$F(\pi/2,\kappa)$ represents a complete elliptic integral of the first kind defined by

$$F(\pi/2,\kappa)=\int_0^{\pi/2}(1-\kappa^2\sin^2 u)^{-1/2}du, \qquad \text{(A3)}$$

which has a logarithmic singularity at $\kappa=1$. The principal part of $F(\pi/2,\kappa)$ at the singular point proves to be given by

$$-\tfrac{1}{2}\ln(1-\kappa^2), \qquad \text{(A4)}$$

as can be seen easily by replacing $\sin^2 u$ in the denominator of (A3) by $1-u^2$. Now (A3) is expanded in a power series by κ^2, followed by integration over u. The series thus obtained is subtracted by the series of $-\frac{1}{2}\ln\times(1-\kappa^2)$, assuming uniform convergence. The result becomes

$$F(\pi/2,\kappa)=-\tfrac{1}{2}\ln(1-\kappa^2)+(\pi/2)\{1+\tfrac{1}{2}(\tfrac{1}{2}-2/\pi)\kappa^2 +\tfrac{1}{2}[2(\tfrac{3}{8})^2-1/\pi]\kappa^4+\tfrac{1}{2}[2(\tfrac{5}{16})^2-2/(3\pi)]\kappa^6+\cdots\}. \qquad \text{(A5)}$$

The above series is now convergent at $\kappa=1$ because the singular part which causes a divergence at $\kappa=1$ has been picked up in the first term.

Substituting (A5) into (A2), we have, after integration,

$$\varphi(x)=(2/\pi)\ln 2-(2/\pi)\ln[1+(1-x)^{1/2}]+1 +\tfrac{1}{4}(\tfrac{1}{2}-2/\pi)x+\tfrac{3}{16}[2(\tfrac{3}{8})^2-1/\pi]x^2 +\tfrac{5}{32}[2(\tfrac{5}{16})^2-2/(3\pi)]x^3+\cdots, \qquad \text{(A6)}$$

using

$$\int_0^{2\pi}\ln[1-x\cos^2(k/2)]dk=4\pi\ln\{[1+(1-x)^{1/2}]/2\}.$$

The series given above reproduces the first three figures correctly for $\varphi(1)$.

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Distribution Functions and Ionization Rates for Hot Electrons in Semiconductors

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The distribution of electrons in a semiconductor at high electric field is governed by a Boltzmann equation which describes the effects of the field, the phonons, and the ionization processes on the electrons. This equation can be converted to an integral equation for the space and energy dependent collision density by performing the angular integrations. The integral equation is solved numerically to obtain alpha, the ionization rate per unit path length. The resulting alpha shows a dependence on field strength $\mathcal{E}$ which is roughly $\exp(-b/\mathcal{E})$ at low fields and $\exp(-c/\mathcal{E}^2)$ at high fields, but there are significant differences from the earlier calculations of Wolff and Shockley. These differences result from the approximations used by the earlier workers to treat the angular dependence. We present graphs of $\log\alpha$ vs $(1/\mathcal{E})$ for a wide range of material parameters. These graphs are useful in interpreting measurements of charge multiplication in terms of the properties of the material supporting the transport process.

I. INTRODUCTION

MEASUREMENTS of the charge multiplication at high electric fields in semiconducting materials can be used to supply basic information about the interaction of hot electrons with the material, provided that one has an adequate model for interpreting the data. A reasonably satisfactory two-part model exists, based on the analogy between the processes taking place in the semiconductor and those taking place in a gas discharge. The first part relates the charge multiplication to the ionization rate per unit path length, α_i,[1,2] and the second part relates this quantity, Townsend's alpha coefficient, to the distribution function for the hot electrons in the material.[3,4]

Calculating the distribution function for the hot electrons is not a particularly simple task, however, and different approximations made at this stage result in different predictions about the field dependence of Townsend's alpha. Wolff,[3] for example, neglects the band structure of the silicon, which was unknown at the time of his calculations, and expands the distribution function in spherical harmonics, retaining only the P_0 and P_1 terms. This procedure is justified at high fields, as Wannier has shown,[5] where the energy loss per collision is so much smaller than the energy gain that the collisions serve to keep the distribution nearly isotropic. The result of retaining only the P_0 and P_1 is a diffusion theory approximation, in which the electrons undergo many collisions in transport from one energy to another. It results in a distribution which, below ionization threshold, is quasi-Maxwellian with a temperature re-

[1] K. G. McKay, Phys. Rev. **94**, 877 (1954).
[2] S. L. Miller, Phys. Rev. **105**, 1246 (1957).
[3] P. A. Wolff, Phys. Rev. **95**, 1415 (1954).
[4] W. Shockley, Czech. J. Phys. **B11**, 81 (1961) and Solid-State Electron. **2**, 35 (1961).
[5] G. H. Wannier, Bell System Tech. J. **32**, 170 (1953).

lated to the mean free path λ, the electric field $\mathcal{E}$ and the optical phonon energy E_R by

$$T_w = Q^2/3E_R, \tag{1.1}$$

where $Q = e\mathcal{E}\lambda$. This, in turn, results in an alpha whose dependence on the field $\mathcal{E}$ is approximately

$$\alpha(\mathcal{E}) \approx a \exp(-c/\mathcal{E}^2).$$

Shockley,[4] on the other hand, also neglects the band structure of the silicon, but argues that the electrons which achieve sufficient energy to cause ionization are those few which have been fortunate enough to escape the collisions which randomize the directions of motion. The distribution function for these electrons is a spike in the direction of the acceleration, and the number in the spike which have survived the transport to energy E without collision is proportional to $\exp(-E/Q)$. By retaining only these electrons in his calculation, Shockley obtains an alpha whose dependence on the field is roughly

$$\alpha \approx c \exp(-E_i/e\mathcal{E}\lambda).$$

A series of careful experiments carried out by Chynoweth[5] and collaborators somewhat before Shockley's prediction did indicate behavior of this sort, and was interpreted at that time in terms of the formula above. Shockley's theory suggested how the coefficient c was related to the properties of the material. One can fit the experimental data to Shockley's formula and obtain thereby the values of mean free path, ionization cross section, etc., on which alpha depends. The values of the parameters obtained by this procedure, however, raise some questions, particularly since it is not clear how valid are the subsequent assumptions which serve to lead from Shockley's conjecture about the importance of the electrons in the spike to his final formula for alpha. Although these questions arise about the final results, the conjecture itself is a particularly attractive one, and one is led to wonder if the fields are, in fact, sufficiently high to maintain the distribution nearly spherically symmetric as Wolff's calculation requires, or are low enough to render the high-energy tail of the distribution sharply peaked in the forward direction as Shockley assumes.

The remaining part of this introduction will consist of an attempt to obtain a qualitative answer to this question. We discover that the fields of interest for the charge multiplication work in many materials are of intermediate strength. We shall thus be forced to consider an approach which relies neither on the spike nor the diffusion approximation. Such an approach can be made via an integral equation for the collision density. This approach forms the subject of the remaining sections of the paper.

