are being actively investigated. After repeated failures the author has given up the first question as being beyond his own mathematical capabilities, but he hopes this paper will stimulate a competent investigation of the question.

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Incoherence, Quantum Fluctuations, and Noise

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An examination is made of the relationship between the uncertainty principle and minimum amplifier noise. First, the concept of coherence is discussed, and an incoherence parameter is defined in terms of the uncertainty that enters into the uncertainty principle. Harmonic oscillator states are examined for coherence. The concept of noise is then discussed and contrasted with incoherence, noise referring to behavior in time of a single system while incoherence involving comparison among members of an ensemble. It is shown, with illustrations, that the two concepts are different, and that an incoherent field of a cavity mode need not exhibit noise. In particular, the zero-point field in a lossless cavity is not noise. The superposition of many incoherent effects, however, usually leads to noise. Spontaneous emission is examined both for coherence and noise. It is shown that the spontaneous emission field of a single molecule is incoherent but does not exhibit noise; the (low order) spontaneous emission from a molecular beam, however, does constitute noise. Spontaneous emission from complex systems is also discussed. The origin of fundamental noise in an amplifier is investigated and is shown to come from spontaneous emission by the amplification mechanism. It is concluded that fundamental noise cannot be determined by a consideration of quantum fluctuations of—or by the application of the uncertainty principle to—the electromagnetic field only, as has been done in several recent articles. The physical significance of the zero-point field is analyzed, and is shown to lie in a formal contribution to spontaneous emission by the mechanism coupled to the field, provided this mechanism is treated quantum mechanically.

INTRODUCTION

In recent years, there has arisen an interest in "fundamental" noise, noise which has been attributed to fundamental physical laws or phenomena, such as the uncertainty principle, quantum fluctuations, or spontaneous emission, and which cannot be eliminated in principle. This interest is due, in large part, to the development of maser amplifiers, in which the noise is so low as to offer the possibility of approaching, indeed, the level of fundamental noise.

There has been discussion[1,2] of fundamental noise that is based mainly on the uncertainty principle or on quantum fluctuation, and, as will be shown, is more or less unsatisfactory. A related unsatisfactory situation exists with respect to the concept of coherence (which is associated with both noise and the uncertainty principle), because of the various different meanings attached to the word "coherent." It is the purpose of the present article to discuss the concept of coherence, and then to analyze the relationship between noise on the one hand, and quantum fluctuations and the uncertainty principle on the other.

Coherence is discussed in Part I, and noise is the chief topic of Part II. Various aspects of spontaneous emission are considered in Part III, and the source of fundamental amplifier noise is discussed in Part IV.

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[1] R. Serber and C. H. Townes, in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960).

[2] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. 124, 1646 (1961).

The physical significance of the zero-point field is studied in Part V.

I

As mentioned above, the word "coherent" is used with various meanings. One hears the expression "coherent oscillator," denoting an oscillator the output of which is a sine wave; here coherence means monochromaticity. The expression "coherent signal" is often used to distinguish information from noise, and coherence implies, in this connection, non-random variation with time but not necessarily monochromatic variation. The word "coherent" is also widely used in physics to indicate correlation between two or more functions of either space or time, (such as its use in the description of two light beams obtained by the splitting of a single beam), although the functions themselves may have some random properties.

The usefulness and propriety of a definition is determined by the nature of the problems in which it is used. For our purposes, it is most convenient to define coherence by means of a consideration related to the uncertainty principle. We consider a dynamical variable, $\varphi(t)$, usually a coordinate or momentum (but not an energy), which may be either quantum mechanical or classical. (In quantum mechanical considerations, the Heisenberg picture will be used.) We then find its expectation value $\langle\varphi(t)\rangle$ and the expectation value of its square $\langle\varphi^2(t)\rangle$. For a quantum mechanical variable, the term "expectation value" needs no explanation; it is the average over a quantum mechanical ensemble. For a classical variable, the term expectation value will denote the average at time t over an ensemble which is appropriate for the situation under discussion. We now define an incoherence parameter

$$\mathcal{I}\equiv\frac{\langle\varphi^2(t)\rangle-\langle\varphi(t)\rangle^2}{\langle\varphi^2(t)\rangle}. \tag{1}$$

