2)$$

when

$$(H-\omega-i\epsilon)|\chi^+(\omega)\rangle = -H_I|N\theta_{\mathbf{k}}\rangle, \qquad (5.3)$$

where

$$H_I = H - H_0. \qquad (5.4)$$

Here, $H_0=\sum_{\mathbf{k}}\omega_{\mathbf{k}}\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{k}}$, the Hamiltonian for a noninteracting $N\theta$ system.

Let us expand the state $|\chi^+(\omega)\rangle$ in terms of the physical states $|\mathbf{V}_1\rangle$, $|\mathbf{V}_2\rangle$ and the noninteraction $|N\theta_{\mathbf{k}}\rangle$ state.

$$|\chi^+(\omega)\rangle = a_1|\mathbf{V}_1\rangle + a_2|\mathbf{V}_2\rangle + \sum_{\mathbf{k}'} f(k')|N\theta_{k'}\rangle. \qquad (5.5)$$

Substituting (5.5) in Eq. (5.3), we get, using (4.3) and (4.4) and the Hamiltonian,

$$\begin{aligned} a_1(m_1-\omega)|\mathbf{V}_1\rangle &+ a_2(m_2-\omega)|\mathbf{V}_2\rangle \\ &+\sum_{\mathbf{k}'} f(k')(\omega'-\omega-i\epsilon)|N\theta_{\mathbf{k}'}\rangle \\ &+\sum_{k'}\frac{f(k')u(k')}{(2\omega'\Omega)^{1/2}}(g_1|1\rangle+g_2|2\rangle) \\ &= -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}(g_1|1\rangle+g_2|2\rangle). \qquad (5.6)\end{aligned}$$

Using the expansions of $|\mathbf{V}_1\rangle$ and $|\mathbf{V}_2\rangle$ (4.8) and (4.18) and comparing coefficients of $|1\rangle$, $|2\rangle$, and $|N\theta_{\mathbf{k}}\rangle$,

$$\begin{aligned} a_1(m_1-\omega)\alpha_1 - a_2(m_2-\omega)\frac{\alpha_1 Z_2}{Z_3} \\ +g_1\sum_{k'}\frac{f(k')u(\omega')}{(2\omega'\Omega)^{1/2}} = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_1, \qquad (5.7)\end{aligned}$$

$$\begin{aligned} -a_1(m_2-\omega)\frac{\beta_2 Z_1}{Z_3} + a_2(m_2-\omega)\beta_2 \\ +g_2\sum_{k'}\frac{f(k')u(\omega')}{(2\omega'\Omega)^{1/2}} = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_2, \qquad (5.8)\end{aligned}$$

and

$$\begin{aligned} a_1 f_1(\mathbf{k}')(m_1-\omega') + a_2 f_2(\mathbf{k}')(m_2-\omega') \\ + f(\mathbf{k}')(\omega'-\omega) = 0. \qquad (5.9)\end{aligned}$$

Substituting $f_1(k')$ and $f_2(k')$ from (4.14) and (4.22) in (5.9), we get

$$\begin{aligned} f(k') = a_1\frac{u(\omega')}{(2\omega'\Omega)^{1/2}}\frac{(m_1-\omega)}{(\omega'-m_1)(\omega'-\omega-i\epsilon)} \\ + a_2\frac{u(\omega')}{(2\omega'\Omega)^{1/2}}\frac{(m_2-\omega)}{(\omega'-m_2)(\omega'-\omega-i\epsilon)}. \qquad (5.10)\end{aligned}$$

Substituting $f(k')$ in (5.7) and (5.8), and taking the limit of $\Omega\to\infty$, we have

$$\begin{aligned} a_1(m_1-\omega)[\alpha_1+g_1^2F(m_1,\omega+i\epsilon)] + a_2(m_2-\omega) \\ \times\left[-\frac{Z_1Z_2}{KZ_3}+g_1g_2F(m_2,\omega+i\epsilon)\right] = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_1, \qquad (5.11)\end{aligned}$$

and

$$\begin{aligned} a_1(m_1-\omega)\left[-\frac{Z_1Z_2}{KZ_3}+g_1g_2F(m_1,\omega+i\epsilon)\right] + a_2(m_2-\omega) \\ \times[\beta_2+g_2^2F(m_2,\omega+i\epsilon)] = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_2. \qquad (5.12)\end{aligned}$$