In the second section, we shall define the space- and energy-dependent collision density and show how Townsend's alpha may be calculated from it. The third section describes the derivation of the integral equation for the density from the Boltzmann equation for the distribution function. Those parts of the derivation furthest removed from the physics are also furthest removed from the main description and are to be found in the Appendix. Certain conservation conditions which will play an important part in the approximations will be emphasized in Sec. IV. Section V contains the results of the calculations and discusses the concepts which emerge. Comparison is made with the earlier work of Wolff and Shockley.

To investigate the critical fields for transition from Shockley's spike to Wolff's diffusion mechanism of electron transport, let us temporarily assume that the distribution function for electrons with speed c and direction cosine μ relative to the field direction is composed of two parts, one part being spherically symmetric and the other part being a spike in the forward direction. This assumption is merely a way of combining the essential features of Shockley's procedure with the essential feature of Wolff's procedure and is not the distribution function which we shall use for quantitative calculations later. This assumed distribution function may be written as

$$f(c,\mu) = A(c) + B(c)\delta(1-\mu), \tag{1.2}$$

and by forcing this into the Boltzmann equation, we can choose the admixture of spiked and spherically symmetric distributions at each speed to best represent the true distribution function. The technique of obtaining $A(c)$ and $B(c)$ is to expand (1.2) as an infinite series of spherical harmonics

$$f(c,\mu) = \sum_l n_l(c) P_l(\mu),$$

$$n_l(c) = \frac{2l+1}{2} \int_{-1}^{1} f(c,\mu) P_l(\mu) d\mu.$$

In such an expansion, n_0 is proportional to the density of electrons at speed c. This expansion converts the integro-differential Boltzmann equation into an infinite set of coupled differential equations for the coefficients $n_l(c)$.[7] The technique is a standard one, and was applied by Wolff[3] to a distribution function which he took to be

$$f(c,\mu) = n_0(c) + \mu n_1(c). \tag{1.3}$$

In our case, the assumption of form (1.2) gives

$$n_2 = (5/3)n_1.$$

This relationship may be used to decouple the first two differential equations from the rest, whereupon they may be combined to give a single equation for the density $n_0(c)$. For constant mean free path λ, the solution which corresponds to Wolff's Maxwellian and to

[6] A. G. Chynoweth, J. Appl. Phys. **31**, 1161 (1960), this paper refers to earlier works.

[7] See reference 5 for details.

Shockley's spike is

$$n_0 = E^{-a} e^{-E/T}, \tag{1.4}$$

with

$$a = (\tfrac{2}{3} + Q/3E_R)^{-1},$$
$$T = Q^2/3E_R + 2Q/3.$$

For $Q = E_R$, this solution is

$$n_0 = E^{-1} e^{-E/Q}, \tag{1.5}$$

which is exactly the shape of the spike proposed by Shockley.

For large Q/E_R, the solution goes over into Wolff's Maxwellian

$$n_0 \approx \exp(-3EE_R/Q^2).$$

This implies that the bulk of the electrons described by (1.2) are in the spherically symmetric part common to both (1.2) and (1.3), rather than in the spike.

We learn from this example that the criterion for the validity of the Shockley spike is that $Q \leq E_R$ or that $e\mathcal{E} \leq E_R/\lambda$. Unfortunately, the fields of interest for the charge multiplication studies are neither this weak nor clearly so strong that $Q \gg E_R$. It is, therefore, highly unlikely that either of the simplifying assumptions (1.2) or (1.3) will be of much use here. We are thus forced to consider an approach which does not depend on having a simple form of angular dependence. This approach will be made via the integral equation governing the collision density, a quantity closely related to $n_0(\epsilon)$. Our solution of the equation will ultimately be numerical, but from a study of the numerical solutions useful information and qualitative concepts can emerge.

II. THE COLLISION DENSITY

Consider a packet of zero-energy electrons released at the plane $Z=0$ in a uniform electric field. The electrons actually released by the ionization process in the semiconductor will have, as we shall see later, small but finite energy, of the order of one-third the band gap or so. Our packet of zero energy electrons is an idealization which should not introduce significant error. As the packet drifts down the field, the electrons in it undergo collisions with phonons and with the valence band electrons. The collision density $M(Z,E)$ will be defined such that the number of collisions of all types which occur in the slab of thickness dZ located at Z and which involve electrons whose energy is in the range dE at E is given by $M(Z,E)dZdE$. Given this collision density, and given the ratio of cross section for ionization to total cross section at energy E,

$$r(E) \equiv \sigma_i(E)/\sigma_{\text{tot}}(E),$$

we can calculate the number of ionizations $n(Z)$ occurring in the slab at Z.

$$n(Z) = \int_0^\infty dE\, M(Z,E) r(E). \tag{2.1}$$

We shall be interested in an $n(Z)$ which represents a density of initial ionizations. By density of initial ionization, we mean that only the first ionizing event after the release of the electron is to be counted. This can be calculated, using (2.1), from a collision density $M(Z,E)$ in which each ionizing event is assumed to remove the electron from the distribution. That is, $M(Z,E)$ will be calculated by solving a Boltzmann equation in which the probability for ionization is replaced by a probability for absorption so that the electron never survives the first ionization.

The average distance at which the first ionization occurs is[8]

$$\bar{Z} = \int_0^\infty n(Z) Z dZ \Big/ \int_0^\infty n(Z) dZ. \tag{2.2}$$

If the cross section for ionization is at all comparable with the cross section for scattering, then the first ionization will occur almost as soon as the electron has sufficient energy to make the process possible. Under these circumstances, the electron will be left with such a small amount of energy after its first collision that we may consider it to have been returned to zero energy. Its next ionizing collision will then occur at a distance $\bar{Z}$ further down the field. Hence, α_i, the number of ionizations per unit length which this single electron causes will be given by[9]

$$\alpha_i = \bar{Z}^{-1}. \tag{2.3}$$

In calculating α by this method, it will not be necessary to know $n(Z)$ at very large values of Z for the following reasons: Firstly, the denominator of (2.2) is the total number of electrons absorbed (i.e., causing a first ionization). Since each electron released in the initial bunch is ultimately absorbed, this denominator can be made unity by normalizing the distribution to the release of a single electron at the plane $Z=0$. This normalization will be adopted. Secondly, one expects the spectral *shape* of $M(Z,E)$ to become Z independent as the electrons reach equilibrium under the heating influence of the field and the cooling influence of the phonons. Once equilibrium is reached, a constant fraction of the electrons will be lost each mean free path and $n(Z)$ will exhibit an exponential decrease. It will therefore be necessary to have $n(Z)$ only for those values of Z between zero and the point at which $n(Z)$ starts to decrease exponentially.[10]

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[8] The zero lower limit arises because zero energy electrons released at $Z=0$ can reach $Z<0$ only if they acquire energy from the phonons. We ignore this possibility here.

[9] There is another method of calculating $\bar{Z}$, hence of α, which can be derived from (2.2). This method, to be described at the end of Sec. III, is of interest in establishing the relation between Wolff's work and Shockley's and also is more convenient numerically at low fields.