$\mathcal{I}$ may vary between 0 and 1, since $\langle\varphi^2(t)\rangle\geqslant\langle\varphi(t)\rangle^2$. If $\mathcal{I}$ is 0, $\varphi(t)$ is said to be (completely) coherent and if $\mathcal{I}$ is 1, $\varphi(t)$ is (completely) incoherent.[3]

For quantum mechanical variables, the above definition of incoherence may be described as *quantum mechanical incoherence in the sense of the uncertainty principle*, for the uncertainty occurring in the latter is just $\mathcal{I}\langle\varphi^2(t)\rangle$. Thus, the incoherence parameter $\mathcal{I}$ is equal to the relative uncertainty. Our concern will be mainly with quantum mechanical dynamical variables. For the reader who is accustomed to think of coherence in terms of correlation, it might be pointed out that there is, in fact, correlation involved in our definition, namely, correlation between members of an ensemble.

It is instructive to apply the definition of coherence to a simple situation. Let us consider a harmonic oscillator and study the coherence of its displacement (or momentum—the two are symmetrical). The quantum mechanical oscillator will be considered first. For simplicity, we use dimensionless quantities for coordinate and momentum, which are defined by the Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\hbar\omega(q^2+p^2), \tag{2}$$

and by the commutation relationship

$$[q,p]=i. \tag{3}$$

The solution of the equations of motion yields

$$\begin{aligned} q(t)&=q(0)\cos\omega t+p(0)\sin\omega t,\\ p(t)&=p(0)\cos\omega t-q(0)\sin\omega t. \end{aligned} \tag{4}$$

The only nonvanishing matrix elements of q and p in the Heisenberg representation are

$$\begin{aligned} q_{n,n+1}&=q_{n+1,n}=[\tfrac{1}{2}(n+1)]^{1/2},\\ p_{n,n+1}&=-p_{n+1,n}=-i[\tfrac{1}{2}(n+1)]^{1/2}. \end{aligned} \tag{5}$$

It is easy to see that if the oscillator is in an energy state, the coordinate is incoherent, since $\langle q\rangle=0$ and, therefore, $\mathcal{I}=1$. If, however, the state of the oscillator consists of a superposition of two or more adjacent energy states, the coordinate is no longer completely incoherent. It can be shown[4] that, for a given energy expectation value $(x+\frac{1}{2})\hbar\omega$, the coherence is a maximum when the oscillator is in the state

$$\psi=\textstyle\sum_{n=0}^{\infty}e^{-\frac{1}{2}x}[x^{(1/2)n}/(n!)^{1/2}]\varphi_n, \tag{6}$$

φ_n being the nth energy state, in which case

$$\mathcal{I}=\tfrac{1}{2}(x+\tfrac{1}{2})^{-1}. \tag{7}$$

Let us consider now an ensemble of classical oscillators, all having the same energy. If the phase of the oscillation is well defined, that is, if it is the same for all members of the ensemble, then $\mathcal{I}=0$, and the oscillation is completely coherent. If, however, the phase is a random variable with all phases being equally probable, then $\mathcal{I}=1$ and the oscillation is completely incoherent. The latter case is evidently the illustration (in classical terms) of the quantum mechanical energy state.

II

We come now to the subject of noise. Although ensembles are useful in the study of statistical properties of noise, they are not essential in the definition or in the understanding of noise. One considers the variation of a coordinate of a single system (which may be a composite of many subsystems) as a function of time. If there is any randomness in this variation, then this randomness is referred to as noise. Thus, the variation

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[3] If φ is increased by a constant value, the incoherence parameter changes. The above definition of coherence is, therefore, meaningless when applied to variables to which the addition of a constant has no physical significance (such as the energy).

[4] I. R. Senitzky, Phys. Rev. **95**, 904 (1954); **95**, 1115 (1954). See also reference 2, Appendix I. φ_n in Eq. (6) should be used either in its real form or multiplied by the phase factor $\exp(in\theta)$.

