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Coupling Constants and Unsubtracted Dispersion Relations

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It is argued that the common occurrence of low-energy resonances and the fact that the cross sections for strongly interacting particles seem to approach nonzero high-energy limits may be related to a physical principle which determines the values of the strong-interaction coupling constants. It is conjectured that, in some cases, these constants may be determined approximately by neglecting inelastic processes and requiring that certain of the low-angular-momentum phase shifts approach $\pm\pi/2$ at high energies. Such a calculation may be made by using unsubtracted dispersion relations for the inverse partial-wave amplitudes. This prescription is illustrated in a few simple models. Our present knowledge of the various pion-nucleon forces is insufficient for a realistic calculation of the pion-nucleon coupling constant f^2. However, a simple calculation involving drastic approximations predicts a value of f^2 close to the experimental value and a low-energy resonance in the (3,3) state. The relationship of our prescription to that of Albright and McGlinn is discussed briefly.

I. INTRODUCTION

IN recent years many physicists have come to believe that the strong-interaction coupling constants are calculable from some basic principle as yet unknown, formulated in the framework of dispersion theory. There are several reasons for this optimism. For one thing, it has become increasingly clear that the differences between weak and strong interactions are basic. A perturbation theory, with coupling constants specified at the start, is applicable to weak interactions but not to strong ones. On the other hand, a surprising number of the properties of strongly interacting particles have been shown to follow from a few basic principles such as Lorentz invariance, unitarity, and causality (i.e., analyticity). It does not require a large stretch of the imagination to suppose that when the analyticity requirements are known more precisely, the strong-interaction constants themselves may be calculable. The experimental fact that these coupling constants are of order one is encouraging, since quantities of order one occur frequently in dispersion theory.

There are two outstanding features of the scattering amplitudes for pairs of strongly interacting particles that may be related to a basic principle that determines coupling constants. The first feature is the common occurrence of low-energy P-wave resonances and of S-wave scattering lengths larger than the expected range of the forces. Such strongly enhanced S waves are known in $\bar{K}$-N scattering, and possibly also in π-π scattering, while low-energy P-wave resonances occur in π-N scattering, and possibly also in π-Λ and π-K scattering. The possible connection between a principle that determines coupling constants and the common occurrence of strongly enhanced amplitudes is illustrated in Secs. III and IV.

The second outstanding fact is the apparent existence of nonzero high-energy limits for the total cross sections. This fact is crucial in leading Chew and Frautschi to formulate the principle that the strong interactions are as strong as they can be consistent with unitarity.[1] These authors argue that constant high-energy cross sections imply "saturation of unitarity,"[2] which in turn implies saturation of forces, since the amplitude for one process is related by analytic continuation to the forces in the "crossed" processes. It is not clear how one applies this principle, but it is clear that the substitution (or "crossing") rule, formulated within dispersion theory, is crucial.

In order to explain the motivation of our calculations we must discuss the following critical question—Does the existence of finite, high-energy cross sections require particular values of the coupling constants? A definitive answer to this question is not likely in the near future, since states involving many particles are important at high energies, and no one knows much about how to include such states in a theory. Hence the question is very much a matter of speculation. In order to illustrate our viewpoint by means of a specific example, we assume that if only π mesons and nucleons existed, the π-π and π-N cross sections would still approach nonzero limits. We further assume that it is legal to consider different possible values of the pion-nucleon and pion-pion interaction constants. If these constants were very small, we would expect the total cross sections to approach zero at high energy. We would also expect that those partial waves most directly coupled to the basic interactions (the P-wave π-N states and the S- and P-wave π-π states) would obtain the largest magnitudes. If the interaction constants were increased, we conjecture that at certain specific values some of these low-angular momentum partial waves would saturate the unitarity condition in the high-energy limit. We define "saturation" to mean that the real part of the scattering phase shift approaches ±90°, which in turn implies a total partial-wave cross section vanishing no more rapidly than as $1/E^2$. High-energy saturation of a few low-j waves might lead to saturation of many states of higher angular momentum because of the connection of the high-j states to states of

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 8, 41 (1962); this paper contains references to earlier works. See also G. F. Chew, in *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, Ltd., Edinburgh, 1960).

[2] M. Froissart, Phys. Rev. 123, 1053 (1961).

three or more particles, two of which are in saturated low-j states. Such a causal mechanism could conceivably lead to saturation of an infinite number of partial waves, and to nonzero total cross sections in the high-energy limit.

A common viewpoint concerning π-N resonances is that the principal forces responsible for the $(\frac{3}{2},\frac{3}{2})$ resonance result from the nucleon pole and from two-particle states, while higher π-N resonances result from the connection of the π-N states to states of three or more particles, two of which are resonating. We are proposing that a similar causal relationship between enhanced partial-wave amplitudes exists in the high-energy limit.

Unfortunately, very little is known about the effects of many-particle states in dispersion theory. However, one can attempt an approximate determination of the coupling constants by neglecting inelastic processes and trying to find the lowest values of the constants that lead to high-energy saturation of any partial-wave state. Even this kind of calculation is difficult without drastic approximations, since the crossing relations couple many amplitudes together. In this paper we consider some situations where the drastic approximation of considering only one type of process and a small number of partial waves may be reasonable enough so that an order-of-magnitude calculation of a coupling constant may be made. In Sec. II the general technique used in the calculations is introduced and illustrated. In Sec. III this technique is applied to a fictitious meson-baryon scattering problem in a kind of scalar meson theory, and in Sec. IV the problem of determining the pseudoscalar πNN interaction constant is considered. The relationship of this work to that of other authors is considered in Secs. V and VI.

II. GENERAL TECHNIQUES AND THE $\pi\Lambda\Sigma$ PROBLEM

At least one subtraction is usually made in dispersion relations for meson-baryon scattering, the subtraction constant being given in terms of the residue of the baryon pole (the coupling constant). The particular method we will use is a modification of the N/D method for a partial-wave amplitude, in which one attempts to avoid the necessity of a subtraction in the dispersion relation for the denominator function, thus allowing a determination of the coupling constant. The fact that this prescription results in high-energy saturation of the partial-wave amplitude is shown later.

