Photon Density and the Gamma-Ray Flux at a Point in an Expanding Universe

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The investigation deals with certain cosmological questions involved in the interpretation of the gamma-ray flux measurements made by the satellite Explorer XI. An expanding universe filled with similar sources of optical radiation is considered and the photon density in the neighborhood of an observer is obtained. Numerical values are given for the density of 1-eV photons when the sources are regarded as blackbodies at a temperature of 5000°K. The necessity of specifying the bandwidth, the model of the universe, and the scale of distance before a density can be stated, is emphasized. The models of the universe considered are those of general relativity and that of the steady-state theory. It is concluded that a photon density can be specified, in the present state of astronomical knowledge, only as lying within wide limits.

The gamma-ray flux observed can be interpreted theoretically by the use of general relativity models in which the density of matter is rather high. An equally satisfactory interpretation is possible by the use of the steady-state theory. It is argued that the objections to the second theory, based on the Explorer XI data, refer only to the hypothesis of the creation of antiprotons and not to the theory itself.

I. INTRODUCTION

A PRELIMINARY report on the measurement of gamma rays by the satellite Explorer XI (1961 ν) has been published by Kraushaar and Clark.[1] Their interpretation of the data makes use of two items of information which involve cosmological theory. One of these is mentioned in the course of the discussion of the production of electron pairs by gamma rays which collide with optical photons. The threshold energy for this process is 10^{12} eV. It is said that "the density of starlight is such that gamma rays of this energy and greater should be only local in origin." The implication appears to be that the term "density of starlight" has a precise meaning and that its numerical value is known. The other item consists of a formula for the anticipated gamma-ray intensity due to all sources in the universe.

The two points in question are special cases of the general problem of finding the density, or the flux density, of radiation which an observer might expect to observe in his neighborhood. As further experiments of the Explorer XI type are carried out, the solution of this problem is likely to acquire increasing importance. The present paper, therefore, gives the underlying theory for the photon density and for the gamma-ray intensity at a point in a uniform model of the universe, whether this is a general relativity model or that of the steady-state theory. In all cases, the origin of the phenomenon is attributed to the entire material content of the universe.

The theory of this paper is an extension of the work done by McVittie and Wyatt[2] on the background radiation at radio and at optical wavelengths produced by all unresolved sources of radiation in the universe. In this earlier paper, Milne's model of the universe was alone considered because the mathematical theory of this model is particularly simple. It has now been possible to extend the theory to any model of the universe and to throw it into a form in which simple numerical integrations can be employed to give the desired results.

II. UNIFORM MODELS OF THE UNIVERSE

We bring together in this section the results of cosmological theory that will be required in the sequel. Detailed expositions will not be attempted because they can be found elsewhere.[3] We begin with results that are valid in both general relativity and the steady-state theory. In subsection (i) are listed those which are peculiar to general relativity, and in (ii), those which apply in the steady-state theory. Finally, in (iii), the numerical values found from observation are summarized.

The metric of space-time in a model universe is taken in its usual form

$$ds^2=dt^2-\frac{R^2(t)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)}{c^2(1+kr^2/4)^2}, \quad (2.1)$$

where s, t are measured in seconds, R in cm, c in cm/sec, and (r,θ,ϕ) are dimensionless space coordinates. The space-curvature constant k has the values $+1$, 0, or -1 according as space is spherical (closed), Euclidean, or hyperbolic (infinite in extent). This will be the meaning of k in this section; in the remainder of the paper k will stand for Boltzmann's constant. The observer O in Fig. 1 is at $r=0$ and makes his observations at time t_0. The figure shows the cross sections of two concentric shells centered at O, of coordinate radii r_i and r_0, respectively, and widths dr_i and dr_0. The distances from O of all points within each shell increase with time because R is assumed to increase with time. Shell r_0 is so close to O at time t_0 that the travel time of the radiation emitted by a source in the shell is, cosmically speaking, negligible. On the other hand, the departure time t_i of

[1] W. L. Kraushaar and G. W. Clark, Phys. Rev. Letters 8, 106, (1962).
[2] G. C. McVittie and S. P. Wyatt, Astrophys. J. 130, 1 (1959).
[3] G. C. McVittie, *General Relativity and Cosmology* (University of Illinois Press, Urbana, Illinois, 1962), Chaps. 8 and 9. G. C. McVittie, *Fact and Theory in Cosmology* (The Macmillan Company, New York, 1962).

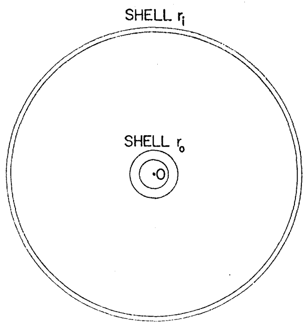

FIG. 1. Schematic shells centered at the observer O.

radiation from a source in shell r_i is less than t_0 and the travel time (t_0-t_i) is not negligible.

All sources of radiation in a shell have, by definition, constant coordinates (r,θ,ϕ). Unless otherwise stated, it is assumed that sources do not vanish, and are not created, as time proceeds. It will also be assumed that their radiative properties are constant in time.