Using the relations (4.23) and defining

$$F(\omega_1,\omega_2,\omega_3) = \frac{1}{4\pi^2}\int_\mu^\infty \frac{ku^2(\omega)d\omega}{(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)}, \qquad (5.13)$$

we can recast (5.11) and (5.12) in the following form:

$$\begin{aligned} a_1(m_1-\omega)[1-g_1^2(m_1-\omega)F(m_1,m_1,\omega+i\epsilon)] \\ + a_2(m_1-\omega)(m_2-\omega)[-g_1g_2F(m_2,m_1,\omega+i\epsilon)] \\ = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_1, \qquad (5.14)\end{aligned}$$

and

$$\begin{aligned} a_1(m_1-\omega)(m_2-\omega)[-g_1g_2F(m_1,m_2,\omega+i\epsilon)] \\ + a_2(m_2-\omega)[1-g_2^2(m_2-\omega)F(m_2,m_2,\omega+i\epsilon)] \\ = -\frac{u(\omega)}{(2\omega\Omega)^{1/2}}g_2. \qquad (5.15)\end{aligned}$$

We can easily solve these simultaneous equations for a_1 and a_2:

$$a_1 = N_1(\omega)/\Delta(\omega) \quad \text{and} \quad a_2 = N_2(\omega)/\Delta(\omega), \qquad (5.16)$$

where

$$\begin{aligned} N_1(\omega) &= -g_1\frac{u(\omega)}{(2\omega\Omega)^{1/2}}(m_2-\omega) \\ &\quad\times[1-g_2^2(m_2-\omega)F(m_2,m_2,\omega+i\epsilon) \\ &\quad+g_2^2(m_1-\omega)F(m_2,m_1,\omega+i\epsilon)] \\ &= -g_1\frac{u(\omega)}{(2\omega\Omega)^{1/2}}(m_2-\omega) \\ &\quad\times[1-g_2^2(m_2-m_1)F(m_2,m_2,m_1)], \qquad (5.17)\end{aligned}$$

and similarly,

$$N_2(\omega)=-g_2\frac{u(\omega)}{(2\omega\Omega)^{1/2}}(m_1-\omega)\times[1-g_1^2(m_1-m_2)F(m_1,m_1,m_2)], \quad (5.18)$$

$$\Delta(\omega)=(m_1-\omega)(m_2-\omega)D(\omega), \quad (5.19)$$

where

$$D(\omega)=\{(1-g_1^2(m_1-\omega)F(m_1,m_1,\omega+i\epsilon)\times[1-g_2^2(m_2-\omega)F(m_2,m_2,\omega+i\epsilon)]-g_1^2g_2^2(m_1-\omega)(m_2-\omega)F^2(m_1,m_2,\omega+i\epsilon)\}. \quad (5.20)$$

We shall later find it useful to rearrange the expression for $D(\omega)$.

Coefficient of $g_1^2g_2^2$ in $D(\omega)$

$$=(m_1-\omega)(m_2-\omega)[F(m_1,m_1,\omega+i\epsilon)F(m_2,m_2,\omega+i\epsilon)-F^2(m_1,m_2,\omega+i\epsilon)], \quad (5.21)$$

which reduces, after small algebraic manipulation, to

$$(m_1-\omega)(m_2-\omega)\Bigg[(m_1-m_2)F(m_2,m_2,m_1)\times\frac{1}{4\pi^2}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{(\omega'-m_1)^2(\omega'-m_2)(\omega'-\omega-i\epsilon)}+(m_2-m_1)F(m_1,m_1,m_2)\times\frac{1}{4\pi^2}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{(\omega'-m_2)^2(\omega'-m_1)(\omega'-\omega-i\epsilon)}\Bigg]. \quad (5.22)$$

Thus,

$$D(\omega)=\Bigg\{1-g_1^2(m_1-\omega)F(m_1,m_1,\omega+i\epsilon)-g_2^2(m_2-\omega)F(m_2,m_2,\omega+i\epsilon)+g_1^2g_2^2(m_1-\omega)(m_2-\omega)\times\Bigg[\frac{1}{4\pi^2}\int_\mu^\infty\frac{d\omega'k'u^2(\omega')}{(\omega'-m_1)^2(\omega'-m_2)(\omega'-\omega-i\epsilon)}\times(m_1-m_2)F(m_2,m_2,m_1)+\frac{1}{4\pi^2}\int_\mu^\infty\frac{d\omega'k'u^2(\omega')}{(\omega'-m_2)^2(\omega'-m_1)(\omega'-\omega-i\epsilon)}\times(m_2-m_1)F(m_1,m_1,m_2)\Bigg]\Bigg\}. \quad (5.23)$$