We consider first a scalar meson theory in which there is no meson-meson interaction and the baryon mass is infinite, although a finite mass difference may exist between different baryons. In such a simple theory only S waves scatter and the static model is correct. The S-wave amplitudes are defined by the formula $T=e^{i\delta}\sin\delta/k$, where δ is the phase shift and k is the meson momentum. The constants $\hbar$ and c are set equal to one throughout the paper. The unitary condition on the right-hand cut is $\mathrm{Im}T^{-1}=-k\eta$, where η is the ratio of the total cross section to the elastic cross section.

We consider an amplitude T that is directly coupled to some one-baryon state so that T contains a direct pole term, i.e., $\lim_{\omega\to\omega_0}T(\omega)=\lambda/(\omega-\omega_0)$. In the conventional N/D method, in which the right-hand cut is the only singularity contained in the D function, an *unsubtracted* dispersion relation for D contains the integral

$$\int_1^\infty \frac{d\omega'\,\mathrm{Im}D(\omega')}{\omega'-\omega}=-\int_1^\infty \frac{d\omega'\,k'N(\omega')\eta(\omega')}{\omega'-\omega}, \qquad (1)$$

where ω is the meson energy and the meson mass is taken to be one. If singularities associated with the crossed channels are neglected so that $N(\omega')$ contains only the simple pole term $\lambda/(\omega'-\omega_0)$, this integral diverges since $\eta(\omega')\geqq 1$. Hence the crossed singularities are absolutely necessary; a subtraction may be avoided only if these singularities are such as to reduce the high-energy behavior of N.

A well-known example in which an unsubtracted dispersion relation is convergent is the static model of π-Λ scattering under the assumptions that the Σ-Λ parity is odd and the only interaction is a scalar $\pi\Lambda\Sigma$ interaction.[3] In this model the direct pole term and "crossed pole term" partially cancel; they may be combined to give a numerator function $N=-2F^2\Delta/(\omega^2-\Delta^2)$, where Δ is the Σ-Λ mass difference and F is the coupling constant. If inelastic processes are neglected, the integral of Eq. (1) converges with such an N, and F^2 may be determined by applying the condition $D(\Delta)=1$ to the solution. This critical value of F^2 leads to a phase shift approaching $-\frac{1}{2}\pi$ at high energy, whereas, if one makes a subtraction and chooses a smaller F^2, the phase shift approaches zero. (A value of F^2 larger than the critical value leads to a ghost.) This equivalence between the "no-subtraction" criterion and the "saturated partial-wave" criterion will be present in all examples considered in this paper, and results from the following facts. The imaginary part of D is fixed by unitarity, so the most convergent behavior of D that is possible occurs when the real part of D becomes small compared to the imaginary part at high energies. Since N is real on the right-hand cut, this condition implies that $T(\infty)$ is imaginary or, equivalently, $\delta(\infty)=\pm\frac{1}{2}\pi$.

Aaron, Vaughn, and Amado have shown that in the cutoff Lee model there is a critical value of the coupling constant, which corresponds to the assumption that the V particle is a θ-N bound state.[4] If the coupling constant is less than the critical value F_c^2, the θ-N (S-wave) phase shift approaches zero at high energies; but if $F^2=F_c^2$, this phase approaches $-\pi$. The fact that in the critical case the phase limit is $-\pi$ rather than

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[3] Richard H. Capps, Phys. Rev. **124**, 945 (1961).

[4] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

$-\frac{1}{2}\pi$ results from the presence of the rapid-cutoff function, which causes the imaginary part of the amplitude to decrease to zero more rapidly than the real part at high energies. It should be noted that phase shifts approaching $\pm\frac{1}{2}\pi$ at infinity do not occur in ordinary potential scattering.[5]

It is unfortunate that one must neglect inelastic processes in order to get simple equations, since these processes are known to be important at high energies, and since their inclusions would change the form of the high-energy limit. However, the principal variations in the amplitudes occur at kinetic energies smaller than the rest energy, so it is hoped that the inclusion of inelastic effects would lead only to a small correction to the calculated F^2. Furthermore, the connection between saturated partial waves and unsubtracted dispersion relations would apply even if inelastic effects were included, because the condition $\text{Re}\delta(\infty)=\pm\frac{1}{2}\pi$ (which corresponds to an imaginary elastic amplitude) implies that the sum of the elastic and inelastic partial wave cross sections is greater than C/ω^2 as $\omega\rightarrow\infty$, where C is a positive constant.

Another feature of the $\pi\Lambda\Sigma$ model which is common to all other models considered in this paper is the occurrence of at least one amplitude characterized by repulsive forces and at least one amplitude characterized by attractive forces. In such a situation one must ask whether or not the special value of the coupling constant determined from the no-subtraction procedure is large enough to imply a bound state in the "attractive" channel.[6] If a strongly bound state (binding energy not small compared to the meson mass) occurs, then the quantitative conclusions of the model cannot be accurate unless the branch cut resulting from the bound state plus one meson is included in the dispersion relations. For simplicity we hope to find examples where no strongly bound state occurs.

This $\pi\Lambda\Sigma$ example is not an ideal example of the no-subtraction procedure in a scalar meson theory, because the Σ-Λ mass difference is essential to the model, and the calculated F^2 depends on this mass difference. It would be much more interesting if F^2 could be calculated in some model not involving a baryon mass difference. Another difficulty is that the no-subtraction procedure cannot be applied to the π-Σ scattering amplitude, and so cannot be a universal prescription in the model.[6] It is not obvious how one should attempt to generalize from the $\pi\Lambda\Sigma$ model to other static S-wave models. The special value of F^2 for the $\pi\Lambda\Sigma$ model corresponds to the physical assumption that the Σ is a π-Λ bound state. There are several other, essentially equivalent prescriptions for calculating this coupling constant, but for more general models these prescriptions are not all equivalent, and may not all be applicable.[7,8] We shall apply the no-subtraction procedure to a more general, static S-wave meson theory in the next section.