The red shift in the spectrum of a source in shell r_i, measured by the observer O, is δ_i, where

$$1+\delta_i=R(t_0)/R(t_i)=R_0/R_i. \tag{2.2}$$

It is convenient to work in terms of a variable

$$Y=R(t)/R_0, \tag{2.3}$$

so that, for shell r_i,

$$1+\delta_i=1/Y_i. \tag{2.4}$$

Radiation, which is emitted by a source in shell r_i at a frequency f_i centered in a frequency interval df_i, arrives at O with frequency f centered in the frequency interval df, where

$$f_i=(1+\delta_i)f=f/Y_i, \quad df_i=(1+\delta_i)df=df/Y_i. \tag{2.5}$$

The arrival time t_0 and the departure time t_i for the radiation from a source in shell r_i are related to one another by the equation of the null-geodesic joining source and observer. The differential form of the equation is

$$(1+kr^2/4)^{-1}R(t)dr=-cdt, \tag{2.6}$$

and its integrated form is

$$c\int_{t_i}^{t_0}\frac{dt}{R(t)}=\int_0^{r_i}\frac{dr}{(1+kr^2/4)}. \tag{2.7}$$

The luminosity distance of the source is

$$D_i=R_0(1+\delta_i)r_i(1+kr_i^2/4)^{-1}. \tag{2.8}$$

The intensity of the source, as it is measured at O, is proportional to the inverse square of the luminosity distance. This variable takes account of the Doppler effect, the change in the rate of arrival of quanta (number effect), and the reduction in the size of the solid angle—three phenomena due to the red shift.

The Hubble "constant" h_1 and the acceleration factor q_0 are functions of t_0 defined by the equations

$$h_1=R_0'/R_0, \quad q_0=-(R_0''/R_0)(R_0/R_0')^2, \tag{2.9}$$

where a prime denotes a derivative with respect to the time. The Hubble constant has the physical dimensions of the reciprocal of a time, whereas q_0 is a pure number. An alternative expression for h_1 is possible in the form of so many km/sec per megaparsec (1 megaparsec $=3.087\times10^{24}$ cm). In these units, it will be denoted by H and

$$H=3.087\times10^{19}h_1. \tag{2.10}$$

The volume of shell r_i at the departure time t_i is

$$d\mathcal{V}_i=4\pi R_i^3r_i^2(1+kr_i^2/4)^{-3}dr_i,$$

and we shall find that many of our formulas will involve the combination $d\mathcal{V}_i/D_i^2$. With the aid of (2.8) we obtain

$$d\mathcal{V}_i/D_i^2=4\pi(R_i/R_0)^4(1+kr_i^2/4)^{-1}R_idr_i. \tag{2.11}$$

But this can be converted, through (2.6) into a formula involving dt_i, the time taken by radiation to cross shell r_i. We have, in fact,

$$d\mathcal{V}_i/D_i^2=-4\pi c(R_i/R_0)^4dt_i.$$

We introduce a dimensionless time variable X by

$$X=h_1t, \tag{2.12}$$

which will be employed in conjunction with Y, defined by (2.3). Thus we have

$$d\mathcal{V}_i/D_i^2=-4\pi(c/h_1)Y_i^4dX_i. \tag{2.13}$$

The foregoing results are valid in general relativity and in the steady-state theory. It is now necessary to deal with each theory separately.

(i) General Relativity

A uniform distribution of permanent and similar sources of radiation is defined by saying that their number density (number per unit volume) in the typical shell r_i, at the departure time t_i, is

$$N_i=N_0(1+\delta_i)^3=N_0/Y_i^3. \tag{2.14}$$

Here N_0 is the number density in shell r_0, i.e., in the, cosmically speaking, neighborhood of O, at the arrival time t_0. Uniformity, therefore, means dependence on the time alone, and independence of spatial position. The density ρ and pressure p of the material content of the model universe whose metric is (2.1) are given by Einstein's field equations. If ρ is in g/cm^3 and p in dyn/cm^2, we have

$$8\pi G\rho=3(R'/R)^2+3kc^2/R^2-\lambda, \tag{2.15}$$

$$8\pi Gp/c^2=-2(R''/R)-(R'/R)^2-kc^2/R^2+\lambda, \tag{2.16}$$

where G is the constant of gravitation and λ, the cosmical constant. Hence the density and pressure are uniform because they vary with time alone. A subgroup of these models contains those in which p/c^2 is regarded as negligibly small compared with ρ throughout the history of the model. We confine attention to these zero-pressure models. If ρ_0 is the density in the neighborhood of O at time t_0, it follows, from (2.15) and (2.16) with $p/c^2=0$, that

$$\lambda/h_1^2=-(3q_0-\sigma_0),\qquad kc^2/(h_1^2R_0^2)=-(q_0+1-\sigma_0), \tag{2.17}$$

where

$$\sigma_0=4\pi G\rho_0/h_1^2. \tag{2.18}$$

It also follows from (2.15), (2.16) that in a zero-pressure model

$$\rho=\rho_0(R_0/R)^3=\rho_0/Y^3. \tag{2.19}$$

Using (2.17) and (2.19) in (2.15) and then converting to Y, X given by (2.3) and (2.12), one finds

$$(dY/dX)^2=[\tfrac{2}{3}\sigma_0+(q_0+1-\sigma_0)Y-(q_0-\tfrac{1}{3}\sigma_0)Y^3]/Y. \tag{2.20}$$