(fluctuation) in time of a coordinate of a thermodynamic system usually exhibits noise. A sinusoidal variation in time is not, of course, noise.

A convenient method of measuring the noise content of a stationary, or steady-state, function of time is offered by the correlation function. Let us consider the (c-number) function $f(t)$ in the interval $-\frac{1}{2}T\leqslant t\leqslant\frac{1}{2}T$, where T is large compared to other times of interest. The correlation function is defined as

$$C_f(\tau)=[f(t)f(t+\tau)]_{\text{av}}=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}dt\, f(t)f(t+\tau). \qquad (8)$$

If we express $f(t)$ as a Fourier integral

$$f(t)=\int_{-\infty}^{\infty}d\omega\, f(\omega)e^{i\omega t},\quad f(-\omega)=f^*(\omega), \qquad (9)$$

and if we define the power spectrum of $f(t)$ as

$$p(\omega)=\lim_{T\to\infty}|f(\omega)|^2T^{-1}, \qquad (10)$$

then, as is well known,[5]

$$C_f(\tau)=4\pi\int_0^{\infty}d\omega\, p(\omega)\cos\omega\tau, \qquad (11)$$

and

$$p(\omega)=\frac{1}{2\pi^2}\int_0^{\infty}d\tau\, C_f(\tau)\cos\omega\tau, \qquad (12)$$

that is, the power spectrum and correlation functions are Fourier transforms of each other.

It is not the purpose of the present article to go into a discussion of the mathematics of randomness or of the properties of noise which may be obtained from the power spectrum. We will make use, however, of the obvious fact that if the power spectrum is a δ function, then $f(t)$ does not contain noise; it is a pure sinusoidal oscillation. (We do not know the phase of the oscillation, since it is not determined by the power spectrum.) In other words, a frequency spread is a necessary condition for the existence of noise.[6]

We apply the above considerations to the harmonic oscillator again. In the classical case, the correlation function for the coordinate is expressed as the product of $q(t)$ and $q(t+\tau)$. In the quantum mechanical case, $q(t)$ and $q(t+\tau)$ are operators, however, and do not commute. We, therefore, take, as the equivalent of the classical quantities, the symmetrized product of the corresponding operators. We then have a *correlation operator*. Using the notation $\{A,B\}=(AB+BA)$, we obtain from Eqs. (4) and (8),

$$C_q(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}dt\,\tfrac{1}{2}\{q(t),q(t+\tau)\}$$
$$=[q^2(0)+p^2(0)]\cos\omega\tau. \qquad (13)$$

For a harmonic oscillator which is in the nth energy state [belonging to the eigenvalue $(n+\frac{1}{2})$], a measurement corresponding to the correlation operator will give

$$(2n+1)\cos\omega\tau, \qquad (14)$$

which yields a power spectrum

$$p_n(\omega')=\frac{2n+1}{4\pi}\delta(\omega'-\omega). \qquad (15)$$

We see, thus, that a harmonic oscillator in an energy state oscillates sinusoidally (hardly a surprise!). Even when the oscillator is in the lowest energy state, $n=0$, the same is true. The motion of the oscillator, therefore, does not exhibit noise.[7]

A question arises concerning quantum (and zero-point) fluctuations. Are these fluctuations noise? Quantum fluctuations are fluctuations in the result of a measurement carried out on members of a quantum mechanical ensemble. Thus, if we consider an ensemble of oscillators which are all in the same energy state, and we measure the coordinate of each oscillator at the same time, the results will fluctuate from oscillator to oscillator. These are the quantum fluctuations, and their presence indicates incoherence, as defined in the present article. The motion of each oscillator, however, is free of noise.

When a single system is under consideration, we see that there is no connection between incoherence (or quantum fluctuations) and noise. There is, nevertheless,

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[5] J. L. Lawson and G. E. Uhlenbeck, *Threshold Signals* (McGraw-Hill Book Company, Inc., New York, 1950), p. 39.