III. THE COUPLED TWO-CHANNEL S-WAVE MODEL

We assume two coupled S-wave meson-baryon scattering amplitudes in the static model, satisfying the crossing relation

$$T_1(-\omega)=A_1T_1(\omega)+(1-A_1)T_2(\omega), \tag{2a}$$

where A_1 is a positive number less than one. This crossing relation is equivalent to the following crossing relation for the amplitude T_2:

$$T_2(-\omega)=-A_1T_2(\omega)+(1+A_1)T_1(\omega). \tag{2b}$$

The forces are represented by poles at zero energy, i.e., $\lim_{\omega\rightarrow 0}\omega T_j=\lambda_j$. The crossing condition implies that the two residues are in the ratio,

$$\lambda_2/\lambda_1=-(1+A_1)/(1-A_1). \tag{3}$$

A physical model described by these equations is the scattering by an isotopic-spin $\frac{1}{2}$ nucleon N of a "pion" of isotopic spin $I_\pi\neq 0$, where the forces result from a scalar coupling of the type $\pi NN'$, N' being a baryon of the same mass as the nucleon with isotopic spin $I_{N'}$ equal to either $I_\pi+\frac{1}{2}$ or $I_\pi-\frac{1}{2}$. In this example, if T_1 and T_2 represent πN scattering in states of isotopic spin $I_\pi+\frac{1}{2}$ and $I_\pi-\frac{1}{2}$, respectively, then $A_1=1/(2I_\pi+1)$. The residue λ_1 is positive if $I_{N'}=I_\pi-\frac{1}{2}$ and negative if $I_{N'}=I_\pi+\frac{1}{2}$. The most interesting special case is symmetric, scalar meson theory, where $I_\pi=1$ and the intermediate baryon N' is the nucleon itself. In this case $A_1=\frac{1}{3}$ and $\lambda_1=F^2$. Another model described by Eqs. (2) and (3) is charged, scalar meson theory. In this case $A_1=0$ and, if the amplitude T_1 describes π^+-p (or π^--n) scattering, $\lambda_1=F^2$.

We write the amplitude in the form $T_j=e^{i\delta_j}\sin\delta_j/k$, where $k=(\omega^2-1)^{1/2}$ is the meson momentum. The elastic unitarity condition on the right-hand cut is

$$T_j^{-1}(\omega+i\epsilon)-T_j^{-1}(\omega-i\epsilon)=-2ik.$$

Since the crossing condition relates two amplitudes of the same type, it is convenient to use the inverse method rather than the conventional N/D method. We assume that T_j is nowhere equal to zero. An unsubtracted dispersion relation for $(\omega T_1)^{-1}$ may be obtained by integrating the function

$$-(i/2\pi)T_1^{-1}(\omega')/[\omega'(\omega'-\omega)]$$

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[5] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).

[6] In the idealized $\pi\Lambda\Sigma$ model where the π, Λ, and Σ are all taken as isotopic singlets, the equation for π-Σ scattering may be obtained by changing the sign of Δ in the equations for π-Λ scattering given in reference 3. If inelastic processes are neglected, the special value of F^2 obtained from an unsubtracted dispersion relation leads to a π-Σ bound state at zero energy. Application of the no-subtraction criterion to π-Σ scattering would lead to a negative F^2 and so is not allowed.

[7] Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **6**, 377 (1961); J. Bernstein and R. Oehme, *ibid.* **6**, 639 (1961); Lu Sun Liu, Phys. Rev. **125**, 761 (1962).

[8] C. H. Albright and W. D. McGlinn, Nuovo cimento **25**, 193 (1962).

around the contour of Fig. 1 in the ω' plane, and taking the limit as the radius R goes to infinity. If use is made of the crossing relation and the elastic unitarity condition, the equation becomes

$$\omega^{-1}T_1^{-1}(\omega)=\frac{1}{\pi}\int_1^\infty\frac{d\omega'\,k'}{\omega'(\omega'-\omega)}+\frac{1}{\pi}\int_1^\infty\frac{d\omega'\,k'X(\omega')}{\omega'(\omega'+\omega)},$$

where the second term results from the left-hand branch cut, and the function X is given by

$$X(\omega)=A_1|T_1(\omega)/T_1(-\omega)|^2+(1-A_1)|T_2(\omega)/T_1(-\omega)|^2. \quad (4)$$

The first (right-hand cut) integral does not converge, but if the integral $\pi^{-1}\int d\omega'\,k'/[\omega'(\omega'+\omega)]$ is added to the first integral and subtracted from the second, the first integral converges and the convergence of the second depends on the high-energy behavior of the quantity $X-1$. If use is made of the crossing relation, this quantity may be written in the form $X(\omega)-1=A_1(1-A_1)|1-R_1(-\omega)|^2$, where R_1 is the complex ratio of the amplitudes, i.e., $T_2(\omega)=R_1(\omega)T_1(\omega)$. The dispersion relation for $T_1^{-1}(\omega)$ then becomes

$$(\omega T_1)^{-1}=\frac{2\omega}{\pi}\int_1^\infty\frac{d\omega'\,k'}{\omega'(\omega'^2-\omega^2)}$$
$$\times\frac{A_1(1-A_1)}{\pi}\int_1^\infty\frac{d\omega'\,k'|1-R_1(-\omega')|^2}{\omega'(\omega'+\omega)}. \quad (5)$$

We define the constant A_2 and the function R_2 by the equations $A_2=-A_1$, and $R_2=R_1^{-1}$. The corresponding relation for T_2^{-1} may then be obtained from Eq. (5) by making the simultaneous substitutions

$$T_1\longrightarrow T_2,\quad A_1\longrightarrow A_2,\quad R_1\longrightarrow R_2. \quad (6)$$

The value of the second integral at $\omega=0$ in Eq. (5) and the corresponding equation for T_2^{-1} determines the residues λ_j, and hence the coupling constant. In the charged scalar theory $A_1=A_2=0$, implying infinite values of λ_j; hence the unsubtracted relations cannot be valid in this theory. Since (when $A_1\neq 0$), $0<A_1<1$ and $A_2=-A_1$, the fact that the second integral in Eq. (5) is positive definite implies that $\lambda_1>0$ and $\lambda_2<0$. Therefore, in the $\pi NN'$ model discussed at the beginning of this section it is possible that the unsubtracted dispersion relations are valid in both channels if the intermediate baryon N' is of isospin $I_{N'}=I_\pi-\frac{1}{2}$. On the other hand, if $I_{N'}=I_\pi+\frac{1}{2}$, neither amplitude can satisfy an unsubtracted relation.

We consider the possibility that both amplitudes satisfy the unsubtracted relations. The second integral in Eq. (5) converges only if $1-R_1(-\omega)$ vanishes as $\omega\longrightarrow\infty$. If we assume that for high ω, $1-R_1(-\omega)$ is bounded by ω^{-n} (where $n>\frac{1}{2}$), then it follows from Eq. (5) that $(\omega T)^{-1}$ becomes imaginary in the limit of high positive energies, i.e., $T^{-1}(\infty+i\epsilon)=-i\omega$. The phase shifts approach the limits, $\delta_1\longrightarrow\frac{1}{2}\pi$ and $\delta_2\longrightarrow-\frac{1}{2}\pi$. Unitarity is saturated in both channels.