[10] The actual rate of exponential decrease can be determined by the spatial integral over this region, the value of $n(Z)$ at the point at which the decrease begins, and the knowledge that the full integral from zero to infinity is equal to unity.

III. THE INTEGRAL EQUATION

There are three types of collisions which are counted by $M(Z,E)$; collisions with acoustic phonons, which we shall take to be elastic, collisions producing optical phonons whose energy we take to be a constant E_R, and collisions with valence electrons, which, for reasons explained earlier, we treat as absorption of the incident electron. The temperature of the lattice will be assumed low enough so that no optical phonons are present. There is no possibility of the electron acquiring energy from the phonon field. Under these conditions, it is useful to regard $M(Z,E)$ as a sum of partial densities $M_n(Z,E)$, where the subscript denotes the number of optical phonons which the electron has emitted prior to the collision counted in $M(Z,E)$. An electron which has energy E after having emitted n phonons of energy E_R has acquired this energy by having drifted a distance

$$Z=(E+nE_R)/e\mathcal{E}$$

along the field. The Z dependence of $M_n(Z,E)$ must, therefore, be a delta function of the above expression, and the full $M(Z,E)$, when expressed in terms of the partial densities, must have the form

$$M(Z,E)=\sum_{n=0}^{\infty} M_n(E)\delta\left(Z-\frac{E+nE_R}{e\mathcal{E}}\right). \tag{3.1}$$

On taking the Laplace transform of this form and multiplying by $\exp(SE/e\mathcal{E})$, we obtain

$$\begin{aligned} M(\eta,E)&\equiv M(S,E)\times\exp(SE/e\mathcal{E})\\ &=\sum_{n=0}^{\infty} M_n(E)\eta^n, \end{aligned} \tag{3.2}$$

where

$$M(S,E)=\int_0^{\infty} dZ\, e^{-SZ}M(Z,E), \tag{3.3a}$$

$$\eta=\exp(-SE_R/e\mathcal{E}). \tag{3.3b}$$

We shall soon derive an equation for $M(\eta,E)$. Equation (3.2) indicates the utility of seeking its solution as a power series in η, since the coefficients of this power series, $M_n(E)$, determine $M(Z,E)$ through (3.1).

To construct the equation for $M(\eta,E)$ we consider the relation between $M(Z,E)$ and the distribution function $f(Z,p,t)$. Let $f(Z,p,t)$ be the probability that a single electron of zero energy released on the plane $Z=0$ at time $t=0$ will be found, at the time t, in the slab dZ with momentum in the range $d\mathbf{p}$. Let $\tau(\mathbf{p})$ be the mean free time between collisions for an electron of momentum $\mathbf{p}$. The total number of collisions which the electron suffers in $dZd\mathbf{p}$ is given by the time integral of the collision rate, which is the product of the collision frequency $\tau^{-1}(\mathbf{p})$ with the probability of the electron being at $(Z,\mathbf{p})$ at time t. Another integration, over all momenta having the energy E, then gives the collision density $M(Z,E)$

$$M(Z,E)=\int_0^{\infty} dt\int d\mathbf{p}\,\delta[E-E(\mathbf{p})]f(Z,\mathbf{p},t)/\tau(\mathbf{p}). \tag{3.4}$$

The time dependence of f is governed by a Boltzmann equation and initial conditions which we assume to be

$$\begin{aligned} &\frac{\partial f}{\partial t}+V_Z\frac{\partial f}{\partial Z}+\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}f+\frac{f}{\tau}\\ &\qquad=\int d\mathbf{p}'\, f(Z,\mathbf{p}',t)F(\mathbf{p}\leftarrow\mathbf{p}')/\tau(\mathbf{p}'), \end{aligned} \tag{3.5}$$

$$f(Z,\mathbf{p},0)=\delta(Z)\delta(E)/A(E),$$

$$A(E)=\int d\mathbf{p}\,\delta[E-E(\mathbf{p})]. \tag{3.6}$$

The initial conditions express the localization of the electron at energy $E=0$ and location $Z=0$. The function $F(\mathbf{p}\leftarrow\mathbf{p}')$ represents the probability that an electron whose momentum is $\mathbf{p}'$ before collision appears at momentum $\mathbf{p}$ after the collision.

The distribution f tends to zero at $t\rightarrow\infty$ because the electron is eventually absorbed. Therefore, if we integrate the Boltzmann equation over all times from $t=0$ to $t=\infty$, using the initial conditions, and then take the Laplace transform of the result, we obtain[11]

$$\begin{aligned} &\left[SV_Z+\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}+\frac{1}{\tau}\right]\tilde{g}(S,\mathbf{p})\\ &\qquad=\frac{\delta(E)}{A(E)}+\int\frac{\tilde{g}(S,\mathbf{p}')F(\mathbf{p}\leftarrow\mathbf{p}')}{\tau(\mathbf{p}')}d\mathbf{p}', \end{aligned} \tag{3.7}$$

where

$$\tilde{g}(S,\mathbf{p})=\int_0^{\infty} dt\int_0^{\infty} dZ\, e^{-SZ}f(Z,\mathbf{p},t). \tag{3.8}$$

Multiplication of (3.7) by $\exp(SE/e\mathcal{E})$ and use of the relations

$$\mathbf{V}=\nabla_{\mathbf{p}}E,\ \dot{\mathbf{p}}=e\mathcal{E}$$

yields

$$\begin{aligned} &[e\mathcal{E}\cdot\nabla_{\mathbf{p}}+1/\tau(\mathbf{p})]g(\eta,\mathbf{p})\\ &\qquad=\frac{\delta(E)}{A(E)}+\int d\mathbf{p}'\, g(\eta,\mathbf{p}')F(\mathbf{p}\leftarrow\mathbf{p}')/\tau(\mathbf{p}')\\ &\qquad\qquad\times\exp[-S(E'-E)/e\mathcal{E}], \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} g(\eta,\mathbf{p})&=\tilde{g}(S,\mathbf{p})\times\exp(SE/e\mathcal{E}),\\ E'&=E(\mathbf{p}'). \end{aligned} \tag{3.10}$$

Comparison of (3.2), (3.3), (3.4), (3.8), and (3.10)

[11] The term $V_zf(Z=0)$ which one might expect to be present actually vanishes because an electron which is able to return to the plane $Z=0$ cannot have a finite energy when it does so. Hence, its velocity at that plane must also be zero.

indicates that

$$M(\eta,E)=\int d\mathbf{p}\,\delta[E-E(\mathbf{p})]g(\eta,\mathbf{p})/\tau(\mathbf{p}). \quad (3.11)$$