[6] From a physical viewpoint (as contrasted to a mathematical one), there may be a subjective element involved in the classification of a function as noise. The essential property of noise is randomness, or unpredictability. Whether a function of time is predictable or not may depend on the length of time it has been under observation. Consider, for instance, a function which is a superposition of a large number of sinusoidal oscillations of different conmeasurate frequencies. This function will be periodic and to an observer whose time scale is large compared to a period, the function will not look like noise. In fact, he can Fourier-analyze it and predict it for future time. If, however, the function is observed for less than a period, predictability cannot be achieved, and the function may look like noise. Thus, the observer's time scale, related to the collection of frequencies constituting the function, determine whether the function is regarded as noise. The infinite time scale involved in Eq. (8) is, of course, an idealization. Generally, a function composed of components whose frequencies are closely spaced with no particular regularity in frequency spacing or in the phases of components, is noise for all practical purpose. A somewhat analogous situation exists in observation of a complicated (classical) system. A small portion of the system may exhibit what appears as random behavior (such as Brownian motion) while the system as a whole behaves in a predetermined manner. It should not be assumed, however, that all the noise functions that we will consider are of this type. Functions which are unpredictable, no matter how long the period of observation, will also be encountered.

[7] The correlation operator of Eq. (13) may also be used for oscillators which are not in an energy state. A measurement of the correlation function will yield $\epsilon\cos\omega\tau$, still the correlation function for a sinusoidal oscillation, but with ϵ now a random variable with respect to the members of the quantum-mechanical ensemble.

a connection between incoherence and noise when the cumulative effect of many systems is under consideration. Let us consider a large group of oscillators, all in the same energy state, and fix our attention on the sum of the coordinates of a subgroup of oscillators, the number in the subgroup being fixed and smaller than the total number of oscillators. Let us now change, either at regular or random intervals, the *identity* of some of the oscillators in the subgroup, removing them from the subgroup and replacing them with others from the remainder of the group. We know from previous discussion that the coordinate of each oscillator is a sinusoidal oscillation of completely random phase. Thus, changing the identity of an oscillator in accordance with the above procedure produces a random change in phase of one of the terms in the sum of the coordinates of the subgroup. It is evident that this sum will be a function of time which exhibits noise, the noise being due to the fact that the oscillation of each oscillator is incoherent. If, however, our original ensemble were composed of coherent oscillators, the sum of the coordinates of the oscillators of the subgroup would be noise-free. We see, then, that incoherence—or quantum fluctuations—leads to noise only when we are dealing with the superposition of effects from many systems, but not when we are dealing with a single system. Loosely speaking, one might say that incoherence on a microscopic scale usually leads to noise on a macroscopic scale.

It is useful, for later purposes, to consider explicitely the electromagnetic field of a single mode of a lossless cavity, and inquire whether its quantum—or zero-point—fluctuations may be regarded as noise. Now, the cavity is a macroscopic system. In quantum mechanics, we usually deal with microscopic systems, and macroscopic systems are generally considered to be composed of a large number of microscopic systems. It is well known, however, that the field of a resonant circuit is a single quantum-mechanical system equivalent to a harmonic oscillator. When this system is in the lowest energy state—or any other state—the field oscillates sinusoidally, and does not exhibit noise.

III

We will now consider spontaneous emission with reference to its coherence and noise properties. We consider first the case of a single atomic system, which we call a molecule, emitting spontaneously into a resonant cavity (with loss[8]). The molecule is considered to have only two energy states, and is initially in the upper state. It is coupled to the field through its electric dipole moment, and its frequency is the same as that of the single cavity mode into which it radiates. The time during which we consider its radiation is long compared to the relaxation time of the cavity but short compared to its lifetime, We may, therefore, use perturbation theory. The lowest order spontaneous emission field in the cavity is given by[9]

$$P^{[1]}(t)=-\frac{\omega}{c}u\int_0^t dt_1\,\gamma^{[0]}(t_1)e^{-(1/2)\beta(t-t_1)}\sin\omega(t-t_1),\quad(16)$$