The scattering amplitude can now be calculated in terms of the state $|N\theta_\omega^+\rangle$:

$$T(\mathbf{k})=\langle N\theta_\mathbf{k}|H_I|N\theta_\omega^+\rangle. \quad (5.24)$$

Using explicit form of H_I and the commutation rules, we have

$$T(\mathbf{k})=\frac{u(\omega)}{(2\omega\Omega)^{1/2}}\langle N|j|N\theta_\omega^+\rangle=\frac{u(\omega)}{(2\omega\Omega)^{1/2}}\langle 0|g_1\psi_1^\dagger+g_2\psi_2^\dagger|N\theta_\omega^+\rangle=\frac{u(\omega)}{(2\omega\Omega)^{1/2}}(g_1a_1+g_2a_2). \quad (5.25)$$

Thus the S-wave amplitude is

$$T(\omega)=\frac{ku^2(\omega)\{g_1^2(m_2-\omega)[1-(m_2-m_1)g_2^2F(m_2,m_2,m_1)]+g_2^2(m_1-\omega)[1-(m_1-m_2)g_1^2F(m_1,m_1,m_2)]\}}{4\pi(m_1-\omega)(m_2-\omega)D(\omega)}. \quad (5.26)$$

Let us denote this expression by

$$T(\omega)=[ku^2(\omega)/4\pi]M(\omega). \quad (5.27)$$

Analytic Properties of $M(\omega)$

It is clear from (5.26) that $M(\omega)$, as a function of the complex energy variable ω, has poles at m_1 and m_2 with residues $-g_1^2$ and $-g_2^2$, respectively, since

$$D(m_1)=[1-g_2^2(m_2-m_1)F(m_2,m_2,m_1)],$$

and

$$D(m_2)=[1-g_1^2(m_1-m_2)F(m_1,m_1,m_2)]. \quad (5.28)$$

Besides these poles, the function $M(\omega)$ has a cut from μ to ∞ in the complex ω plane, coming from the functions $F(m_i,m_j,\omega)$. To calculate the jump across the cut, we use the explicit expression for $F(m_i,m_j,\omega)$. Then we have

$$\text{Im}\Delta(\omega)=-\frac{ku^2(\omega)}{4\pi}\{g_1^2(m_1-\omega)[1-(m_2-m_1)\times g_2^2F(m_2,m_2,m_1)]+g_2^2(m_2-\omega)\times[1-(m_1-\omega)g_1^2F(m_1,m_1,m_2)]\} \quad (5.29)$$

$$=-[ku^2(\omega)/4\pi]N(\omega), \quad (5.30)$$

where $N(\omega)$ is the numerator function in the expression,

$$M(\omega)=N(\omega)/\Delta(\omega). \quad (5.31)$$

Thus we have

$$\text{Im}M(\omega)=[ku^2(\omega)/4\pi]|M(\omega)|^2. \quad (5.32)$$

$M(\omega)$ thus has poles and cuts as required by mass spectrum and unitarity. There are no other poles as long as $D(\omega)$ has no zeros, which is certainly true if g_1^2 and g_2^2 are sufficiently small.

VI. AN ILLUSTRATION OF THE USE OF THE HEISENBERG FIELDS ψ_1 AND ψ_2

We observed in Sec. III that ψ_1 and ψ_2 describe in an asymptotic sense the particles V_1 and V_2, respectively. We shall now show that it is possible to use these operators, in the reduction formulas of the standard formalism,[6] as the field operator for V_1 and V_2 particles, respectively. This is instructive in view of the fact that $\psi_1^\dagger$ and ψ_2 do not anticommute, and hence the states $|1\rangle$ and $|2\rangle$ do not represent bare V_1 and V_2 states. We shall first define in- and out-fields, and then later illustrate the assertion that $\psi_1(t)$ and $\psi_2(t)$ can indeed be used as field operators for V_1 and V_2, by showing that their use does not contradict the Hamiltonian formalism used till now.