Let us now assume that the probability $F(\mathbf{p}\leftarrow\mathbf{p}')$ depends only on the initial and final energy of the electron. Furthermore, we assume that it consists of two terms, one describing the elastic scattering by acoustic phonons and the other describing the emission of an optical phonon of energy E_R. Then

$$F(\mathbf{p}\leftarrow\mathbf{p}')=[\delta(E-E')\sigma_A(E')/\sigma_T(E') +\delta(E+E_R-E')\sigma_0(E')/\sigma_T(E')]/A(E), \quad (3.12)$$

where E and E' are the energies associated with $\mathbf{p}$ and $\mathbf{p}'$, and where σ_A, σ_0, and σ_T are the acoustic, optical, and total cross sections for electrons of energy E'. Inserting this into (3.9) and using (3.11) gives

$$[e\mathcal{E}\cdot\nabla_p+1/\tau(\mathbf{p})]g(\eta,\mathbf{p}) =A^{-1}(E)\{\delta(E)+[\sigma_A(E)/\sigma_T(E)]\times M(\eta,E) +\eta\times[\sigma_0(E+E_R)/\sigma_T(E+E_R)]\times M(\eta,E+E_R)\}.$$

The differential operator which acts on g can be removed by a series of integrations which are given in the Appendix. This gives $g(\eta,\mathbf{p})$ in terms of an integral over the expression on the right. Finally, the entire expression is multiplied by $\delta[E-E(\mathbf{p})]/\tau(\mathbf{p})$ and integrated to give, using (3.11),

$$M(\eta,E)=\int_0^\infty dE'\,T(E,E') \times\{\delta(E')+[\sigma_A(E')/\sigma_T(E')]M(\eta,E') +\eta[\sigma_0(E'+E_R)/\sigma_T(E'+E_R)]M(\eta,E'+E_R)\}. \quad (3.13)$$

The kernel $T(E,E')$ which is given in Appendix A, is a rather complicated expression which arises from carrying out the indicated operations but its meaning is simple. If one considers an electron with initial momentum $\mathbf{p}'$ and final momentum $\mathbf{p}$, then there is some trajectory in space which the electron would travel while undergoing this change of momentum. The probability that the electron can traverse the trajectory without an intervening collision and, having reached momentum $\mathbf{p}$, then collide for the first time since leaving $\mathbf{p}'$, depends on the mean free time $\tau(\mathbf{p})$ and electron momentum at each point of the trajectory. It can be calculated when the trajectory is known. The kernel $T(E,E')$ is just this probability, averaged over all initial momenta compatible with an initial energy E', averaged over final momenta compatible with a final energy E, and averaged over all allowed trajectories compatible with the initial and final momenta. The content of Eq. (3.13) is that the collision density at E is caused by electrons whose previous collision (or whose initial release into the system) left them with some other energy E'. Each collision leaving the electron at E' is weighted by the probability of the direct transport from energy E' to energy E. The number of collisions leaving the electron at E' is determined by the collision density at E', for acoustic phonon collision, and by the collision density at $E'+E_R$ for optical phonon emission.

Although (3.13) is a consequence of the original integro-differential Boltzmann equation, this new form is a significant advance over the original. The advance was made possible by the assumption that the scattering probability depends only on energy. Because of this single assumption, it has been possible to reduce a partial integro-differential equation in several variables to a single integral equation in one variable. Similar reductions are possible under less stringent requirements on the scattering cross section, but they lead to systems of coupled integral equations which are only slightly more tractable than the original.[12] The stringent restriction on scattering cross section results in an enormous simplification of the problem.

There is one last point to be made here. Having found that $M(Z,E)$ has the form (3.1), we may calculate $n(Z)$ and $\bar{\alpha}$ using (3.1) in (2.1) and (2.2) as

$$n(Z)=e\mathcal{E}\textstyle\sum_{n=0} M_n(Ze\mathcal{E}-nE_R)r(Ze\mathcal{E}-nE_R), \quad (3.14)$$

and

$$(\bar{\alpha})^{-1}=(1/e\mathcal{E})[\bar{E}+\bar{n}E_R], \quad (3.15)$$

where

$$\bar{E}=\int_0^\infty dE\,Er(E)\sum_{n=0}^\infty M_n(E), \quad (3.16)$$

$$\bar{n}=\int_0^\infty dE\,r(E)\sum_{n=0}^\infty nM_n(E). \quad (3.17)$$

Expression (3.15) may be interpreted as meaning that an electron which is absorbed at an average energy $\bar{E}$ after having emitted an average number $\bar{n}$ phonons of energy E_R will have drifted a distance $(\bar{\alpha})^{-1}$ down the field. This statement forms the starting point of Shockley's calculation. The average number $\bar{n}$ of phonons emitted is also given by

$$\bar{n}=\int_0^\infty dE\,M(\eta=1,E)[1-r(E)].$$

Inserting this into (3.15) gives

$$(\bar{\alpha})^{-1}=\frac{1}{e\mathcal{E}}\int_0^\infty dE\,M(\eta=1,E) \times\{Er(E)+E_R[1-r(E)]\}. \quad (3.18)$$

The first spherical harmonic equation which relates the current density $n_1(E)$ to the particle density $n_0(E)$ can, for the case of a spatially independent density and a constant mean free path, be integrated exactly so as to

[12] H. S. Wilf, J. Math. Phys. 1, 225 (1960).

give the total current in terms of an integral over the particle density. This integration, expressed in terms of collision density, and normalized to the absorption of a single particle, shows that the expression on the right of (3.18) is the total current. Wolff's calculation of α, as the ratio of absorption rate to current, is thus identical with (3.18). This underlines the point that conceptually Shockley's theory and Wolff's theory are the same, and that the differences in the two theories arise solely from their differing descriptions of the distribution function.

IV. THE ITERATIVE SOLUTION

We seek the solution of (3.13) as a power series in η of the form (3.2). This series may be inserted into (3.13) and the various powers of η separated, but the result is an integral equation for $M_0(E)$ and a set of integral equations for $M_n(E)$ in terms of $M_{n-1}(E)$. It is possible to bypass the situation of handling integral equations, if, as is commonly assumed,[3,4] the acoustic phonon scattering is negligible at energies above the optical phonon energy. Let us assume this to be the case. Then

$$\sigma_a(E)/\sigma_{\text{tot}}(E)=1, \quad E<E_R,$$
$$=0, \quad E>E_R.$$

Having made this assumption, we may insert (3.2) into (3.13) and separate powers of η to obtain

$$M_0(E)=T(E,0)+\int_0^{E_R} dE'\, T(E,E')M_0(E'), \tag{4.1a}$$

$$M_n(E)=\int_0^{E_R} dE'\, T(E,E')M_n(E')+\int_0^{\infty} dE'\, T(E,E') \times[1-r(E'+E_R)]M_{n-1}(E'+E_R), \tag{4.1b}$$

where

$$r(E)=[\sigma_{\text{tot}}(E)-\sigma_0(E)]/\sigma_T(E) =\sigma_i(E)/\sigma_{\text{tot}}(E). \tag{4.1c}$$