where

$$\gamma^{[0]}(t)=\tilde{\gamma}\begin{pmatrix}0 & e^{-i\omega t}\\ e^{i\omega t} & 0\end{pmatrix},\quad(16a)$$

the unperturbed dipole moment along the electric field, and the bracketed superscript indicates the perturbation theory order. The spontaneous emission field $P^{[1]}(t)$ becomes stationary (within the limits of perturbation theory) after an initial transient period given by the cavity relaxation time. Since $\langle\gamma^{[0]}(t)\rangle=0$, we have, from Eq. (16)

$$\langle P^{[1]}(t)\rangle=0,\quad(17)$$

which yields $\mathscr{g}=1$, and shows that the spontaneous emission field is completely incoherent. We look next at the correlation operator for the first order field. If we ignore the initial transient period, then

$$\{P^{[1]}(t),P^{[1]}(t+\tau)\}=(4\omega^2u^2\tilde{\gamma}^2/c^2\beta^2)\cos\omega\tau.\quad(18)$$

We see that the correlation operator is that for a pure sinusoidal oscillation. The lowest order spontaneous emission field from a single molecule may not, therefore, be regarded as noise.

Let us consider now a molecular beam passing through the cavity, the molecules entering in an excited state as in a molecular beam maser, but with conditions such that the amplification by the beam is negligible.[10] We now have a situation in which each molecule produces a sinusoidal field during its transit time, but due to the incoherence of the field of each molecule these fields have random phases. [Strictly speaking, Eq. (17) implies that the fields produced by different molecules *with each in a different cavity* have random phases. It is quite obvious, however, that if the molecules are in uncorrelated states[11] when they enter the cavity, their (lowest order) spontaneous emission fields are uncorrelated. A formal method of exhibiting the absence of correlation of the spontaneous emission fields of two molecules in the same cavity is to consider the expectation value of the product of these two fields. Equations (16) and (16a) show it to be zero.] In analogy

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[8] A general quantum-mechanical method of analyzing a lossy cavity (or lossy harmonic oscillator) is given by I. R. Senitzky, Phys. Rev. **119**, 670 (1960), and the behavior of a molecule in a lossy cavity is described by I. R. Senitzky, *ibid.* **115**, 227 (1959).

[9] The electric field of the single cavity mode under consideration is written as $\mathbf{E}=-4\pi c\mathbf{u}(\mathbf{r})P(t)$, where $\mathbf{u}(\mathbf{r})$ is a normalized function describing the spatial variation of the field and $P(t)$ is a quantum-mechanical operator. For simplicity, $\mathbf{u}(\mathbf{r}_m)$, where $\mathbf{r}_m$ is the position of the molecule, is assumed constant in time while the molecule is in the cavity.

[10] A molecular beam amplifier is analyzed in detail by I.R. Senitzky, Phys. Rev. **127**, 1638 (1962).

[11] The correlation of molecular states, as well as the interaction between molecules and a lossless cavity, is discussed by I. R. Senitzky, Phys. Rev. **111**, 3 (1958).

with the case of the subgroup of incoherent harmonic oscillators considered previously, we see that the result constitutes noise. It can be shown that the frequency spread of this noise is just the bandwidth of each individual oscillation of finite duration (if the molecular transit times are equal).[12]

In the case of large amplification by the molecular beam, we have a maser amplifier or oscillator. Let us consider the latter, in which oscillation has built up "spontaneously."[13] In the steady state, a molecule passing through the cavity sees a strong field and emits mostly induced emission, which sustains the field at its high steady-state level. The field in the cavity, as has been verified experimentally,[14] is a sinusoidal oscillation with a high degree of purity. It is not possible to follow analytically the complete history of the system of molecules and field up to the steady state (perturbation theory is not applicable, of course). It is, however, possible, from very general considerations (see reference 11), to show that,

$$\langle P(t)\rangle = 0, \tag{19}$$

in other words, the field is incoherent. This should be hardly surprising—if one bears in mind that coherence involves an ensemble average—since all that incoherence implies in this case is that in an ensemble of maser oscillators the phase of the steady-state oscillation is unpredictable. (The same is true for any self-starting oscillator.) Here we have another instance of maximum uncertainty in the sense of the uncertainty principle—but no significant noise. [There is always some (low order) spontaneous emission from the beam, but the difference between $\mathcal{S}$ and zero can be made insignificant.]