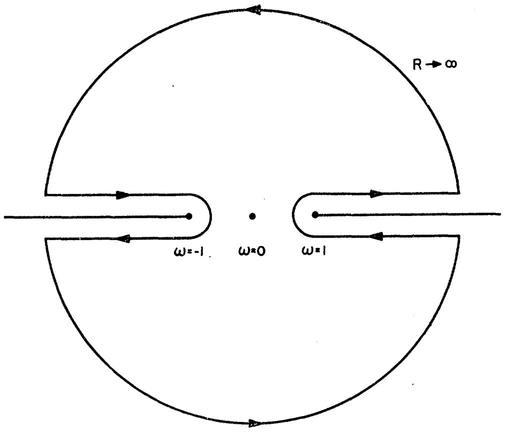

FIG. 1. Contour used in deriving static-model dispersion relation.

It is clear from the condition $T^{-1}(\infty)=-i\omega$ that the contribution from the upper semicircle in the integration around the path of Fig. (1) *does not vanish* in the limit $R\longrightarrow\infty$, but contributes a term $-\frac{1}{2}i$. This contribution is exactly cancelled by the contribution of the lower semicircle, however, so the omission of this effect in Eq. (5) is valid.

However, we have not shown that amplitudes exist which satisfy the two unsubtracted relations and the crossing condition. As in the $\pi\Lambda\Sigma$ problem, we must investigate the possibility that the special value of the coupling constant implied by the no-subtraction procedure is sufficiently great that a bound state is predicted in the channel with the positive residue (channel one). We shall investigate the bound-state problem by finding an approximate solution. The common approximation procedure of evaluating the left-hand cut in Born approximation would lead to a divergent integral in Eq. (5), so we modify this procedure slightly by setting $R(-\omega)$ equal to the Born-approximation value for values of $|\omega|$ less than a cutoff energy ω_0 and equal to one for larger values of $|\omega|$. The cutoff energy ω_0 may then be adjusted for consistency, i.e., the integrals

$$\int_1^\infty d\omega'\,k'|1-R_j(-\omega')|^2\omega'^{-2}$$

obtained from the R_j functions calculated from the solutions to Eqs. (5) and (6) should be about the same as the corresponding integrals calculated from the originally assumed R_j functions. The values of the residues of T_1 and T_2 at $\omega=0$ obtained by this procedure automatically satisfy Eq. (3). This follows from Eqs. (5) and (6) and the fact that in Born approximation

$$\lambda_1A_1(1-A_1)|1-R_1(-\omega)|^2$$
$$=\lambda_2A_2(1-A_2)|1-R_2(-\omega)|^2=4\lambda_1A_1/(1-A_1).$$

The residues of the poles in this approximation are given by

$$\frac{\lambda_1}{1-A_1}=-\frac{\lambda_2}{1-A_2}$$

$$=\frac{\pi}{4A_1}\{\ln[\omega_0+(\omega_0{}^2-1)^{1/2}]-(\omega_0{}^2-1)^{1/2}/\omega_0\}^{-1}.$$

We now limit consideration to values of A_1 that are less than $\frac{1}{2}$, since only these values correspond to interesting physical models. If $A_1\leqq\frac{1}{2}$, it may be shown that in our approximation a bound-state zero occurs in T_1^{-1} in the region $0<\omega<1$ unless $\omega_0>4$. Such a large value of ω_0 does not satisfy the above-mentioned consistency requirement, however, because the ratio $R(-\omega)$ computed from the approximate solutions for T_1 and T_2 becomes approximately equal to one at energies much lower than ω_0. Hence it appears that the no-subtraction criterion leads to bound states in this model.

Because of the omission of bound-state effects in Eqs. (5) and (6), the predictions resulting from these equations are not expected to be accurate quantitatively. However, these equations do illustrate the fact that in certain models the no-subtraction criterion may be applicable to several amplitudes simultaneously. Furthermore, the relation between this criterion and the possible existence of strongly enhanced amplitudes may be seen by comparing the two integrals in Eq. (5). At $\omega=1$ the integrals are of the same order of magnitude, and if the force is attractive ($\lambda_j>0$) they are of opposite sign. Hence T_j^{-1} is reduced significantly from its value in Born approximation. Whether or not this enhancement of T_j is sufficient to cause a bound state depends critically on the relative coefficients of these two terms. If a generalization of this technique is applicable to some real physical system (such as the coupled $\bar{K}$-N and π-Y systems, where the S-wave amplitudes are known to be large but the nature of the principal forces is not known), one might expect the occurrence of strongly enhanced amplitudes, but could not predict whether or not bound states should occur without knowing the strength of the left-hand cut fairly accurately.

It is instructive to write down the form of $(\omega T)^{-1}$ resulting if a subtraction is made at $\omega=0$. The equation is

$$(\omega T_j)^{-1}=\lambda_j{}^{-1}-\frac{2\omega}{\pi}\int_1^\infty\frac{d\omega'\,k'}{\omega'(\omega'^2-\omega^2)}$$

$$-\frac{A_j(1-A_j)\omega}{\pi}\int_1^\infty\frac{d\omega'\,k'|1-R_j(-\omega')|^2}{\omega'^2(\omega'+\omega)}.\quad(7)$$

The coupling constant is no longer restricted to a special value. If $R(-\infty)\neq1$, $(\omega T)^{-1}$ is proportional to $\ln\omega$ at high energies; Eq. (7) plus the crossing relation lead to the behavior

$$\lim_{\omega\to\infty}(\omega T_j)^{-1}=(4/\pi)[A_j/(1-A_j)]\ln\omega.$$

If λ_1 is positive and sufficiently small so that no zero occurs in the T_j^{-1}, the behavior of the phase shifts at infinity is $\delta_1(\infty)=\pi$; $\delta_2(\infty)=-\pi$.