In complete analogy with the discussion in Sec. II, we can define the asymptotic fields, $\psi_1^{\text{in}}(t)$ and $\psi_2^{\text{in}}(t)$, which shall anticommute with each other:

$$\psi_1^{\text{in}}(t)=\psi_1(t)-\int_{-\infty}^{\infty} S_R(m_1, t-t')f_1(t')dt', \qquad (6.1)$$

and

$$\psi_2^{\text{in}}(t)=\psi_2(t)-\int_{-\infty}^{\infty} S_R(m_2, t-t')f_2(t')dt', \qquad (6.2)$$

where $S_R(m, t-t')$ is the retarded Green's function of the equation

$$(-i(d/dt)+m)\psi(t)=f(t). \qquad (6.3)$$

The asymptotic condition can be written down as

$$\lim_{t\to\infty}(\Psi|\psi_i(t)|\Phi)=(\psi|\psi_i^{\text{in}}(t)|\phi). \qquad (6.4)$$

We need not use the wave functions $f(x)$ as fields $\psi(t)$ have no space dependences. The operators $\psi_1^{\dagger\,\text{in}}(0)$ and $\psi_2^{\dagger\,\text{in}}(0)$ can now be used as the creation operators for the V particles in the remote past, as they obey

$$\{\psi_1^{\dagger\,\text{in}}(0),\psi_1^{\text{in}}(0)\}=\{\psi_2^{\dagger\,\text{in}}(0),\psi_2^{\text{in}}(0)\}=1,$$

and

$$\{\psi_1^{\dagger\,\text{in}}(0),\psi_2^{\text{in}}(0)\}=0. \qquad (6.5)$$

In the same way it is possible to define $\psi_1^{\dagger\,\text{out}}(0)$ and $\psi_2^{\dagger\,\text{out}}(0)$, which can be used as creation operators in the distant future.

We can now check the consistency of this section with Hamiltonian formalism by direct calculation.

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=\langle 0|\psi_1\psi_1^{\dagger\,\text{in}}|0\rangle=\langle 0|\{\psi_1,\psi_1^{\dagger\,\text{in}}\}|0\rangle. \qquad (6.6)$$

The extra term introduced by the anticommutator vanishes. We now use the asymptotic condition (6.4):

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=\lim_{t\to-\infty}\langle 0|\{\psi_1,\psi_1^\dagger(t)\}|0\rangle e^{im_1t} \qquad (6.7)$$

$$=-\int_{-\infty}^{\infty}\frac{d}{dt}\{\langle 0|\{\psi_1,\psi_1^\dagger(t)\}|0\rangle\theta(-t)e^{im_1t}\}dt$$

$$=-i\int_{-\infty}^{\infty}\langle 0|\{\psi_1,f_1^\dagger(t)\}|0\rangle\theta(-t)e^{im_1t}dt +\langle 0|\{\psi_1,\psi_1^\dagger\}|0\rangle. \qquad (6.8)$$

We introduce the complete set of in-states and use the time translation property,

$$f(t)=e^{iHt}f(0)e^{-iHt}, \qquad (6.9)$$

and then integrate over time when

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=\frac{1}{Z_1}+\sum_s\frac{\langle 0|\psi_1|s^+\rangle\langle s^+|f_1^\dagger|0\rangle}{s-m_1}. \qquad (6.10)$$

Using the fact that the current operators do not have any matrix elements between vacuum and single-particle states, we have

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=\frac{1}{Z_1}+\frac{1}{\pi}\int_\mu^\infty\frac{ku^2(\omega)}{4\pi}\frac{L_1(\omega)K_1^*(\omega)}{\omega-m_1}, \qquad (6.11)$$

where

$$K_1(\omega)=\langle 0|f_1|N\theta_\omega^+\rangle[(2\omega\Omega)^{1/2}/u(\omega)], \qquad (6.12)$$

and

$$L_1(\omega)=\langle 0|\psi_1|N\theta_\omega^+\rangle[(2\omega\Omega)^{1/2}/u(\omega)]. \qquad (6.13)$$