So far we have considered exceedingly simple systems, the harmonic oscillator and the two-level molecule, systems which are characterized by a single frequency. Let us now consider the opposite extreme, a very complex multifrequency system with close energy level spacing (or even a continuum), one that is described thermodynamically, for instance. The spontaneous emission from such a system (when it is excited) consists of the superposition of many frequencies with no particular regularity in the frequency spacing and in the phases of the oscillations of the different frequencies. In accordance with our previous discussion, such an output is clearly noise. If the state of the system is described by a superposition of energy states in which the superposition constants are completely determined, then a coordinate of the system may show much coherence, which merely means that in an ensemble of identical systems, all in the same superposition of energy states, the coordinate exhibits approximately the same time variation for all systems, so that we have "correlated noise." Usually, however, we do not know, and cannot control, the exact values of the superposition constants. If we know only their absolute values, our system may be pictured as being a member of an ensemble in which the absolute values of the superposition constants are the same for all members but the phases of the superposition constants have random (but constant in time) values, all phases being equally likely. An average system of such an ensemble is often said to be in a mixture of energy states rather than in a superposition of energy states. The coordinates of such a system are obviously completely incoherent. One may go further yet in the lack of knowledge about the state of the system, and specify only the *ensemble average* of the absolute value of the superposition constants. (In this case, the average system is also said to be in a mixture of states.) This last specification is the one used in thermodynamics, and likewise, of course, leads to incoherence.[15] A large number of simple systems may be considered to form a single complex system; the above considerations then apply.

IV

We come now to the matter of noise in an amplifier. Consider a type of amplifier which may be regarded as being composed of three coupled systems: (1) the electromagnetic field of a single cavity mode (or a harmonic oscillator); (2) a loss mechanism (this may or may not be essential for the desired amplification properties, but cannot be eliminated); (3) an amplification mechanism, which is a source of power.

Let us examine the sources of noise. In accordance with our previous discussion, the electromagnetic field is a simple single-frequency system and does not produce noise. The loss mechanism is a complex system which does produce noise when in an excited state, namely, thermal noise. This can be made arbitrarily small, in principle, by cooling the loss mechanism to a

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[12] The output of a single molecule has a frequency spread, but is not noise. The random superposition of the outputs of the individual molecules of the beam supplies the randomness which converts the combined output into noise, the frequency spread remaining the same. The analysis is analagous to that of the shot effect, with additional randomness coming from the random phases. For the (realistic) case of a dispersion in transit times, see reference 10.

[13] If we regard all the molecules as a single system, self-starting oscillation could also be called spontaneous emission. From the point of view of perturbation theory, it is a high-order behavior. Our present interest is in low-order spontaneous emission, and this is what we will have in mind in our further discussion of—and reference to—spontaneous emission.

[14] J. P. Gordon, in *Quantum Electronics* edited by C. H. Townes (Columbia University Press, New York, 1960).

[15] Another, more refined, method of regarding these three types of specification is in terms of an ensemble of ensembles. The superposition of energy states is the usual quantum-mechanical ensemble—let use call it sub-ensemble—in which each system is in the same quantum mechanical state; the second case, above, is an ensemble of these sub-ensembles, the phases of the superposition constants for each sub-ensemble being random but the absolute values of the constants being the same; and the thermodynamic system may be regarded as one of a super-ensemble of the ensembles. Three types of averaging are thus possible. The first gives the quantum mechanical expectation value and the third gives the thermodynamic value. One may omit the second case and go directly from the sub-ensemble to the super-ensemble.

sufficiently low temperature. The amplification mechanism is either a collection of many simple systems (such as a molecular beam) or a complex system. It will produce spontaneous emission—which, as the previous discussion shows, is noise—if it is in an excited state. But the amplification mechanism must always be in an excited state, since it is a source of power. Thus, spontaneous emission from the amplification mechanism is the fundamental limiting noise in an amplifier (of the type considered. We will not discuss the question whether this type includes all amplifiers.)