IV. THE PION-NUCLEON INTERACTION

Because of the pseudoscalar nature of the pion-nucleon interaction, the π-N scattering partial waves most directly connected to the nucleon pole are the P waves. In order that the P-wave amplitudes do not contain kinematic singularities at the π-N threshold, it is necessary that these amplitudes be defined in terms of the phase shifts by the formula

$$T_j=\rho(W)e^{i\delta_j}\sin\delta_j/k^3,\quad(8)$$

where k is the particle momentum in the center-of-mass system, and where $\rho(W)$ is some function of the total center-of-mass system energy W that is not singular at $k=0$. The unitarity condition is $\mathrm{Im}(T_j^{-1})\cdots=-k^3\times\rho^{-1}(W)\eta(W)$, where η is the ratio of the total cross section to the elastic cross section. The P-wave problem is essentially relativistic since, in the static approximation (in which ρ^{-1} is replaced by a constant) the integral over the right-hand cut in the dispersion relation for the inverse amplitude diverges even if one subtraction is made.

One tries to choose $\rho(W)$ in such a way as to avoid kinematic singularities at energies below the π-N threshold. In the case of the $j=\frac{1}{2}$, $I=\frac{1}{2}$ amplitude T_{11} an appropriate choice is $\rho(W)=(W+M)^2-1$, where M is the nucleon mass and the meson mass is taken as unity. The contribution of the real nucleon state is then a simple pole in the W plane. Frautschi and Walecka have discussed possible choices of $\rho(W)$ for the various P-wave amplitudes.[9] The significant fact for the present discussion is that an appropriate choice of $\rho(W)$ behaves as W^2 at high energies. With such a choice the high energy behavior of $\mathrm{Im}(T_j^{-1})$ implied by the elastic unitarity condition is linear, and one may use a technique similar to that used in the S-wave model of Sec. III. We again assume that there are no zeros in the T_j, and use the inverse method. If no subtraction is made in a dispersion relation for $[(W-M)T_{11}]^{-1}$, the integral over the right-hand cut is logarithmically divergent; the integral is

$$\int_{M+1}^\infty dW'\,k'^3\rho^{-1}(W')\eta(W')/[(W'-M)(W'-W)].\quad(9)$$

We make this term convergent by subtracting the integral

$$\int_{M+1}^\infty dW'\,k'^3\rho^{-1}(W')\eta(W')/[(W'-M)(W'+W)],$$

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[9] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

and then adding this term to the contribution from the left-hand cut. The assumption that no subtractions are necessary again becomes a condition on the behavior of the left-hand cut at high negative energies.

The P-wave case differs from the S-wave case in that contributions to the integrals are important at distances of order M from the π-N threshold. One cannot argue that the effect of neglecting inelastic effects is small. Furthermore, there are many contributions to the left-hand cut which cannot be evaluated accurately at the present time; there is the π-π cut, and contributions to both the π-π and π-N cuts from many partial waves. The convergence of the left-hand cut integral involves a sum over infinitely many amplitudes rather than the ratios of a few amplitudes. For these reasons an accurate calculation of f^2 cannot be made at present. However, one can at least make a very simple calculation of f^2 which does not involve the introduction of arbitrary parameters by making the static approximation,[10] abruptly cutting off all integrals at the energy of the nucleon mass and following a procedure similar to that of Sec. III. The answer will be reasonable only if the actual total contributions of the left-hand cuts to the various P-wave amplitudes are of the same order of magnitude as the contributions of only the P waves. Since the π-N, P-wave amplitudes are known to be large, this condition may be true, although we are unable to present any convincing arguments for its validity. We regard it only as a possibility that is sufficiently interesting so that its consequences should be investigated. Of course, one could calculate the P-wave contributions to the left-hand cut more accurately by treating these amplitudes relativistically. This complicates the calculation greatly without making it accurate, however, so we stick to the simple static model.

Salzman and Salzman, using the measured value of f^2, have calculated the approximate position of the (3,3) P-wave resonance in a static cutoff model.[11] (This work is discussed more fully in Sec. V.) Frautschi and Walecka improved this calculation by including a contribution from π-π forces and treating the nucleon relativistically, thereby avoiding the necessity of a cutoff.[9] They also predicted a resonance, though not at the right energy. The present calculation differs from those of these two references in that we do not assume a value of f^2 but attempt to calculate it as well as the resonance position. (Of course our handling of the high-energy region is less accurate than that of reference 9.) Important short-range forces are omitted in all these calculations, so that an accurate calculation of the resonance position is not expected. The well-known relation between the width of the resonance, its position, and the coupling constant may be shown from the effective range approximation to any one of these models, so we will not discuss it further.

The cutoff function in our model serves a dual purpose. First, it allows us to replace the function $\rho^{-1}(W)$ by a constant. Secondly, it leads to the convergence of the left-hand cut integral and thus implies a convergence assumption similar to that made in Sec. III; i.e., the various contributions to the left-hand cut are assumed to cancel sufficiently at high energy so that the no-subtraction procedure is valid. Because of the first role of the cutoff function and the fact that in a relativistic theory the energy at which $\rho^{-1}(W)$ decreases significantly from its threshold value is of order M, the cutoff energy must be taken of order M. We take the cutoff at $\omega=M$, where ω is the meson energy.

We write $T_j=e^{i\delta_j}\sin\delta_j/k^3$. If no subtractions are made and the elastic unitarity condition is assumed, the dispersion relations for $(\omega T)^{-1}$ in the cutoff static model may be written in the form

$$(\omega T_j)^{-1}=\frac{-2\omega}{\pi}\int_1^M \frac{d\omega'\,k'^3}{\omega'(\omega'^2-\omega^2)} +\frac{1}{\pi}\int_1^M \frac{d\omega'\,k'^3[X_j(\omega')-1]}{\omega'(\omega'+\omega)}, \tag{10}$$

where the term linear in X_j represents the contribution from the left-hand cut. One can express the quantities X_j-1 in terms of the various amplitude ratios by using the well-known static model crossing relations,

$$T_j(-\omega)=\sum\nolimits_i A_{ji}T_i(\omega), \tag{11}$$

where

$$A=(1/9)\begin{array}{cccc} (3,3) & (3,1) & (1,3) & (1,1) \\ \end{array}$$
$$A=(1/9)\left[\begin{array}{rrrr} 1 & 2 & 2 & 4 \\ 4 & -1 & 8 & -2 \\ 4 & 8 & -1 & -2 \\ 16 & -4 & -4 & 1 \end{array}\right]$$