This equation determines Y at each value of X and, therefore, of t. Thus Y_i in (2.13) is known in terms of X_i. It is convenient to write (2.13) entirely in terms of Y in the form

$$d\mathcal{V}_i/D_i^2=-4\pi(c/h_1)(Y_i^4)Y_i^{1/2}[\tfrac{2}{3}\sigma_0+(q_0+1-\sigma_0)Y_i-(q_0-\tfrac{1}{3}\sigma_0)Y_i^3]^{-1/2}dY_i. \tag{2.21}$$

Among the zero-pressure models there is a class, defined by certain combinations of the values of the constants q_0 and σ_0, which possess an initial instant at which $R=0$. But $Y=0$ when $R=0$ and, of course, $Y=1$ at the instant t_0. Thus, the lapse of time from the initial moment to t_0 is found by integrating (2.20). This time lapse is called the "age" of the model universe. If the reciprocal of the Hubble constant, which is also a time interval, is denoted by T_0, then

$$t_0/T_0=\int_0^1 Y^{1/2}[\tfrac{2}{3}\sigma_0+(q_0+1-\sigma_0)Y-(q_0-\tfrac{1}{3}\sigma_0)Y^3]^{-1/2}dY, \tag{2.22}$$

and so

$$t_0=AT_0\ \text{sec}, \tag{2.23}$$

where A is the value of the definite integral in (2.22).

(ii) The Steady-State Theory

In this theory it is necessarily the case that h_1 is an absolute constant and that, in (2.1),

$$R=R_0e^{h_1(t-t_0)},\quad k=0.$$

It is also the case that new galaxies appear, because of the creation of matter process, at such a rate that their number density is constant in all shells and, therefore,

$$N_i=N_0. \tag{2.24}$$

The original formulation of the theory by Bondi and Gold[4] dealt only with this point and contained no theory of dynamics, including gravitation. However, Hoyle[5] has attempted to remedy this defect and his version of the steady-state theory shows that the density of matter is also constant and is related to the Hubble constant in such a way that

$$\sigma_0=4\pi G\rho/h_1^2=1.5. \tag{2.25}$$

The exponential character of R means that

$$q_0=-1, \tag{2.26}$$

and

$$dY/dX=Y. \tag{2.27}$$

Hence (2.13) now becomes

$$d\mathcal{V}_i/D_i^2=-4\pi(c/h_1)Y_i^3dY_i. \tag{2.28}$$

The age of the model may again be defined as the lapse of time required for Y to increase from zero to unity. Since R is zero when $t=-\infty$, this time lapse is infinitely long.

(iii) Observational Results

The data on the red shifts and apparent magnitudes of the brighter members of galaxies in clusters indicate that the value of H probably lies between 75 km/sec/megaparsec and 150 km/sec/megaparsec, or somewhat more. The differences arise because of uncertainties in the scale of distance, H being inversely proportional to the distance scale. We write

$$H=75F\ \text{km/sec/megaparsec}, \tag{2.29}$$

where F is a factor whose upper and lower limits will be taken to be 1 and 2. If F is not equal to 1, all distances must be multiplied by $1/F$. The time $1/h_1$ becomes, by (2.10) and (2.29),

$$T_0=1/h_1=4.12\times10^{17}F^{-1}\ \text{sec}=13\times10^9\ F^{-1}\text{yr}. \tag{2.30}$$

We shall adopt Oort's[6] determination of the average density of matter in the, cosmically speaking, neighborhood of O, who is now identified as a terrestrial observer. That this is what Oort means by "the universe" is clear from the fact that he does not take account of galaxies whose apparent magnitudes exceed 18. His distance scale corresponds to $F=1$. The average density he finds, $\bar{\rho}_0$, and the average mass-to-light ratio for galaxies, μ, that he employs are, respectively,

$$\bar{\rho}_0=3.1\times10^{-31}\ \text{g/cm}^3,\quad \mu=21. \tag{2.31}$$

It is unlikely that $\bar{\rho}_0$ is in error by a factor of 10—probably a factor of 2 or 3 would be realistic—but in order to allow for all possibilities we shall introduce a

[4] H. Bondi and T. Gold, Monthly Notices Roy. Astron. Soc. **108**, 252 (1948).
[5] H. Bondi, *Cosmology* (Cambridge University Press, New York, 1960), 2nd ed., Chap. 12.
[6] J. H. Oort, *La Structure et l'Evolution de l'Univers* (Stoops, Brussels, 1958), p. 163.