Following a procedure similar to (6.6)–(6.11) we can get the following:

$$\langle 0|\psi_2|\mathbf{V}_2\rangle=\frac{1}{Z_2}+\frac{1}{\pi}\int_\mu^\infty\frac{L_2(\omega)K_2^*(\omega)}{(\omega-m_2)}\frac{ku^2(\omega)}{4\pi}d\omega, \qquad (6.14)$$

$$\langle 0|\psi_1|\mathbf{V}_2\rangle=\frac{1}{Z_3}+\frac{1}{\pi}\int_\mu^\infty\frac{L_1(\omega)K_2^*(\omega)}{(\omega-m_2)}\frac{ku^2(\omega)}{4\pi}d\omega, \qquad (6.15)$$

and

$$\langle 0|\psi_2|\mathbf{V}_1\rangle=\frac{1}{Z_3}+\frac{1}{\pi}\int_\mu^\infty\frac{L_2(\omega)K_1^*(\omega)}{(\omega-m_1)}\frac{ku^2(\omega)}{4\pi}d\omega, \qquad (6.16)$$

where

$$K_2(\omega)=\langle 0|f_2|N\theta_\omega^+\rangle[(2\omega\Omega)^{1/2}/u(\omega)] \qquad (6.17)$$

and

$$L_2(\omega)=\langle 0|\psi_2|N\theta_\omega^+\rangle[(2\omega\Omega)^{1/2}/u(\omega)]. \qquad (6.18)$$

We are thus required to calculate the functions $K_i(\omega)$ and $L_i(\omega)$, and again we follow the dispersion-theoretic

[6] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **6**, 319 (1957).

technique:

$$K_1(\omega)=\langle 0|f_1|N\theta_\omega{}^+\rangle\frac{(2\omega\Omega)^{1/2}}{u(\omega)}$$

$$=\lim_{t\to-\infty}\langle 0|[f_1,\alpha_k{}^\dagger(t)]|N\rangle e^{i\omega t}\frac{(2\omega\Omega)^{1/2}}{u(\omega)}$$

$$=\langle 0|[f_1,\alpha_k{}^\dagger]|N\rangle\frac{(2\omega\Omega)^{1/2}}{u(\omega)}$$

$$+\sum_s\frac{\langle 0|f_1|s^+\rangle\langle s^+|j^\dagger|N\rangle}{s-\omega-i\epsilon}. \quad (6.19)$$

Making use of the expression (4.32) for f_1, the commutation relations, and the current matrix elements, we obtain

$$K_1(\omega)=-\left(\frac{g_1}{Z_1}+\frac{g_2}{Z_3}\right)+\frac{1}{\pi}\int_\mu^\infty\frac{K_1(\omega')e^{-i\delta(\omega')}\sin\delta(\omega')}{\omega'-\omega-i\epsilon}d\omega' \quad (6.20)$$

as

$$\frac{u(\omega)}{(2\omega\Omega)^{1/2}}\langle 0|j|N\theta_\omega{}^+\rangle=\frac{ku^2(\omega)}{4\pi}M(\omega)=e^{i\delta(\omega)}\sin\delta(\omega). \quad (6.21)$$

The equation for $K_1(\omega)$ is the standard form of integral equation discussed by Omnes[7] and its solution can be written down directly. If we define

$$\rho(\omega)=\frac{P}{\pi}\int_\mu^\infty\frac{\delta(\omega')}{\omega'-\omega}d\omega', \quad (6.22)$$

the solution may be written in the form,

$$K_1(\omega)=-\left(\frac{g_1}{Z_1}+\frac{g_2}{Z_3}\right)+e^{\rho(\omega)+i\delta(\omega)}\times\frac{1}{\pi}\int_\mu^\infty\left(-\frac{g_1}{Z_1}-\frac{g_2}{Z_3}\right)\frac{e^{-\rho(\omega')}\sin\delta(\omega')}{\omega'-\omega-i\epsilon}d\omega'. \quad (6.23)$$

Using the results (B5) and (B7) from the Appendix, we get

$$K_1(\omega)=-\left(\frac{g_1}{Z_1}+\frac{g_2}{Z_3}\right)\left[1-\frac{1}{2\pi i}\frac{1}{Q(\omega)}\times\int_{C_1}\frac{Q(z)}{z-\omega-i\epsilon}dz\right] \quad (6.24)$$

$$=-\left(\frac{g_1}{Z_1}+\frac{g_2}{Z_3}\right)\frac{Q(\infty)}{Q(\omega)}=-g_1\frac{Q(m_1)}{Q(\omega)}, \quad (6.25)$$

[7] R. Omnes, Nuovo cimento **8**, 316 (1958).