If $R_{j,k}$ is defined by $T_j(\omega)=R_{j,k}(\omega)T_k(\omega)$, and the symbol $\mathcal{R}_{j,k}$ is used to denote $R_{j,k}(-\omega)$, the equations for X are

$$X_{33}(\omega)-1 =(2/9)^2|1+\mathcal{R}_{11,33}-\mathcal{R}_{13,33}-\mathcal{R}_{31,33}|^2 +(2/9)\{|1-\mathcal{R}_{13,33}|^2+|1-\mathcal{R}_{31,33}|^2\}, \tag{12a}$$

$$X_{31}(\omega)-1 =-(8/81)|1+\mathcal{R}_{13,31}-\mathcal{R}_{11,31}-\mathcal{R}_{33,31}|^2 -(4/9)|1-\mathcal{R}_{33,31}|^2+(2/9)|1-\mathcal{R}_{11,31}|^2, \tag{12b}$$

$$X_{11}(\omega)-1 =(4/9)^2|1+\mathcal{R}_{33,11}-\mathcal{R}_{13,11}-\mathcal{R}_{31,11}|^2 -(4/9)\{|1-\mathcal{R}_{13,11}|^2+|1-\mathcal{R}_{31,11}|^2\}. \tag{12c}$$

The equation for $X_{13}-1$ may be obtained by reversing the subscripts 31 and 13 in Eq. (12b). Because of the factor k'^3 in the integrand, the first term in Eq. (10)

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[10] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[11] G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957); This paper will be referred to by the symbol SS.

reaches its maximum magnitude, not at $\omega=1$ (as in the S-wave case) but at a higher energy. Therefore, if the enhancement of the attractive (3,3) amplitude is not quite sufficient for binding, a low-energy resonance will occur.

The coupling constant is to be determined from the well-known equations for the residues of the four P-wave amplitudes,

$$\lim_{\omega\to 0}\omega T_j=c_jf^2,$$
$$c_{33}=4/3,\quad c_{11}=-8/3,\quad c_{31}=c_{13}=-2/3. \tag{13}$$

It is seen from Eqs. (10) and (12a) that $X_{33}-1$ is of a definite sign which, fortunately, is positive. We find approximate solutions to the dispersion equations by using a technique similar to that used in Sec. III; we replace the various ratios $R_{j,k}(-\omega)$ at all energies by their Born approximation values. Since we have used an abrupt cutoff rather than an ω^{-2} cutoff, there is no need to introduce a further cutoff in this approximation. This procedure satisfies the following two requirements:

Consistency. The π-N coupling constants computed from the dispersion relations for the four P-wave amplitudes are all the same. This follows from Eqs. (10), and (13) and the fact that in Born approximation,

$$\begin{aligned}(X_{33}-1)&=-2(X_{11}-1)\\&=-\tfrac{1}{2}(X_{31}-1)=-\tfrac{1}{2}(X_{13}-1)=1.\end{aligned} \tag{14}$$

Approximate crossing. This approximation to the second integral in Eq. (10) is not expected to be accurate for appreciably negative values of ω. Hence one can expect crossing symmetry to be satisfied by the approximate solutions only for small values of $|\omega|$. In the neighborhood of zero energy, we expand the amplitudes in powers of ω, i.e., $T_j=c_jf^2\omega^{-1}+\beta_j+\cdots$. We assume that T_{13} and T_{31} are identical, since their equations are the same. Equation (13) implies that crossing is satisfied to order ω^{-1}. The four crossing relations of Eq. (11) are identical when applied to the constant terms of T_j, and lead to the condition $2\beta_{33}=\beta_{31}+\beta_{11}$. It may be shown that in our approximation procedure $\beta_{31}=0$ and $\beta_{11}=2\beta_{33}$, so crossing is satisfied to zero order in ω.

The values of T_j^{-1} resulting from this approximation can be written in the form

$$(\omega T_j)^{-1}=\pi^{-1}[B_1+(4/3c_j)B_2], \tag{15}$$

where the c_j are given in Eq. (13), and the functions B_1 and B_2 are given by the equations

$$B_1=-2\omega\left[P-\frac{1}{\omega^2}\tan^{-1}P-\frac{k^3}{2\omega^2}\ln\frac{P+k}{P-k}\right]-i\pi k^3/\omega, \tag{16a}$$

$$\begin{aligned}B_2=\tfrac{1}{2}MP+(\omega^2-\tfrac{3}{2})\ln(M+P)-\omega P\\+\frac{1}{\omega}\tan^{-1}P-\frac{k^3}{\omega}\ln\frac{M+\omega}{M\omega-Pk+1},\end{aligned} \tag{16b}$$

where $P=(M^2-1)^{1/2}$ and $k=(\omega^2-1)^{1/2}$. In the unphysical region on the real axis $|\omega|<1$ these functions are most easily represented if one defines the quantity κ by the relation $k=i\kappa=i(1-\omega^2)^{1/2}$ and writes

$$\frac{k^3}{\pi}\ln\frac{P+k}{P-k}-ik^3=-\frac{2\kappa^3}{\pi}\tan^{-1}\left(\frac{P}{\kappa}\right), \tag{17a}$$

$$k^3\ln\frac{M+\omega}{M\omega-Pk+1}=\kappa^3\tan^{-1}\left(\frac{P\kappa}{M\omega+1}\right). \tag{17b}$$

There are no bound states in this approximation, as none of the inverse amplitudes have any zeros anywhere. It is easily verified from Eqs. (16) and (17) that the T_j^{-1} have no zeros along the real axis in the region $-1<0<1$. The behavior of the T_j^{-1} off the real axis may be investigated from Eq. (10) and the approximations for X_j [Eq. (14)]. The quantities $\mathrm{Im}[(\omega T_{11})^{-1}]$ and $\mathrm{Im}[(\omega T_{33})^{-1}]$ have no zeros off the real axis. On the other hand, $\mathrm{Im}[(\omega T_{31})^{-1}]$ is zero along the imaginary axis (but nowhere else off the real axis) in our approximation, but $\mathrm{Re}[(\omega T_{31})^{-1}]$ is positive everywhere along the imaginary axis.