factor η such that the density ρ_0 in a model universe is related to $\bar{\rho}_0$ by

$$\eta = \rho_0/\bar{\rho}_0. \tag{2.32}$$

Oort's method for finding the average density implies that a change of distance scale by the factor $1/F$ causes the density to be multiplied by F^2 and not by F^3. The reason is that, though all volumes are multiplied by $1/F^3$, the mass of each galaxy is also multiplied by $1/F$ because masses are determined in such a way that they are proportional to the adopted distances. Therefore, a density which is, of course, a mass divided by a volume, is multiplied by F^2. The value of σ_0 corresponding to $\bar{\rho}_0$ is

$$\bar{\sigma}_0 = 4\pi G\bar{\rho}_0/h_1^2 = 0.044. \tag{2.33}$$

We shall assume that, in all our models, σ_0 has the property of being independent of the distance scale and, therefore, (2.32) may also be written as

$$\eta = \sigma_0/(0.044). \tag{2.34}$$

The density $\bar{\rho}_0$ refers to the matter present in galaxies both in the form of luminous matter (stars) and of non-luminous (interstellar gas clouds). This is the matter that has so far been observed and studied by astronomers and, in the present writer's view, is the material which is the subject of cosmology. However, there are some cosmologists who like to assert that this material constitutes only a small fraction of that which is present in the universe. The argument is that, if there were additional material in the form of some kind of unobservable intergalactic matter, its dynamical effects would become noticeable only if the average density exceeded 10^{-27} g/cm^3. Such a point of view would seem to make of cosmology the study of a universe which has been imagined by the investigator rather than an investigation of the universe which astronomers see around them. We shall, therefore, reject the high-density hypothesis until its proponents can adduce some observational evidence for the existence of the excess material.

The acceleration factor q_0 is determinable in principle from the red shift data. If it is assumed that there are no secular changes[7] in the optical emission of galaxies, the estimates of q_0 run from $+0.5$ to as high as $+3$, with $+1$ as a favored value at present. It will be noticed that the value $q_0 = -1$ predicted by the steady-state theory does not fall fall within these limits.

III. THE DENSITY OF PHOTONS

It will be assumed that the "density of starlight" means the density of photons of some given energy, in the neighborhood of the observer, due to the radiation of all sources in a model universe. To fix ideas, consider photons of energy 1 eV which means radiation whose frequency and wavelengths are, respectively,

$$f_1 = 2.418\times 10^{14}\text{ cps},\quad \lambda_1 = 12396\text{ Å}. \tag{3.1}$$

This radiation is in the infrared. The theory we give below, however, applies not only to these photons but to those of other energies as well, if the numerical values of certain constants are altered appropriately.

It is well known in the theory of radiation that the density of photons can be specified only when the bandwidth employed is known. Consider a source in shell r_i in Fig. 1 that emits radiation of frequency f_i centered in an infinitesimal bandwidth df_i. Let the energy emitted isotropically by the source be

$$dL_i = bB(f_i)df_i\text{ W}, \tag{3.2}$$

where $B(f_i)$ is the energy distribution function for the source, and b is a normalizing factor. This flow of energy arrives at O and, if it is to contribute to the f_1-photon density, f_i and df_i must be related to f_1 and df_1 by (2.5). Thus in terms of the received frequency and bandwidth

$$dL_i = bB(f_1/Y_i)df_1/Y_i. \tag{3.3}$$

Let N_i be the number density of sources in shell r_i, whose volume is $d\mathcal{V}_i$. Then the flux of f_1 photons at the observer, due to the radiation from all sources in the shell, is, by (2.14),

$$\begin{aligned} dE_i &= bB(f_1/Y_i)df_1(N_i/Y_i)d\mathcal{V}_i/(4\pi D_i^2)\\ &= bB(f_1/Y_i)df_1(N_0/Y_i^4)d\mathcal{V}_i/(4\pi D_i^2)\text{ W cm}^{-2}. \end{aligned} \tag{3.4}$$

When f_1 corresponds to a frequency in or near the optical range of frequencies used in astronomy, the constant b can be eliminated as follows: Let f_0', f_0 be the upper and lower limits of optical frequencies and suppose that the same function B applies to this range as it does to f_1. Consider a typical source near to the observer, so that its red shift is zero. Then the emission of this source is

$$L_0 = b\int_{f_0}^{f_0'} B(f)df = bI_0\text{ W}, \tag{3.5}$$

where I_0, of course, stands for the value of the definite integral. Hence with the aid of (2.21) we have

$$\begin{aligned} dE_i = &-(N_0L_0/I_0)(c/h_1)\\ &\times B(f_1/Y_i)df_1Z(Y_i)dY_i\text{ W/cm}^2, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} Z(Y) = Y^{1/2}[&\tfrac{2}{3}\sigma_0 + (q_0+1-\sigma_0)Y\\ &-(q_0-\tfrac{1}{3}\sigma_0)Y^3]^{-1/2}. \end{aligned} \tag{3.7}$$

The energy density is, therefore, dE_i/c J/cm^{-3}, and it consists of photons of energy hf_1. Hence it is possible to define the number of f_1 photons per unit volume and per unit bandwidth as

$$\begin{aligned} dP_i &= dE_i/(chf_1df_1)\\ &= -[N_0L_0/(I_0hf_1)](1/h_1)B(f_1/Y_i)Z(Y_i)dY_i\\ &\qquad f_1\text{ photons cm}^{-3}\text{ (cps)}^{-1}. \end{aligned} \tag{3.8}$$

[7] A. Sandage [Astrophys. J. 134, 916 (1961)] in an illustrative investigation, shows that secular changes of luminosity might reduce q_0 to $+0.2$.