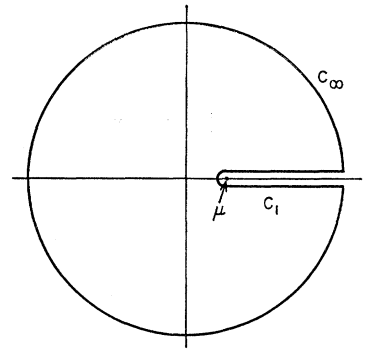

FIG. 1. Contours in the ω plane.

where we have used Eq. (C9) in the last step. C_1 is a contour in ω plane defined in Fig. 1. A reduction formula similar to (6.19) can also be obtained for the function $L_1(\omega)$:

$$L_1(\omega)=\sum_s\frac{\langle 0|\psi_1|s^+\rangle\langle s^+|j^\dagger|N\rangle}{s-\omega-i\epsilon}\frac{(2\omega\Omega)^{1/2}}{u(\omega)}. \quad (6.26)$$

As only $|\mathbf{V}_1\rangle$ and the $|N\theta_\omega{}^+\rangle$ states contribute, we get

$$L_1(\omega)=-\frac{g_1}{m_1-\omega}+\frac{1}{\pi}\int_\mu^\infty\frac{e^{-i\delta(\omega')}\sin\delta(\omega')L_1(\omega')d\omega'}{\omega'-\omega-i\epsilon}. \quad (6.27)$$

Again we can utilize the results from Appendix B5 and B7 and write down a solution of this integral equation:

$$L_1(\omega)=-\frac{g_1}{m_1-\omega}+e^{\rho(\omega)+i\delta(\omega)}\frac{1}{\pi}\int_\mu^\infty e^{-\rho(\omega')}\times\sin\delta(\omega')\frac{g_1}{\omega'-m_1}\frac{d\omega'}{\omega'-\omega-i\epsilon} \quad (6.28)$$

$$=-\frac{g_1Q(m_1)}{(m_1-\omega)Q(\omega)}. \quad (6.29)$$

Similarly,

$$K_2(\omega)=-\frac{g_2Q(m_2)}{Q(\omega)}\text{ and }L_2(\omega)=-\frac{g_2Q(m_2)}{(m_2-\omega)Q(\omega)}. \quad (6.30)$$

Substituting (6.25), (6.29), and (6.30) in (6.11), (6.14), and (6.15), we obtain

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=\frac{1}{Z_1}+\frac{g_1{}^2Q^2(m_1)}{\pi}\int_\mu^\infty\frac{ku^2(\omega)}{4\pi}\times\frac{1}{|Q(\omega)|^2}\frac{1}{(\omega-m_2)^2}d\omega, \quad (6.31)$$

$$\langle 0|\psi_2|\mathbf{V}_2\rangle=\frac{1}{Z_2}+\frac{g_2{}^2Q^2(m_2)}{\pi}\int_\mu^\infty\frac{ku^2(\omega)}{4\pi}\times\frac{1}{|Q(\omega)|^2}\frac{1}{(\omega-m_1)^2}d\omega, \quad (6.32)$$

and

$$\langle 0|\psi_1|\mathbf{V}_2\rangle=\langle 0|\psi_2|\mathbf{V}_1\rangle$$

$$=\frac{1}{Z_3}+g_1g_2Q(m_1)Q(m_2)\frac{1}{\pi}\int_\mu^\infty \frac{ku^2(\omega)}{4\pi}\times\frac{1}{|Q(\omega)|^2}\frac{d\omega}{(\omega-m_1)(\omega-m_2)}. \quad (6.33)$$

These integrals have been evaluated in the Appendix C, with the results expected:

$$\langle 0|\psi_1|\mathbf{V}_1\rangle=1,\quad \langle 0|\psi_2|\mathbf{V}_1\rangle=0,$$

and

$$\langle 0|\psi_1|\mathbf{V}_2\rangle=0,\quad \langle 0|\psi_2|\mathbf{V}_2\rangle=1. \quad (6.34)$$

These are identical with the conditions which we imposed on the eigenstates in Sec. IV. Thus, we see that the use of the asymptotic condition in the form (6.4) is perfectly consistent with the Hamiltonian formalism, at least for the matrix elements calculated above.