One might, perhaps, argue that the amplification mechanism could be a simple system and would, therefore, not emit noise. Most real macroscopic systems (here the field of a single mode seems to be in a class by itself) are not simple systems, however, although we may sometimes idealize them as such. Their spontaneous emission is, therefore, noise.

It is not possible to give a quantitative expression for the spontaneous emission from a general amplification system. The details of each specific situation must be considered. The spontaneous emission depends not only on the amplification system but also on the loss mechanism. In the case of a molecular beam amplifier, this calculation has been made, and the result exhibits an explicit dependence, both on the properties of the amplification system and on the losses.[10] The dependence on the latter follows from the fact that the lowest order spontaneous emission power from a molecule in a cavity is a function of cavity loss.[16] It is clear that the limiting noise cannot be derived from an application of the uncertainty principle to the electromagnetic field, as is done in references 1 and 2.[17]

V

An intriguing question related to the subject under discussion is that pertaining to the physical significance of the zero-point field. It is very weak, of course. Can it be amplified? Louisell, Yariv, and Siegman,[2] in their analysis of fundamental noise in a parametric amplifier, arrive at the result that the zero-point field of a lossless cavity is amplified together with the signal, and call it noise.

We have already seen that the zero-point field of a lossless cavity is not noise. We may go further, however, and inquire whether it can be amplified. Let us try to amplify the zero-point field of a lossless cavity[18] by sending a "bunch" of molecules, of the two-level type considered previously, through the cavity. (The analysis for the "bunch" is simpler than that for a molecular beam, since all the molecules of the "bunch" are exposed to the same field; essentially similar considerations, however, apply to the beam.) We consider conditions such that perturbation theory applies. The lowest order field is the zero-point field itself. The time average of the square of the electric field cavity is in its lowest energy state is given by[11]

$$P^{[0]2} = \hbar\omega/8\pi c^2. \tag{20}$$

This expression may be considered either as a multiple of the unit operator, as an eigenvalue, or as an expectation value. (In the latter case, time averaging is not necessary, since ensemble averaging gives the same result.)

The lowest order field that is due to the molecules is given by the operator

$$\begin{aligned} P^{[1]}(t) &= \sum P_m{}^{[1]}(t), \\ P_m{}^{[1]}(t) &= -\frac{\omega}{c}u\int_0^t dt_1\,\gamma_m{}^{[0]}(t_1)\sin\omega(t-t_1), \end{aligned} \tag{21}$$

P_m being the field due to each molecule. Applying the same reasoning as that used previously for the molecule in the lossy cavity, we can see that there is no correlation between the first order fields of different molecules. The second order field is given by[11]

$$P^{[2]}(t) \approx [\mathcal{E}_0(t)/2\hbar\omega]P^{[0]}(t)\sum I_m, \tag{22}$$

where

$$\mathcal{E}_0(t) = 2\pi\omega^2 u^2 \tilde{\gamma}^2 t^2,$$

and I_m is a diagonal matrix in the energy representation of the unperturbed molecule, with eigenvalues -1 and $+1$ for the lower and upper states of the mth molecule, respectively. The second order field has the formal appearance of an induced field, for, if $P^{[0]}(t)$ were a classical driving field, $P^{[2]}(t)$ would indeed be the induced field, reenforcing the driving field if the molecule is in the upper state (emission), and diminishing the driving field if the molecule is in the lower state (absorption). In the present instance, however, there can be little justification for saying that spontaneous emission is induced by the zero-point field. In the first place $P^{[2]}$ is a higher order term than $P^{[1]}$, where the zero-point field does not even occur; and in the second

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[16] The dependence of spontaneous emission on cavity loss was first discussed by E. M. Purcell, Phys. Rev. **69**, 681 (1946).