This is, of course, the contribution to the number density of photons near the observer provided by the sources in shell r_i. Thus adding the contributions for all shells from the observer outwards, we have

$$P_1=[N_0L_0/(I_0hf_1)](1/h_1)\int_0^1 B(f_1/Y)Z(Y)dY \quad f_1 \text{ photons cm}^{-3}\,(\text{cps})^{-1}, \tag{3.9}$$

assuming that the model universe is of the kind which starts from $Y=0$.

Further progress depends not only on the choice of some specific model but also on a knowledge of the form of the function B. Ideally this energy distribution function should be chosen as the average function, determined from observation, for a variety of types of galaxies. The function should apply in the optical range of frequencies and in the infrared also. Such an empirical determination is not yet available. We shall, therefore, make the rough approximation that our sources radiate like blackbodies at a temperature of 5000°K. Now for a blackbody at temperature T we may take

$$B(f)=f^3(e^{hf/kT}-1)^{-1},$$

since multiplicative constants have been dealt with through the factor b. The letter k now stands for Boltzmann's constant. Let

$$y=cT/f,$$

and let y_0', y_0 be the values of y corresponding to the limits of frequency, f_0' and f_0. With the aid of the variable y, the integral I_0 in (3.5) can be evaluated from the tabulated values of Allen's function[8] $\mathfrak{F}_\nu/\mathfrak{F}_{\nu m}$. Allen writes $\nu=f$ and his definition of $\mathfrak{F}_{\nu m}$ is

$$\mathfrak{F}_{\nu m}=(2\pi h/c^2)(kT/h)^3(0.3544)^{-3}(e^{1/(0.3544)}-1)^{-1}. \tag{3.10}$$

Also, for a fixed value of T, we have

$$df=-cTy^{-2}dy,$$

and so

$$I_0=[c^3T/(2\pi h^2)]\mathfrak{F}_{\nu m}\int_{y_0}^{y_0'}(\mathfrak{F}_\nu/\mathfrak{F}_{\nu m})y^{-2}dy. \tag{3.11}$$

Moreover

$$B(f_1/Y)=f_1^3(e^{hf_1/kTY}-1)^{-1}Y^{-3}.$$

Hence, using (3.11) also, we may write (3.9) as

$$P_1=C\int_0^1(e^{x_1/Y}-1)^{-1}Y^{-3}Z(Y)dY \quad f_1 \text{ photons cm}^{-3}\,(\text{cps})^{-1}, \tag{3.12}$$

[8] C. W. Allen, *Astrophysical Quantities* (The Athlone Press, London, 1955), p. 101.

where

$$C=(N_0L_0/h_1)[x_1^2/(ckT^2)](0.3544)^3(e^{1/(0.3544)}-1)J_0^{-1},$$
$$x_1=hf_1/(kT), \tag{3.13}$$
$$J_0=\int_{y_0}^{y_0'}(\mathfrak{F}_\nu/\mathfrak{F}_{\nu m})y^{-2}dy.$$

Numerical values must now be calculated. The combination N_0L_0 can be found from Oort's data which imply that $F=1$. Let M_0 be the mass of an average galaxy and let L_0 be its absolute, or intrinsic, luminosity. The corresponding quantities for the sun are $M_\odot$ and $L_\odot$. The mass-to-light ratio for the galaxy is

$$\mu=(M_0/M_\odot)/(L_0/L_\odot),$$

and therefore

$$N_0L_0=(N_0M_0/\mu)(L_\odot/M_\odot).$$

Now N_0M_0 is clearly equal to $\bar{\rho}_0$ g/cm^3, the average density of matter near the observer, and so

$$N_0L_0=(\bar{\rho}_0/\mu)(L_\odot/M_\odot).$$

Since $L_\odot=3.86\times10^{26}$ W and $M_\odot=1.99\times10^{33}$ g, Oort's data (2.31) give $N_0L_0=2.86\times10^{-32}$ erg/sec. Now we want to allow for possible changes in the distance scale. We have seen that one effect of the factor F is to multiply the density by F^2. But the mass-to-light ratio is also affected. The luminosities with which we are concerned being absolute, they must be deduced from the observed luminosities by using the inverse square law of diminution of brightness with respect to distance. Therefore, a luminosity is multiplied by F^2 and, since masses are multiplied by $1/F$, the quantity μ is multiplied by $1/F^3$. Thus N_0L_0 is multiplied by F^5. We may also wish to allow for changes of opinion regarding the density which are not due to alterations in the distance scale. This can be done by multiplying N_0L_0 by the factor η of (2.32). The assumption that σ_0 is independent of F means that (2.34) may also be used. Hence the general value of N_0L_0 is

$$N_0L_0=2.86\times10^{-32}\eta F^5$$
$$=2.86\times10^{-32}(\sigma_0/0.044)F^5. \tag{3.14}$$