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APPENDIX A

It has been proved in the text that the $N-\theta$ scattering amplitude can be put into the following form:

$$M(\omega)=\frac{g_1^2(m_2-\omega)D(m_1)+g_2^2(m_1-\omega)D(m_2)}{(m_1-\omega)(m_2-\omega)D(\omega)}. \quad \text{(A-1)}$$

We can now define a function $Q(\omega)$ such that

$$M(\omega)=[(\omega_0-\omega)/(m_1-\omega)(m_2-\omega)][1/Q(\omega)], \quad \text{(A-2)}$$

where ω_0 is the zero of the numerator in $M(\omega)$. As $M(\omega)$ has poles at m_1 and m_2 with residues $-g_1^2$ and $-g_2^2$, respectively, we have

$$g_1^2Q(m_1)=(\omega_0-m_1)/(m_2-m_1),$$

and

$$g_2^2Q(m_2)=(\omega_0-m_2)/(m_1-m_2). \quad \text{(A-3)}$$

Also, it is evident that

$$Q(\omega)=D(\omega)/[g_1^2D(m_1)+g_2^2D(m_2)]. \quad \text{(A-4)}$$

The function $Q(\omega)$ has a cut from $\mu\to\infty$. The jump across this cut can be evaluated from the unitarity relation,

$$\text{Im}M(\omega)=[ku^2(\omega)/4\pi]|M(\omega)|^2 \quad \text{for} \quad \omega>\mu, \quad \text{(A-5)}$$

whence it follows that

$$\text{Im}Q(\omega)=[ku^2(\omega)/4\pi][(\omega-\omega_0)/(\omega-m_1)(\omega-m_2)] \quad \text{for} \quad \omega>\mu. \quad \text{(A-6)}$$

Although we have obtained the explicit form for $Q(\omega)$ in terms of $D(\omega)$, it is sometimes more convenient to use the form,

$$Q(\omega)=Q(\infty)+\frac{1}{4\pi^2}\int_\mu^\infty \frac{k'u^2(\omega')(\omega'-\omega_0)}{(\omega'-m_1)(\omega'-m_2)}\frac{d\omega'}{(\omega'-\omega-i\epsilon)}, \quad \text{(A-7)}$$

which follows from (A-6) and the fact that $Q(\infty)$ is finite. This gives an important relation:

$$Q(\omega_0)-Q(\infty)=F(m_1,m_2). \quad \text{(A-8)}$$

Now

$$D(\infty)=(1-g_1^2F(m_1,m_1))(1-g_2^2F(m_2,m_2))-g_1^2g_2^2F(m_1,m_2). \quad \text{(A-9)}$$

Using (4.25), we get

$$D(\infty)=\alpha_1\beta_2-(1-K)\alpha_1\beta_2=K\alpha_1\beta_2. \quad \text{(A-10)}$$

Another form can be obtained using the forms of $D(m_1)$ and $D(m_2)$:

$$1-g_1^2F(m_1,m_1)=D(m_2)-g_1^2F(m_1,m_2)$$

and

$$1-g_2^2F(m_2,m_2)=D(m_1)-g_2^2F(m_1,m_2). \quad \text{(A-11)}$$

Substituting in (A-9), we get

$$D(\infty)=D(m_1)D(m_2)-[g_1^2D(m_1)+g_2^2D(m_2)]F(m_1,m_2). \quad \text{(A-12)}$$

Using (A-12), (A-8), and (A-9), we get

$$Q(\omega_0)=D(m_1)D(m_2)/[g_1^2D(m_1)+g_2^2D(m_2)]. \quad \text{(A-13)}$$

APPENDIX B

Here we calculate certain integrals which are required for explicitly writing down the solution of the Omnes[7] type equations encountered in Sec. VI.

As defined in the text,

$$\rho(\omega)+i\delta(\omega)=\frac{1}{\pi}\int_\mu^\infty \frac{\delta(\omega')d\omega'}{\omega'-\omega-i\epsilon}. \quad \text{(B-1)}$$

Since

$$e^{i\delta(\omega)}\sin\delta(\omega)=\frac{ku^2(\omega)}{4\pi}\frac{(\omega_0-\omega)}{(m_1-\omega)(m_2-\omega)}\frac{1}{Q(\omega)}, \quad \text{(B-2)}$$

we have

$$\delta(\omega)=-(1/2i)[\ln Q(\omega)-\ln Q(\omega^*)], \quad \text{(B-3)}$$

where use has been made of the fact that $\delta(\omega)$ is real and $Q^*(\omega)=Q(\omega^*)$. It follows that

$$\rho(\omega)+i\delta(\omega)=-\frac{1}{2\pi i}\int_{C_1}\frac{\ln Q(z)}{z-\omega-i\epsilon}dz \tag{B-4}$$

$$=-\frac{1}{2\pi i}\oint\frac{\ln Q(z)dz}{z-\omega-i\epsilon}+\frac{1}{2\pi i}\int_{C_\infty}\frac{\ln Q(z)dz}{z-\omega-i\epsilon}$$

$$=-\ln Q(\omega)+\ln Q(\infty).$$

Thus,

$$e^{\rho(\omega)+i\delta(\omega)}=Q(\infty)/Q(\omega), \tag{B-5}$$

and

$$e^{-\rho(\omega)}\sin\delta(\omega)=-[\mathrm{Im}Q(\omega)/Q(\infty)]. \tag{B-6}$$