This is still a set of integral equations which must be solved sequentially. Note, however, that if there were no acoustic phonons, that is, if the first integral on the right of (4.1a, b) were not present, then this set would yield a simple recursion for determination of $M_n(E)$ from $M_{n-1}(E)$. It is highly unlikely that the presence of acoustic phonons at low energy can influence the shape of the distribution at those higher energies important to the ionization process. This suggests that we consider the set (4.1) at energies E well above the phonon energy. For these energies, we may regard the acoustic phonon scatter as occurring at zero energy, so that $T(E,E')$ may be replaced, for $E'<E_R$, by $T(E,0)$. If we make this approximation, the set (4.1) becomes

$$M_0(E)=T(E,0)[1+S_0], \tag{4.2a}$$

$$M_n(E)=T(E,0)S_n+\int_0^{\infty} dE'\, T(E,E') \times[1-r(E'+E_R)]M_{n-1}(E'+E_R), \tag{4.2b}$$

where

$$S_n=\int_0^{E_R} M_n(E')dE'. \tag{4.2c}$$

We may integrate Eqs. (4.2a, b) from zero to E_R. This will convert the $M_n(E)$ on the left to an S_n, which makes it possible to solve each equation for the S_n it contains.

$$S_0=(1-I)^{-1}, \tag{4.3a}$$

$$S_n=(1-I)^{-1}\int_0^{E_R} dE\int_0^{\infty} dE'\, T(E,E') \times[1-r(E'+E_R)]M_{n-1}(E'+E_R), \tag{4.3b}$$

$$I=\int_0^{E_R} dE\, T(E,0). \tag{4.3c}$$

Having found the S_n, we may substitute them back into (4.2) and write the resulting equations as

$$M_0(E)=(1-I)^{-1}T(E,0), \tag{4.4a}$$

$$M_n(E)=\int_0^{\infty} dE'\, \mathfrak{F}(E,E') \times[1-r(E'+E_R)]M_{n-1}(E'+E_R), \tag{4.4b}$$

$$\mathfrak{F}(E,E')=T(E,E')+M_0(E)\int_0^{E_R} T(Z,E')dZ. \tag{4.4c}$$

This result (4.4) also is simply interpreted. It states that the effect of including acoustic phonons which scatter at low energy is to augment the probability that an electron follows a trajectory directly from E' to E, $T(E,E')$, by another term which is the probability that an electron follows a trajectory from E' to all energies low enough for acoustic phonon scatter and then proceeds, like a newly released electron, to energy E.

These considerations of the effect of acoustic phonon scatter become superfluous if one is not interested in the spatial variation of $M(Z,E)$. In that case, one would be working with the spatial integral of $M(Z,E)$, i.e., with

$$M(\eta=1, E)=\textstyle\sum_{n=0}^{\infty} M_n(E)\equiv M(E). \tag{4.5}$$

Summing the Eqs. (4.1a) and (4.1b) gives

$$M(E)=T(E,0)+\int_0^{E_R} dE'\, T(E,E')M(E') +\int_0^{\infty} dE'\, T(E,E')[1-r(E'+E_R)]M(E'+E_R). \tag{4.6}$$

In the region $E>E_R$, this may be written approximately as

$$M(E)=T(E,0)\left[1+\int_0^{E_R} dE'\, M(E')\right]$$
$$+\int_0^{\infty} dE'\, T(E,E')[1-r(E'+E_R)]M(E'+E_R).$$

This equation in turn may be rewritten exactly as

$$M(E)=f(E)\left[1-\int_0^{E_R} f(E')dE'\right]^{-1}, \qquad (4.7)$$

where $f(E)$ satisfies

$$f(E)=T(E,0)+\int_0^{\infty} dE'\, T(E,E')$$
$$\times[1-r(E'+E_R)]f(E'+E_R). \qquad (4.8)$$

The equation for $f(E)$ is exactly what one would have obtained by ignoring the scattering caused by acoustic phonons. Equation (4.7) says that $M(E)$ has the same shape as $f(E)$ and differs only in normalization. The normalization usually can be ignored or can be adjusted by some condition on the number of absorbed particles, and is, therefore, unimportant. The normalization factor of (4.7) may be interpreted in the following way: Suppose each collision with an acoustic phonon to result in the absorption of the electron. Then (4.8) is the equation for the exact collision density. Using this exact density, the number of electrons absorbed in acoustic phonon collisions will be

$$N=\int_0^{E_R} f(E)dE.$$

Thus, for each electron released at zero energy, there are N absorbed at an energy close to zero. The net source of electrons for flow to higher energies is then 1-N. Dividing by 1-N would then give a distribution containing a single electron. This division by 1-N is indicated by Eq. (4.7).

Superfluous though these considerations may be when one is not interested in the spatial dependence of $M(Z,E)$, there is one matter about which one must be especially careful, both in Eq. (4.4) and in Eq. (4.8). This is the normalization of the kernel $T(E,E')$. We have asserted (but not yet proved) that T is the probability that an electron released at energy E' will suffer its next collision at some energy E. Since the electron will eventually collide at some, energy, the assertion implies that the kernel will satisfy the normalization

$$\int_0^{\infty} dE\, T(E,E')=1. \qquad (4.9)$$

That it does so will be proved in Appendix B. It is this normalization, in fact, which indicates that T is the probability we have claimed it to be. This normalization must be maintained scrupulously in any approximation scheme using integral techniques, for fictitious absorption or sources appear once this condition is relaxed, and the effect of these fictitious sources or absorptions can, in the case of small true absorption, completely alter the solutions one obtains.

V. RESULTS

Equations (4.4) make possible the calculation of $M_n(E)$ once the kernel $T(E,E')$ has been calculated. This kernel contains all the effects of the band structure on the motion of the electrons, and may, in principle, be calculated, given $E(p)$ and $\tau(p)$ by means of Eq. (A.6) of the Appendix. Needless to say, this calculation in principle becomes nearly impossible in practice except for the simplest of band structures, the quasi-free electrons. We can, assuming constant mean free path λ,[13] evaluate $T(E,E')$ in this case, to the extent of expressing T as a finite integral over a single parameter. $T(E,0)$ can be evaluated exactly for this case and is

$$T(E,0)=Q^{-1}\exp(-E/Q). \qquad (5.1)$$

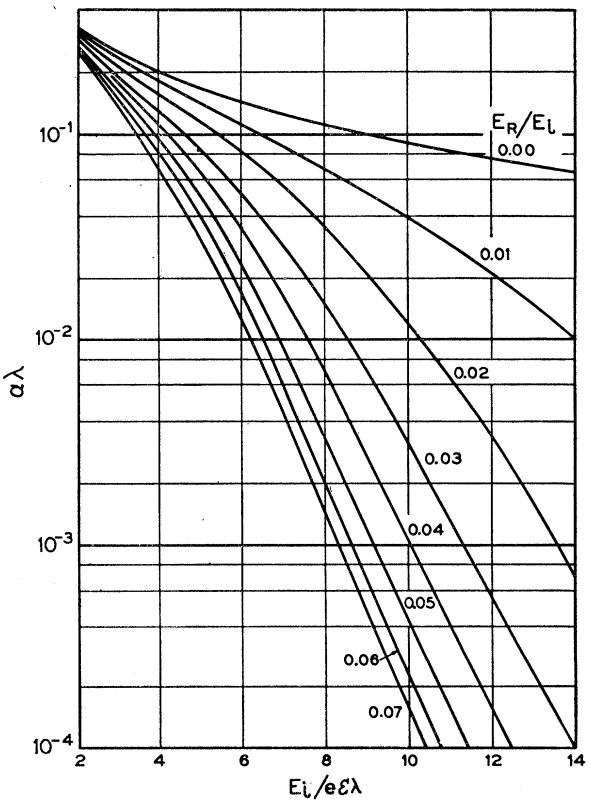

FIG. 1. Dependence of $\alpha\lambda$ on $E_i/e\mathcal{E}\lambda$ for various values of E_R/E_i.