Thus the constant C of (3.13) may be written with the aid of (2.30) as

$$C=2.01\times10^{-9}x_1^2\eta F^4/(T^2J_0). \tag{3.15}$$

We assume that $T=5000$°K and that the range of optical wavelengths runs from 3200 Å to 10 000 Å. Since $y=cT/f=\lambda T$, the limits for the integral J_0 are easily calculable. Moreover, the value of x_1 for 1-eV photons follows from (3.1). Thus

$$x_1=2.3213,\quad y_0'=0.16,\quad y_0=0.50, \tag{3.16}$$

and a numerical integration then gives

$$J_0=3.2952. \tag{3.17}$$

The final result is that

$$P_1=1.31\times10^{-16}\eta F^4\int_0^1 (e^{2.3213/Y}-1)^{-1}Y^{-3}Z(Y)dY$$

$$1 \text{ eV photons cm}^{-3} (\text{cps})^{-1}. \quad (3.18)$$

The value of the "Hubble distance" employed by Kraushaar and Clark corresponds to $F=1$, and their density of matter in intergalactic space is 10^{-5} proton/cm^3. This means that $\rho_0=1.67\times10^{-29}$ g/cm^3. Hence $\eta=1670/31$ and $\sigma_0=2.37$. Complete specification of a model universe also requires a value of q_0. The most likely model consistent with these values is the general relativity model discussed in Sec. II(i) with the added restriction that the cosmical constant is zero.[9] Hence by (2.17), $q_0=\sigma_0/3$, and the model is specified by

$$\sigma_0=2.37, \quad q_0=0.79. \quad (3.19)$$

This is a high-density model, with a density 54 times Oort's. It has a positive space-curvature constant [k in (2.17)] and, therefore, space is of finite extent. The age is relatively short because the constant A in (2.23) has the value 0.60, a result found by the integration of (2.22) with the values (3.19) of the constants. Numerical evaluation of the integral in (3.18) yields 0.0485. Hence with $\eta=1670/31$ and $F=1$, we have

$$P_1=3.44\times10^{-16} \quad 1 \text{ eV photons cm}^{-3} (\text{cps})^{-1}. \quad (3.20)$$

A model which, in the present writer's opinion, fits the astronomical data better than the one just considered is defined by

$$q_0=1.02, \quad \sigma_0=0.060. \quad (3.21)$$

Thus $\eta=15/11$. This model has a negative cosmical constant and the space-curvature constant is negative, which means that space is hyperbolic and infinite in extent. The age is $0.75T_0$ and, thus, is longer than that of any model in which the cosmical constant is zero and $q_0\geqslant0.5$. The maximum age for such models is $0.67T_0$ and occurs in the Einstein–de Sitter universe ($q_0=0.5$, $\sigma_0=1.5$). We introduce (3.21) into (3.7) and then evaluate the integral in (3.18); it turns out to be 0.0524. Hence with $\eta=15/11$ and $F=1$, there comes

$$P_1=9.38\times10^{-18} \quad 1 \text{ eV photons cm}^{-3} (\text{cps})^{-1}. \quad (3.22)$$

Therefore, the change of model universe has reduced the value of P_1 for 1-eV photons by a factor of 37. Moreover, if F is different from unity we have, instead of (3.20) and (3.22),

$$\left.\begin{array}{l}P_1=3.44\times10^{-16}F^4\\ P_1=9.38\times10^{-18}F^4\end{array}\right\} \quad 1 \text{ eV photons cm}^{-3} (\text{cps})^{-1}, \quad (3.23)$$

in the two models, respectively.

In order to find the density of 1-eV photons, it is necessary to multiply P_1 by some appropriate bandwidth. It must be remembered that the foregoing theory presupposes that the bandwidth is infinitesimal. In practice this may be taken to mean that the bandwidth should not exceed, say 5% of f_1. This amounts to 1.21×10^{13} cps. The density of 1-eV photons would thus be 4.2×10^{-3} cm^{-3} for (3.20) and 1.1×10^{-4} cm^{-3} for (3.22). These figures are merely illustrative and show that the photon density at a particular frequency probably lies in the range 10^{-3} to 10^{-4} cm^{-3}. This is true even for 1-eV photons, the total amount of which benefits by the inclusion of red-shifted photons that were emitted by the sources as blue, yellow, ultraviolet, etc., light and even as x-rays.

(ii) The Steady-State Theory

The formula for P_1 predicted by this theory may be obtained by the preceding method, the formulas (2.24) to (2.28) being used instead of the general relativity relations. The result is

$$P_1=C\int_0^1 \{(e^{x_1/Y}-1)^{-1}Y^{-3}\}Y^2dY$$

$$f_1 \text{ photons cm}^{-3} (\text{cps})^{-1}, \quad (3.24)$$

where C, x_1 have the same meanings as in (3.13). Because of (2.25), we shall say that $\eta=1.5/0.044$ in the steady-state universe. Numerical computations for the values (3.16) of the constants show that the integral in (3.24) has the value 0.0335. Hence

$$P_1=1.50\times10^{-16}F^4 \quad 1 \text{ eV photons cm}^{-3} (\text{cps})^{-1}. \quad (3.25)$$

Comparison of this with (3.23) shows that the steady-state theory predicts a value of P_1 equal to 0.44 of that found for the first, high-density, general relativity model and 16 times that of the second. Kraushaar and Clark's density of intergalactic protons leads to $\sigma_0=2.37$, whereas $\sigma_0=1.5$ in Hoyle's version of the steady-state theory.