APPENDIX C

Here we evaluate some integrals encountered in Sec. VI.

$$I_1=\frac{1}{\pi}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{4\pi}\frac{1}{|Q(\omega')|^2}\frac{1}{(\omega'-m_1)^2} \tag{C-1}$$

$$=\frac{1}{\pi}\int_\mu^\infty\frac{\mathrm{Im}Q(\omega')d\omega'}{[Q(\omega')]^2}\frac{(\omega'-m_2)}{(\omega'-m_2)(\omega'-\omega_0)}. \tag{C-2}$$

Making use of (A-6), we have

$$I_1=-\frac{1}{2\pi i}\int_{C_1}\frac{dz}{Q(z)}\frac{(z-m_2)}{(z-m_1)(z-\omega_0)} \tag{C-3}$$

$$=-\left[\frac{(m_1-m_2)}{(m_1-\omega_0)}\frac{1}{Q(m_1)}+\frac{(\omega_0-m_2)}{(\omega_0-m_1)}\frac{1}{Q(\omega_0)}-\frac{1}{Q(\infty)}\right]. \tag{C-4}$$

Making use of (A-3), we get

$$I_1=-\frac{1}{g_1^2Q^2(m_1)}+\frac{1}{g_1^2Q(m_1)Q(\omega_0)}+\left[\frac{1}{Q(\omega_0)}-\frac{1}{Q(\infty)}\right]. \tag{C-4$'$}$$

Using (A-8), we find

$$I_1=-\frac{1}{g_1^2Q^2(m_1)}+\frac{1}{Q(\omega_0)}\left[\frac{1}{g_1^2Q(m_1)}+\frac{F(m_1,m_2)}{Q(\infty)}\right]. \tag{C-5}$$

A minor manipulation gives the result,

$$I_1=-[1/g_1^2Q^2(m_1)][1-(1/Z_1)]. \tag{C-6}$$

Similarly,

$$I_2=\frac{1}{\pi}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{4\pi}\frac{1}{|Q(\omega')|^2}\frac{1}{(\omega'-m_2)^2},$$

$$I_2=-\frac{1}{g_2^2Q^2(m_2)}\left[1-\frac{1}{Z_2}\right], \tag{C-7}$$

and

$$I_3=\frac{1}{\pi}\int_\mu^\infty\frac{k'u^2(\omega')d\omega'}{4\pi}\frac{1}{|Q(\omega')|^2}\frac{1}{(\omega'-m_1)(\omega'-m_2)}$$

$$=-\frac{1}{2\pi i}\int_{C_1}\frac{dZ}{Q(Z)(Z-\omega_0)}$$

$$=-\frac{1}{2\pi i}\oint\frac{dz}{Q(z)(z-\omega_0)}+\frac{1}{2\pi i}\int_{C_\infty}\frac{dz}{(z-\omega_0)Q(z)}.$$

Using (A-8), we have

$$I_3=-\left[\frac{1}{Q(\omega_0)}-\frac{1}{Q(\infty)}\right]=\frac{F(m_1,m_2)}{Q(\omega_0)Q(\infty)},$$

and, from (A-13),

$$I_3=[F(m_1,m_2)/Q(m_1)Q(m_2)][1/K\alpha_1\beta_2]. \tag{C-8}$$

Another result which is useful, is proved using (4.25):

$$\left(\frac{g_1}{Z_1}+\frac{g_2}{Z_3}\right)=\frac{g_1}{K\alpha_1\beta_2}[\beta_2+g_2^2F(m_2,m_1)]$$

$$=\frac{g_1}{K\alpha_1\beta_2}[1-(m_2-m_1)g_2^2F(m_2,m_2,m_1)]$$

$$=\frac{g_1Q(m_1)}{Q(\infty)},$$

$$\left(\frac{g_2}{Z_2}+\frac{g_1}{Z_3}\right)=\frac{g_2Q(m_2)}{Q(\infty)}. \tag{C-9}$$