The value of f^2 computed by evaluating ωT_j at $\omega=0$ from Eqs. (13), (15), (16b), and (17b) is

$$f^2=\tfrac{3}{4}\pi[\tfrac{1}{2}MP-\tfrac{3}{2}\ln(M+P)+P/M]^{-1}. \tag{18}$$

For $M=6.7$ this leads to $f^2=0.12$, about 1.5 times the measured value. There is a resonance in the (3,3) state at the energy $\omega=1.3$. The phase shift δ_{33} increases to $\sim 142°$ at $\omega\sim 1.7$, levels off, then decreases slowly toward 90°. If the cutoff were increased, so that f^2 would correspond with the measured value, the resonance energy would be higher, but still below the experimental position. It is interesting to note that if one varies the nucleon mass M (i.e., the cutoff), it is the coupling constant $G^2=f^2M^2/4$ (rather than f^2 or f^2M) that is nearly constant in Eq. (18).

This static model calculation of f^2 is not realistic, since many important effects are omitted. However, the general consistency of the calculation (the facts that the residues are of the right signs and approximately correct magnitudes) leads us to hope that in a more sophisticated treatment the no-subtraction criterion will still be applicable and will still predict the existence of the (3,3) resonance.

If additional forces were considered, or if a more accurate solution to Eq. (10) were obtained, it is unlikely that the different P-waves would still saturate at the same value of the coupling constant f^2. However, if the conjecture of Sec. I is correct, i.e., that the saturation of one partial wave leads to the saturation of other partial waves because of the presence of many particle intermediate states, then it is not necessary for our viewpoint that several partial waves saturate simultaneously in a calculation in which inelastic effects are neglected. Presumably, the best value of the coupling constant calculable in such an "elastic" model is the lowest value that leads to saturation in any partial-wave state.

The expressions for T_j^{-1} in Eqs. (16) are not meaningful at energies approximately equal to or greater than M. The singularities at $\omega=M$ are artificial, resulting from the use of the abrupt cutoff. In fact, use of the abrupt cutoff in the first integral of Eq. (10) actually destroys the condition $\delta(\infty)=\pm\frac{1}{2}\pi$, but this condition would be fulfilled if a cutoff were used which behaved as ω^{-2} at high energies.

V. RELATION TO CONVENTIONAL STATIC MODELS

In the conventional static model of π-N scattering, one assumes a fixed value of the coupling constant, and solves subtracted dispersion relations for $(\omega T_j)^{-1}$. These equations may be written in the form

$$(\omega T_j)^{-1}=\lambda_j^{-1}-\frac{2\omega}{\pi}\int_1^\infty\frac{d\omega'\,k'^3\rho^{-1}(\omega')}{\omega'(\omega'^2-\omega^2)}-\frac{\omega}{\pi}\int_1^\infty\frac{d\omega'\,k'^3\rho^{-1}(\omega')[X_j(\omega')-1]}{\omega'^2(\omega'+\omega)},\quad(19)$$

where the $\rho^{-1}(\omega)$ is the cutoff function and the $X(\omega)$ are given in Eqs. (12). Although it is not necessary to include a contribution from the left-hand cut in the first integral, we have done so in order to make the energy dependence more transparent. The large variations in ωT_j that can occur at low energies result from the first integral.

It is difficult to improve the mathematical accuracy of the solutions to the unsubtracted relations by iterating, since the values of f^2 obtained from the three different relations are not likely to be equal in each iteration. On the other hand, one could assume a fixed value of f^2, iterate the subtracted relations in order to find an accurate solution, and then increase f^2 and repeat the procedure, hoping to find a critical f^2 that leads to the condition $[(\omega T)^{-1}(\infty)]=0$ for one or more of the amplitudes. Such an iteration procedure has been performed by Salzman and Salzman, using a Gaussian cutoff function.[11] The fact that they did not discover a critical value of f^2 may be due to the fact that they had no reason to look for it, and thus examined carefully only the physical value $f^2=0.08$ and values of the cutoff parameter P less than or equal to 7. For $f^2=0.08$ and $P=7$, the asymptotic limits of $(\omega T_{33})^{-1}$ and $(\omega T_{11})^{-1}$ are $\sim0.27\lambda_{33}^{-1}$ and $\sim0.32\lambda_{11}^{-1}$, respectively.[11] In view of the results of Sec. IV it appears likely that one or both of these amplitudes would approach zero at a somewhat larger value of the coupling constant (or cutoff).

SS do find that for a value of the cutoff between 4 and 5, the amplitude $T_{13}=T_{31}$ changes its asymptotic behavior. The quantity $(\omega T_{31}/\lambda_{31})^{-1}$ does not approach zero at high energies, however, but rather approaches a value greater than one for cutoffs higher than this singular cutoff.[12] Nothing strange happens to T_{11} and T_{33} at this point; SS point out that T_{13} and T_{31} are coupled rather weakly to the amplitudes T_{11} and T_{33}.

The use of a cutoff behaving as ω^{-2} at high energies leads to a high-energy behavior quite different from that obtained with a more rapid cutoff, although in both cases the first integral term in Eq. (19) vanishes at infinity. If the cutoff is rapid, a noncritical value of f^2 leads to $(\omega T)^{-1}$ approaching a nonzero constant at high energy. The first integral term in Eq. (19) is artificially large at energies slightly above the cutoff energy. If this term is important at low energies, it will still be large at energies comparable to the cutoff energy squared, so that the amplitudes approach their asymptotic limits very slowly. On the other hand, an ω^{-2} cutoff leads to a logarithmic asymptotic behavior of $(\omega T)^{-1}$ for noncritical values of f^2. The amplitude approaches its asymptotic behavior more rapidly, because the principal-part integral $\int d\omega'\,g(\omega')/(\omega'^2-\omega^2)$ is small if g is a nearly constant function. At the critical coupling constant, (if one exists), $X(\omega)$ approaches one and the second integral term in Eq. (19) approaches $-\lambda^{-1}$ as $\omega\rightarrow\infty$.

Wanders has considered in detail the simpler, fictitious example of neutral pseudoscalar meson theory, in which there are only two P-wave amplitudes, characterized by angular momentum $\frac{3}{2}$ and $\frac{1}{2}$.[13] He assumes the elastic unitarity condition and a cutoff of the form $(C^2+1)/(C^2+\omega^2)$, and exhibits a representation for the S-matrix elements having the proper analytic properties and satisfying the crossing relations exactly. Wanders discusses in particular detail a simple family of solutions corresponding to a range of values of the coupling constant f^2.[14] There is a largest value f_c^2 of the coupling constant in this family. If $f^2<f_c^2$ the limiting values of the phase shifts are $\delta_3(\infty)=\pi$ and $\delta_1(\infty)=0$, and it is easily shown that when $f^2=f_c^2$, $\delta_3(\infty)=\frac{1}{2}\pi$ and $\delta_1(\infty)=-\frac{1}{2}\pi$. [We have defined $\delta(1)$ to be 0.]