[13] W. Shockley, Bell System Tech. J. **30**, 990 (1951) and F. Seitz, Phys. Rev. **73**, 550 (1948).

The $M_0(E)$ corresponding to this may be calculated from (4.4) as[14]

$$M_0(E)=Q^{-1}\exp[-(E-E_R)/Q]. \quad (5.2)$$

We can also obtain useful closed form expressions which approximate the true $T(E,E')$ for this situation. These expressions are useful in programming these equations for the high-speed digital computer. The evaluation and approximation are matters of detail which may be found in Appendix C.

We have programmed Eqs. (4.4), (4.5), (3.18), and (3.14) for the IBM 7090 using the approximate $T(E,E')$ described in Appendix C. The machine was instructed to iterate (4.4b) until the total number of electrons absorbed had converged, or for a predetermined number of iterations, whichever occurred first. We have obtained, in this way the functions $n(Z)$, $M(E)$ and α for a wide range of the relevant variables. The results of the calculations of α appear as Fig. 1, whose meaning and use we should like to explain.

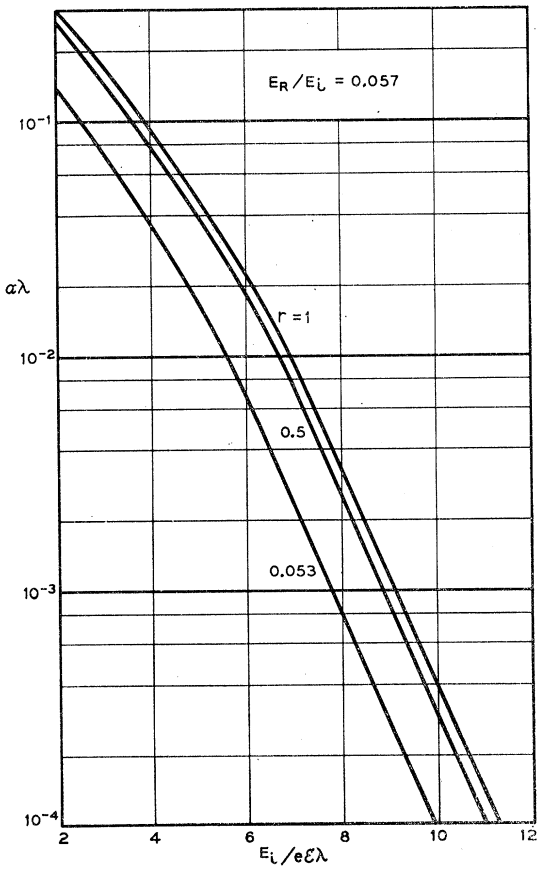

FIG. 2. Dependence of $\alpha\lambda$ on r for $E_R/E_i=0.057$.

[14] This expression (5.2) is exactly the $M_0(E)$ corresponding to no acoustic phonons and hence to an infinite mean free path for electrons whose energy is less than E_R. One may ask why it was necessary to bring in the acoustic phonons at all if the effect of their presence is the same as the effect of their absence. The answer simply, is that by including them, the task of evaluating $T(E,E')$ is made significantly easier.

TABLE I. $\alpha\lambda$ vs r for $E_R/E_i=0.057$, $E_i/Q=6.6$.

r	$\alpha\lambda$
1.0	1.4×10^{-2}
0.5	1.15×10^{-2}
0.25	0.9×10^{-2}
0.05	0.36×10^{-2}

Figure 1 is essentially a universal plot of logα vs reciprocal electric field. Choice of the ionization threshold E_i and photon mean free path λ determines the horizontal entry corresponding to a given electric field $e\mathcal{E}$. The ratio of phonon energy E_R to ionization energy E_i selects a particular curve, and the product $\alpha\lambda$ which the material should display in the given field may be read on the vertical axis, whereupon α is determined. These curves correspond to a situation in which the ionization cross section rises abruptly at energy E_i to a value equal to the cross section for phonon emission. The abrupt rise has been chosen to approximate the behavior of the ionization cross section near threshold, where the actual cross section should be increasing about quadratically with increasing energy.[3,4] Our choice of a high-energy limit equal to the cross section for phonon emission is quite arbitrary. However, other calculations in which the high-energy limit was varied (these are summarized in Table I and Fig. 2) indicate that alpha depends on the value of the cross section above threshold so weakly that the curves we have just described will be useful for an ionization cross section equal to or arbitrarily greater than one-fourth the cross section for phonon emission.

The parallel straight lines in the lower portion of the figure correspond in slope to the predictions of Shockley's theory although they give a much larger value of alpha. This occurs because Shockley calculates as though each electron which collides at intermediate energy returns to zero energy before attempting another flight to high energy. He, therefore, badly underestimates the number of high-energy electrons by discarding all those which scatter and continue upwards in energy.[15] The concave downward curves in the upper part of the figure have a form suggesting Wolff's theory but again, the fit is not precise.

There are several conclusions to be drawn from the information presented in Table I. This table shows $\alpha\lambda$ vs r, where r is the ratio of the ionization cross section to the total cross section above threshold. The important result is the relative insensitiveness of $\alpha\lambda$ to r. The first conclusion is that it will be very difficult to obtain r from charge multiplication measurements alone. Such quantities as "numbers of phonons per ionization" which have been taken as a parameter in past studies of charge multiplication and which have been adjusted to fit the charge multiplication data may have to be re-examined.

[15] This same conclusion has been reached by Moll and Meyer [J. J. Moll and N. I. Meyer, Solid-State Electron. 3, 155 (1961)].

The second conclusion concerns the nature of the process by which the electrons reach high energy. If the electrons which are at high energy are those few which have had a fortuitously long flight since their last collision, then $M(E)$, the rate at which electrons collide at energy E, depends on the chances of avoiding all collisions, and is independent of whether a given collision produces a phonon or ionization. Thus, $M(E)$ will be independent of r and the ionization rate, essentially r times the collision rate, will be proportional to r. Alpha will then be directly proportional to r, as Shockley's theory predicts.[4]

If, on the other hand, electrons at high energy have suffered many collisions, between each of which they gained a small net amount of energy (i.e., have gradually diffused upwards in energy) then $M(E)$ at high energy will be inversely proportional to r at high energy for the following reasons: The rate at which electrons are fed to high energy depends solely on the difficulty of this diffusion. The rate at which they are removed from high energy is the ionization rate. In the steady state, these rates will be equal. Hence an increase of r must be compensated by a decrease of $M(E)$ to keep the ionization rate equal to the supply rate, which is independent of r. Said differently, an increase of the ionization probability depopulates the high-energy region of electrons. Under these circumstances, alpha, essentially an ionization rate, tends to be r independent also. This feature appears in Wolff's work.[3]

Our finding that $\alpha\lambda$ is fairly insensitive to r indicates then that a diffusion type mechanism is operative in controlling the flow of electrons to higher energy. The collision density $M(E)$ corresponding to the calculations reported in Table I appear in Fig. 3, where the depression of $M(E)$ at high energy relative to low energy may be seen to increase with r.