[17] Serber and Townes *do* attribute the noise to spontaneous emission from the molecules in their analysis of a molecular amplifier. (They do not, actually, speak of noise but of "fluctuations." The context of their discussion, as well as Prof. Townes' remark that follows the article, indicates, however, that they are referring to noise.) Their signal-to-noise ratio for the amplified field bears close resemblance to expressions which follow from the uncertainty principle applied to the unamplified field. This would seem consistent with their claim that the uncertainty principle determines the ultimate signal-to-noise ratio. However, the relationships on which their argument rests are true only for the case of a lossless cavity—the only case they analyze. For a generalization of their problem which includes cavity losses, see I. R. Senitzky, Phys. Rev. **123**, 1525 (1961).

[18] The amplification mechanism is a macroscopic system composed of a large number of microscopic systems and driven by an external source. A treatment which starts from fundamental principles applied to all the microscopic systems is, of course, very difficult and probably unnecessary. The extent to which simplifications may be made and a phenomenological description introduced needs examination. It is unlikely, however, that a purely macroscopic view of the amplification mechanism may be maintained.

place, it is impossible to diminish (absorb) the zero-point field. $P^{[2]}$ may not, therefore, be considered apart from $P^{[1]}$. The physical meaning of $P^{[1]}$ and $P^{[2]}$ is clarified in a calculation of the second order spontaneous emission energy for a single molecule. The electric field energy comes from two parts: $P^{[1]2}$, which is proportional to the unit operator, and $2P^{[0]}P^{[2]}$, which is proportional to I_m, with the same proportionality constant.[11] The energy is, therefore, proportional to the operator (I_m+1), which makes the spontaneous emission energy zero, as it must be, when the molecule is in the lower state. We see, thus, that it is impossible to attach physical significance to the role of the zero-point field. It plays a purely formal role in contributing one of the two terms that are needed for a correct expression of spontaneous emission. The spontaneous emission cannot be said to be an *amplification* of the zero-point field, since the spontaneous emission fields of different molecules are uncorrelated (unless the molecules happen to be initially in a correlated state), and yet the zero-point field is essential for the proper expression of spontaneous emission.

Now, spontaneous emission is not only a quantum-mechanical phenomenon, but it is a fundamental phenomenon in classical electrodynamics. If the molecule is treated quantum-mechanically, however, the above analysis shows that the field must also be treated quantum-mechanically in order to obtain the correct spontaneous emission expression. Likewise, if the field is treated quantum-mechanically, one can see easily that the molecule must be treated quantum-mechanically. It is reasonable to generalize this statement and say that if the field is to be treated quantum-mechanically, or if the zero-point field enters into consideration, then the amplifying mechanism must also be treated quantum-mechanically. The zero-point field will then play its role in contributing toward a correct expression for the spontaneous emission from the amplifying mechanism (and no other role).

A possible omission in the article of Louisell, Yariv, and Siegman now suggests itself. They did not treat the amplification mechanism quantum-mechanically; they, therefore, obtained terms which, when appearing alone, give the effect of amplification of the zero-point field. Had they included some essential quantum-mechanical properties of the amplification mechanism, they would probably have obtained an expression for spontaneous emission which includes all zero-point field effects. (Had we ignored the $P^{[1]2}$ term and considered only the $P^{[0]}P^{[2]}$ term, we, too, would have obtained "amplification" of the zero-point field for molecules in the upper state.)

The above discussion of the significance of the zero-point field refers to a lossless cavity. We could have carried out an entirely parallel discussion for a lossy cavity, and arrived at the same conclusions. It should be pointed out, however, that the formal description of the zero-point field in a lossy cavity is different from that in a lossless cavity. In the former, as discussed in detail elsewhere,[8] the zero-point field is no longer that of the harmonic oscillator but rather the field generated by the zero-point oscillation of the loss mechanism. It is not a purely sinusoidal oscillation, and comes under our definition of noise; but it is not noise which may be amplified. Its role is identical to that of the zero-point field of the lossless cavity: It acts (formally) in such a manner as to give a correct expression for spontaneous emission from the system coupled to the field in a formalism in which both the field and the system coupled to it are treated quantum-mechanically.

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