It may be shown that for $f^2<f_c^2$ Wanders' solution involves at least one T-matrix zero, and thus is not equivalent to the solution that may be obtained from subtracted dispersion relations for T^{-1}. For the critical value of f^2 it is not clear whether or not Wanders' solution is free of T-matrix zeros. There is no closed expression for the critical coupling constant in this model, but for extremely large values of the cutoff C, f_c^2 is proportional to $C^{-2}\ln C$. A pair of S-matrix zeros occur in the first and fourth quadrant for the $j=\frac{3}{2}$

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[12] It may be seen from Eqs. (19) and (12) that this kind of behavior is impossible for the amplitude T_{33} and extremely unlikely for T_{11}, since $X_{33}-1$ is positive definite and the two terms of $X_{11}-1$ with largest coefficients are negative definite.

[13] G. Wanders, Nuovo cimento **23**, 817 (1962).

[14] This is the family characterized by the choice of a constant value for the function $\bar{\beta}$ defined in Eq. (3.3) of reference 13.

amplitude; it is tempting to interpret these as representing a resonance. However, for the value of the cutoff $C=6.7(2/\pi)$ (which is roughly equivalent to an abrupt cutoff of 6.7) we have shown that if $f^2=f_c^2$ the phase shift δ_3 does not pass through 90°; if $f^2<f_c^2$, δ_3 passes through 90° only at an energy above the cutoff. A resonance occurs below the cutoff energy only if the cutoff is much larger than $6.7(2/\pi)$, and this resonance is not at low energy. It is interesting to note that if the approximation procedure of Sec. IV is applied to this two-channel case, the amplitudes may be represented by Eq. (10), but the value of X_3-1 in Born approximation is 2, so that the enhancement of the $j=\frac{3}{2}$ amplitude is only about half as great as the enhancement of the (3,3) amplitude in the four-channel case.

Wilson, using techniques similar to those of Wanders, has studied the static model for symmetric, scalar meson theory, with neglect of inelastic effects.[15] (This corresponds to the choice $A_1=\frac{1}{3}$ in Sec. III.) Wilson exhibits a family of amplitudes satisfying the elastic unitarity condition and the crossing and analyticity requirements exactly. However, his solutions differ from the solutions to Eq. (7) in that they involve at least one T-matrix zero. There is a maximum value of f^2 (corresponding to a residue of T of $\sim\frac{1}{3}$ in the isotopic spin $\frac{3}{2}$ channel) in the simple family of solutions given by Wilson, but this value does not correspond either to a singular behavior at infinity or to the emergence of a bound-state pole on the physical sheet.

VI. DISCUSSION AND CONCLUSIONS

The prescription given in this paper for calculating coupling constants is a double prescription. In general, there are an infinite number of sets of amplitudes that satisfy the analyticity, unitarity, and crossing requirements representing some idealized physical situation. One generally assumes a particular method of solution, such as the inverse method or N/D method, which restricts the solutions to a one-parameter set, this parameter being interpretable as a coupling constant. In this paper we use the inverse method. The condition $\mathrm{Re}[\omega^{-1}T^{-1}(\infty)]=0$ then is used to limit the solution to one, corresponding to a particular coupling constant. A similar situation occurs in the model of Wanders discussed in Sec. V, where a one-parameter family of solutions is presented, and the critical coupling constant corresponds to one member of this family.[13]

Recently, Albright and McGlinn have proposed that coupling constants may be determined from the prescription that partial-wave S-matrix elements have a minimum number of zeros.[8] This prescription is related to that of the author for the $\pi\Lambda\Sigma$ problem in Sec. II and in reference 3. It may be shown from the equations of reference (3) that S, the S-matrix element for π-Λ scattering, is negative for values of ω^2 just less than Δ^2 (the energy of the Σ pole). If the coupling constant f^2 is less than the critical value f_c^2, then $S(\omega^2\rightarrow-\infty)=S(\omega^2\rightarrow\infty)=1$, and S has a zero in the region $-\infty<\omega^2<\Delta^2$. As f^2 approaches f_c^2 this zero moves toward $\omega^2=-\infty$. At $f^2=f_c^2$ the zero vanishes; $S(\omega^2\rightarrow-\infty)=S(\omega^2\rightarrow\infty)=-1$, and $\delta(\omega\rightarrow\infty)=-\frac{1}{2}\pi$. If $f^2>f_c^2$, then $S(\omega^2\rightarrow-\infty)=S(\omega^2\rightarrow\infty)=1$ and a ghost pole occurs in the region $-\infty<\omega^2<\Delta^2$. The prescription of Albright and McGlinn is also related to ours in the static model of neutral pseudoscalar meson theory, for in Wanders' model the critical value of f^2 corresponds to the vanishing of S-matrix zeros for the two P-wave amplitudes at $\omega^2=-\infty$.[13]

One cannot hope to test the validity of the no-subtraction conjecture by examining experimental data for evidence of a π-N P-wave phase shift approaching 90°. Unfortunately, the general shape of a P-wave amplitude satisfying a subtracted dispersion relation may not be significantly different from that predicted by an unsubtracted relation at energies below the nucleon mass. On the other hand, if one considers systems that interact strongly in S waves, such as the coupled $\bar{K}$-N and π-Y systems, high kinetic energies are presumably not important in the dispersion relations. When something is known about the origin of the forces in these systems, it may be that experimental observations concerning the persistence of the amplitudes as the energy is increased will provide evidence for or against the no-subtraction conjecture.

If the π-Y interactions are related to the π-N interaction by some type of global symmetry, it is not known in what manner "renormalization effects" caused by coupling of the π-Y and $\bar{K}$-N channels lead to a breaking of the symmetry with respect to the physical interaction constants. If the conjecture of Sec. I is correct, this coupling leads to such adjustments in the π-Y interaction constants as are necessary for the π-Y cross sections to approach nonzero high-energy limits.

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[15] K. Wilson, thesis, California Institute of Technology (unpublished). The author wishes to thank Professor C. Goebel for notes concerning this work.