In summary, the situation describing the distribution of electrons at fields of interest in charge multiplication studies is this: They are in a distribution which is too sharply peaked for the P_0 and P_1 approximation of Wolff to describe, yet their transport to high energies is essentially diffusive rather than ballistic as Shockley had assumed. The stress laid by Wolff on the effect of the ionization cross section on the distribution function, and the stress laid by Shockley on the especial importance of electrons traveling along the field direction, are complementary; neither alone is sufficient to determine the distribution of electrons.

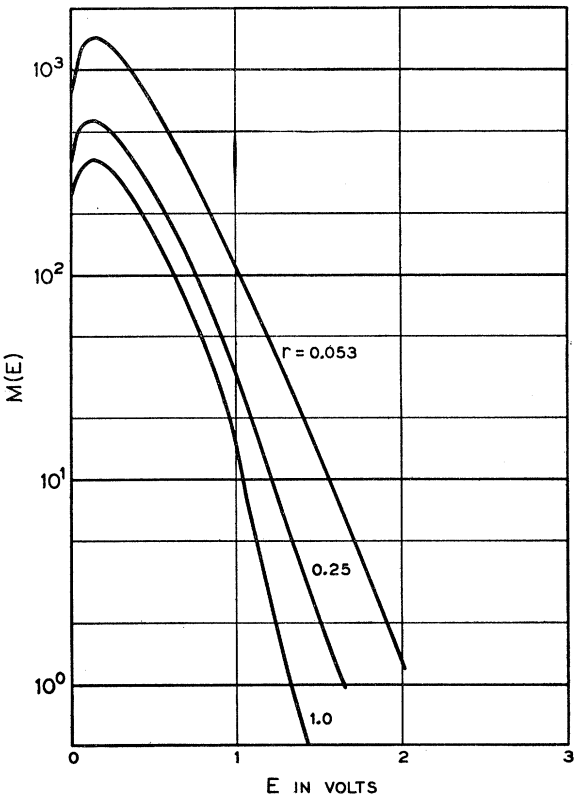

FIG. 3. Energy distributions $M(E)$ calculated for the parameters indicated in Table I.

ACKNOWLEDGMENTS

The author wishes to record his indebtedness to Dr. E. I. Blount for the short proof in Appendix B which replaces an overly long one which had been originally devised. Thanks are due too to Dr. P. A. Wolff who pointed out an important simplification which eliminated some superfluous work in passage from (4.2) to (4.4). Dr. P. A. Wolff, Dr. E. O. Kane, Dr. R. A. Logan, Dr. C. A. Lee, and Dr. A. G. Chynoweth are to be thanked for their critical reading of the manuscript. Special thanks are due Dr. Chynoweth and Dr. Wolff for guidance and stimulation through the entire work.

APPENDIX A. THE INTEGRATION OF THE DIFFERENTIAL EQUATION

The equation following (3.12) has the form

$$[e\mathcal{E}\cdot\nabla_p+1/\tau(p)]g(\mathbf{p})=Q(E)/A(E). \quad \text{(A1)}$$

Define an attenuation coefficient $\chi(\mathbf{p})$ by

$$\chi(\mathbf{p})=\chi(p_x,p_y,p_z)=\frac{1}{e\mathcal{E}}\int_0^{p_z} dW\,\tau^{-1}(p_x,p_y,W), \quad \text{(A2)}$$

so that

$$e\mathcal{E}\cdot\nabla_p\chi(\mathbf{p})=e\mathcal{E}\partial\chi(\mathbf{p})/\partial p_z=1/\tau(\mathbf{p}).$$

Then the substitution into (A1) of

$$g(\mathbf{p})=f(\mathbf{p})\exp[-\chi(\mathbf{p})] \quad \text{(A3)}$$

gives

$$\partial f(\mathbf{p})/\partial p_z=Q(E)[e\mathcal{E}A(E)]^{-1}\exp\chi(\mathbf{p}),$$

where $E=E(p)$. Integrating this from $-\infty$ to p_z and ex-

pressing the result in terms of g gives

$$g(\mathbf{p})=\frac{1}{e\mathcal{E}}\int_{-\infty}^{p_z}dp_z'\,\frac{Q(E')}{A(E')}\exp[\chi(\mathbf{p}')-\chi(\mathbf{p})], \quad \text{(A4)}$$

$$\mathbf{p}=p_x,p_y,p_z, \quad \text{(A5a)}$$

$$\mathbf{p}'=p_x,p_y,p_z', \quad \text{(A5b)}$$

$$E'=E(\mathbf{p}'). \quad \text{(A5c)}$$

We may multiply (A4) by $\delta[E-E(p)]/\tau(p)$ and integrate over p. Using the definition (3.11) then gives

$$M(\eta,E)=\frac{1}{e\mathcal{E}}\int d\mathbf{p}\,\frac{\delta[E-E(p)]}{\tau(p)}\times\int_{-\infty}^{p_z}dp_z'\,\frac{Q(E')}{A(E')}\exp[\chi(\mathbf{p}')-\chi(\mathbf{p})].$$

Finally, we elevate E' to the status of an independent variable by inserting $\delta[E'-E(p')]$ into the integrand and integrating over E'. This yields

$$M(\eta,E)=\int_0^\infty T(E,E')Q(E')dE',$$

with

$$T(E,E')=\frac{1}{e\mathcal{E}}\int d\mathbf{p}\,\delta[E-E(\mathbf{p})]/\tau(\mathbf{p})\times\int_{-\infty}^{p_z}dp_z'\,\delta[E'-E(\mathbf{p}')]\times A^{-1}(E')\times\exp\left[-\frac{1}{e\mathcal{E}}\int_{p_z'}^{p_z}dW\,\tau^{-1}(p_x,p_y,W)\right]. \quad \text{(A6)}$$

APPENDIX B. PROOF OF THE NORMALIZATION CONDITION

We asserted that T satisfied the normalization condition

$$\int_0^\infty dE\,T(E,E')=1. \quad \text{(B1)}$$

The proof of this follows directly from (A6) as follows. The E integration removes a delta function from the $\mathbf{p}$ integrand. We can then write $(e\mathcal{E})^{-1}\tau^{-1}(\mathbf{p})\times\exp[\cdots]$ as the derivative of the exponential with respect to p_z so that

$$\int_0^\infty dE\,T(E,E')=-\int dp_x dp_y dp_z\int_{-\infty}^{p_z}dp_z'\,\delta[E'-E(\mathbf{p}')]A^{-1}(E')\times\frac{d}{dp_z}\left\{\exp\left[-\frac{1}{e\mathcal{E}}\int_{p_z'}^{p_z}dW\,\tau^{-1}(p_x,p_y,W)\right]\right\}. \quad \text{(B2)}$$