When the density of photons of some energy other than 1 eV is required, or when the temperature of the sources is different from 5000°K, the computations in this section must be repeated with the appropriate values of C and of x_1. Logically, of course, all frequencies in the optical range should be considered and the total photon density of all such frequencies be calculated. This would involve lengthy numerical computations because it would be first necessary to integrate (3.6) numerically, for each value of Y_i, over the whole range of optical frequencies. Then a second

[9] The observational reasons for supposing that this constant is zero are based on data available up to 1932 when these reasons were stated by Einstein. The accumulation of astronomical data since that time requires a reconsideration of the question. This has been done by G. C. McVittie [J. Ind. and Appl. Math. (to be published)] and it leads to the conclusion that agreement with today's data is best obtained through a nonzero (negative) cosmical constant.

numerical integration over Y would have to be carried out. As an alternative to this laborious procedure, it might be sufficient to calculate P_1 at each of a number of frequencies, throughout the finite range, and then add the results. This extension of the present work is not attempted here chiefly because the assumption that a galaxy radiates like a blackbody at a particular temperature is a crude one. Observations now in progress should supply, in or year or two, more realistic energy distribution functions B for the spectra of galaxies. When these become available, the total density of all photons could be calculated by the method outlined above.

(iii) Conclusions

It is clear from the foregoing that, when the density of starlight is interpreted to mean the density of photons of some particular frequency (or energy), wide variations are possible. The bandwidth itself can easily introduce a factor of 10 or more. But apart from this question, which is one of definition, there are other sources of uncertainty. We do not know with certainty what specific model of the universe to choose. We have seen that a change of model can introduce a factor of 37 in the photon density. We are also uncertain about the scale of distance, which can introduce factors of up to 16. And, as has already been stated, our empirical knowledge of the function B is still scanty. The blackbody approximation has, however, served to show how much the photon density depends on the model universe and on the scale of distance.

This investigation reveals how very desirable a measurement from an artificial satellite of the 1-eV photon density would be.[10] Those models and those values of F which gave values that were too high or too low might thus be rejected. It is conceivable that the experiment might permit us to decide between the general relativity and the steady-state models, though the closeness of the values (3.23) and (3.25) suggests that no very sanguine hopes should be entertained.

IV. THE FLUX OF GAMMA-RAYS

Kraushaar and Clark's experiment suggests that the average directional flux of gamma rays in the neighborhood of the earth lies in the range

$$J=(7.35\pm 3.65)\times 10^{-4}\ \mathrm{cm^{-2}\ sr^{-1}\ sec^{-1}}. \qquad (4.1)$$

The question arises whether this flux could be due to gamma rays originating throughout the universe by π^0-producing collisions between ordinary cosmic rays and gas. The source strength for this process is

$$S=jn\sigma m\ \mathrm{cm^{-3}\ sr^{-1}\ sec^{-1}}, \qquad (4.2)$$

where j is the directional cosmic-ray intensity, n is the gas density in protons/cm^3, σ is the cosmic ray–proton cross section in cm^2, and m is the average number of gamma rays produced per interaction. We give below a theory for J in a uniform model of the universe based on a source strength of type (4.2).

(i) General Relativity

Consider again the shell r_i of Fig. 1 and let $4\pi S_i$ be the source strength of gamma rays in the shell in cm^{-3} sec^{-1}. The density of the matter in the shell is ρ given by (2.19) and so

$$n=\rho_0(1.67\times 10^{-24})^{-1}Y^{-3}. \qquad (4.3)$$

Presumably σ and m have the same values in shell r_i as they do in the observer's neighborhood. The same cannot be said with certainty of j; therefore, we shall write $j(t_i)$, thus making j depend on the departure time. The number of gamma rays produced per second in the shell and emitted isotropically is

$$4\pi S_i d\mathcal{V}_i\quad \gamma\ \text{rays sec}^{-1}.$$

But these gamma rays consist of radiation and they will presumably have a frequency spectrum. Let us, therefore, say that the gamma rays with frequencies in the range $f_i\pm df_i/2$ are equivalent to an isotropic energy emission of

$$(4\pi S_i d\mathcal{V}_i)bB(f_i)df_i\ \mathrm{W},$$

where b is a normalizing constant. At the observer this flux of energy will be reckoned to be of frequency f lying in the bandwidth df, which are given by (2.5). Also the intensity at the observer is reduced by the factor $1/(4\pi D_i^2)$. Thus the flux of energy at the observer is

$$dE_i=[4\pi S_i d\mathcal{V}_i/(4\pi D_i^2)]bB(f/Y_i)df/Y_i\ \mathrm{W\ cm^{-2}},$$

so that, by (4.2), (4.3), (2.21), (3.7), and $T_0=1/h_1$,

$$dE_1=-\frac{4\pi\rho_0\sigma m}{1.67\times 10^{-24}}(cT_0)j(t_i)bB(f/Y_i)dfZ(Y_i)dY_i\quad \mathrm{W\ cm^{-2}}.$$

Summing for all shells from the observer outwards we have

$$E=\frac{4\pi\rho_0\sigma m}{1.67\times 10^{-24}}(cT_0)\int_0^1 j(t_i)bB(f/Y_i)dfZ(Y_i)dY_i\quad \mathrm{W\ cm^{-2}}.$$

But the Explorer XI apparatus has a very wide bandwidth. Let us, therefore, suppose that it records all gamma rays in the (finite) range of frequency f_0 to f_0'. This means that the differential $B(f/Y_i)df$ in the formula for E must be replaced by

$$\mathcal{B}(f_0,f_0',Y_i)=\int_{f_0}^{f_0'} B(f/Y_i)df. \qquad (4.4)$$

Moreover, by (2.3), t_i can be expressed as a function of

[10] The possibility of disentangling the cosmical radiation from that due to local sources is discussed in Sec. VI of McVittie and Wyatt (reference 2).

Y_i. Hence we may write

$$E=\frac{4\pi\rho_0\sigma m}{1.67\times10^{-24}}(cT_0)\int_0^1 j(Y)b\mathcal{B}(f_0,f_0',Y)Z(Y)dY \quad \text{W cm}^{-2}. \quad (4.5)$$

One assumption made by Kraushaar and Clark is that j is independent of t_i, and therefore of Y_i. This assumption by itself is insufficient to obtain J from E. It is also necessary to know $\mathcal{B}$. A speculative assumption is that B is independent of frequency which would mean that $\mathcal{B}$ could be written as $\mathcal{B}_0(f_0'-f_0)$, where $\mathcal{B}_0$ was some constant. The two assumptions together give, with the aid of (2.23),

$$J=E/[4\pi b\mathcal{B}(f_0'-f_0)]$$
$$=\left(\frac{\rho_0\sigma m j_0}{1.67\times10^{-24}}\right)(cT_0)A \quad \gamma \text{ rays cm}^{-2}\text{ sr}^{-1}\text{ sec}^{-1}, \quad (4.6)$$

where j_0 is the local value of j and A is the value of the definite integral in (2.22). This is a formula of the same form as that quoted by Kraushaar and Clark: The first bracket is their S, the second is what they call the "Hubble distance" and their α is presumably A. But this last point is not certain since they do not give the theory of α. It is said to have the value 0.4 and to be "the factor which corrects for the relativistic solid angle contraction of the receding matter." This, however, is only one of the effects produced by the receding matter, the others being the Doppler shift and the number effect. All three are taken into account in the theory of the luminosity distance D. It is, of course, possible to find models in which $A=0.4$: for example, the model in which $q_0=3$, $\sigma_0=9$, yields $A=0.42$ and the model defined by $q_0=4$, $\sigma_0=12$ has $A=0.38$. But these models have values of q_0 which are far too large and densities of matter which are much too high.

In order to compare the prediction (4.6) with the observed values (4.1) we follow Kraushaar and Clark in taking $j_0=0.3$ cm^{-2} sr^{-1} sec^{-1}, $\sigma=4\times10^{-26}$ cm^2, $m=2$ and $F=1$ in (2.30). We also use (2.32) in the form $\rho_0=\eta(3.1\times10^{-31})$ g/cm^3. Then

$$J=0.55\times10^{-4}\eta A. \quad (4.7)$$

In order to obtain the observed values (4.1) it follows that the model must be such that

$$6.7\leqslant\eta A\leqslant 20. \quad (4.8)$$

Clearly the low-density model (3.21) in which $\eta=15/11$ and $A=0.75$ has too small a value of $A\eta$. In the high-density model (3.19), we have $A=0.60$ and $\eta=1670/31$ so that $A\eta=32$, thus overshooting the mark. Two models which do lie in the required range are the following: in the first model $q_0=0.4$ and $\sigma_0=1.2$, which give $A=0.70$ and $A\eta=19$. In the second model $q_0=1.15$ and $\sigma_0=0.45$, with $A=0.68$ and $A\eta=7$. The density in the first model is equivalent to 5×10^{-6} proton/cm^3, and in the second to 1.9×10^{-6} proton/cm^3. Therefore, it is possible to find general relativity models in which (4.7) represents the observations without using quite as high a density of intergalactic protons as Kraushaar and Clark have assumed. Nevertheless, the densities in these models are on the high side compared with the Oort value ($\sigma_0=0.044$).

(ii) The Steady-State Theory

Secular changes of j are not now permissible and, of course, the density, and, therefore, n must be constant. Hence S_i is constant for all shells. Proceeding as before with the aid of (2.28) one finds that, for B independent of frequency

$$J=\frac{\rho_0\sigma m j_0}{1.67\times10^{-24}}(cT_0)\int_0^1 Y^2dY$$
$$=\frac{\rho_0\sigma m j_0}{1.67\times10^{-24}}(cT_0)(1/3).$$

Thus the constant A of (4.6) is now 1/3. Also by (2.25), we have $\sigma_0=1.5$. Thus (4.7) may again be used with $\eta=1.5/0.044$ and $A=1/3$ which give $A\eta\simeq11$. This value falls within the limits given by (4.8) and, therefore, the steady-state theory gives as much of an acceptable interpretation of the data as does general relativity. Kraushaar and Clark's reference to the steady-state theory merely constitutes an argument against the creation of antiprotons as well as of protons. The creation of antiprotons is an addition to the steady-state theory which finds no place in the most recent exposition of the theory by Bondi.[5]

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