The order of the p_z integration and p_z' is to be interchanged so that p_z' ranges over all values, and p_z is confined to values larger than p_z'. Using (A5b) we have

$$\int_0^\infty dE\,T(E,E')=\frac{-1}{A(E')}\int d\mathbf{p}'\,\delta[E'-E(\mathbf{p}')]\times\int_{p_z'}^\infty dp_z\frac{d}{dp_z}\exp\left(-\frac{1}{e\mathcal{E}}\int_{p_z'}^{p_z}\frac{dw}{\tau}\right)=A^{-1}(E')\int d\mathbf{p}'\,\delta[E'-E(\mathbf{p}')]=1.$$

APPENDIX C. EVALUATION OF $T(E,E')$

We may evaluate $T(E,E')$ for the case of constant mean free path and quasi-free electrons, by which is meant

$$E(p)=(p-p_0)^2/2m, \quad \text{(C1a)}$$

$$\tau=\lambda/v=m\lambda/|\mathbf{p}-\mathbf{p}_0|. \quad \text{(C1b)}$$

One introduces variables R, C, θ, and φ by means of $p_z-p_z'=R$, $p_x-p_{0x}=C\sin\theta\cos\varphi$, $p_y-p_{0y}=C\sin\theta\cos\varphi$, $p_z-p_{0z}=C\cos\theta$, $dp_z'=-dR$, $d\mathbf{p}=C^2dCd\varphi d(\cos\theta)$. The φ integration in (A6) may be performed and the delta functions may be used to eliminate the C and $\cos\theta$ integrations with the result

$$T(E,E')=[(4\pi m(mE/2)^{1/2}/e\mathcal{E}\lambda)A(E,E')]/A(E'), \quad \text{(C2)}$$

$$A(E,E')=\int_{|C-C'|}^{C+C'}\frac{dR}{R}\exp\left\{-\frac{1}{e\mathcal{E}\lambda m}\times\int_{\mu C-R}^{\mu C}dW\,[C^2(1-\mu^2)+W^2]^{1/2}\right\}, \quad \text{(C3)}$$

$$C=(2mE)^{1/2}, \quad \text{(C4a)}$$

$$C'=(2mE')^{1/2}, \quad \text{(C4b)}$$

$$\mu=(R^2+C^2-C'^2)/2RC. \quad \text{(C4c)}$$

The integral in the exponent may be performed and one then may use (C4c) to eliminate μ. The exponential then takes the form

$$\exp\{-(2e\mathcal{E}\lambda m)^{-1}f(C,C',R)\} \quad \text{(C5a)}$$

$$f=\left(\frac{C+C'}{2}\right)\left[R+\frac{(C-C')^2}{R}\right]+\left[\frac{C^2+C'^2}{2}-\left(\frac{C^2-C'^2}{2R}\right)^2-\frac{R^2}{4}\right]\times\ln\left(\frac{C+C'+R}{C+C'-R}\right). \quad \text{(C5b)}$$

Then using (C1a) in (3.6)

$$A(E)=8\pi m(mE/2)^{1/2}, \tag{C6}$$

so that

$$T(E,E')=(2\mathcal{E}e\lambda)^{-1}(E/E')^{1/2}A(E,E'). \tag{C7}$$

We need $T(E,E')$ in the limit $E'\to 0$. In this limit, $C'\to 0$, $R\to C$ so that $f\to C^2/2$. Then in this limit,

$$A(E,E'\to 0)=\int_{C-C'}^{C+C'}\frac{dR}{R}e^{-E/e\mathcal{E}\lambda}$$
$$=2(E'/E)^{1/2}e^{-E/e\mathcal{E}\lambda},$$

so that

$$T(E,E'\to 0)=(e\mathcal{E}\lambda)^{-1}e^{-E/e\mathcal{E}\lambda}. \tag{C8}$$

It is possible to approximate the expression for $T(E,E')$ in closed form. The approximation to be described is motivated by Shockley's observation that an electron which is scattered so as to be headed in the direction of the field will be much more likely to reach a higher energy than is an electron which is scattered so as to be headed against the field. This latter electron must be slowed, turned around, and reaccelerated before it reaches higher energy. It traverses a longer path and is more likely to suffer a collision in route.

The function $f(C,C',R)$ is proportional to the path length along a trajectory which carries the electron from $|\mathbf{p}'|=C'$ to $|\mathbf{p}|=C$. The parameter R labels the various trajectories which are consistent with this transport. It turns out that the shortest trajectories are labeled by the smallest value of R. Since the shortest trajectories are the one responsible for most of the transport, an error in the length of the longer trajectories will have little effect on the value of the kernel. This suggests the utility of expanding $f(C,C',R)$ as a Taylor series about the smallest value of R contributing to the integral. This we may do, and, retaining only linear terms, we obtain

$$f(C,C',R)\cong|C^2-C'^2|-2CC'\left|\ln\frac{C}{C'}\right|+\left(\frac{2CC'}{C-C'}\ln\frac{C}{C'}\right)R.$$

This approximation allows the integral in (C3) to be performed with the result

$$A(E,E')\cong\exp\left(-\left|\frac{E-E'}{Q}\right|+\frac{(EE')^{1/2}}{Q}\left|\ln\frac{E}{E'}\right|\right)$$
$$\times\left\{E_1\left(\frac{(EE')^{1/2}}{Q}\left|\ln\frac{E}{E'}\right|\right)\right.$$
$$\left.-E_1\left(\frac{(EE')^{1/2}}{Q}\times\left|\ln\frac{E}{E'}\right|\times\left|\frac{E^{1/2}+(E')^{1/2}}{E^{1/2}-(E')^{1/2}}\right|\right)\right\}, \tag{C9}$$

where E_1 is the tabulated exponential integral

$$E_1(x)=\int_x^\infty e^{-t}dt/t.$$

This result, though unappealing, is at least in closed form. A more careful study of $f(C,C',R)$ reveals that our approximation overestimates the path lengths associated with large R, and, hence, underestimates $A(E,E')$. The effect is small, but it means that the approximate $T(E,E')$ will no longer satisfy the normalization (B1). Therefore, it is necessary to divide by the normalization, which means that we calculate

$$T(E,E')=T_a(E,E')\Big/\int_0^\infty dE\,T_a(E,E'), \tag{C10}$$

where T_α is the approximate T. The effect of this normalization is to shorten the trajectories which we have overestimated, and also to increase the number of electrons transported along the shorter trajectories. This latter effect is equivalent to the situation in which the scattering were no longer spherically symmetric but exhibited, instead, a slight bias for scatter along the field. The error introduced in this way is expected to be small, however.