Electrodynamic Properties of a Quantum Plasma in a Uniform Magnetic Field

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A quantum-mechanical derivation is given of the conductivity tensor of a free electron gas in the presence of a dc magnetic field and of an arbitrary electromagnetic disturbance. The results obtained are applied to study some simple problems on the dispersion and attenuation of sound waves in metals in the presence of a magnetic field. In particular, it is shown that both the velocity and attenuation of longitudinal acoustic waves travelling parallel to the magnetic field exhibit oscillations as a function of the magnetic field under suitable conditions.

I. INTRODUCTION

A GREAT deal of attention has been paid recently to the study of the response of an electron gas to an electromagnetic disturbance that varies both in space and in time. This interest has been motivated by experimental work on the absorption of electromagnetic radiation and of sound waves by metals.[1] The response of an electron gas to an electromagnetic field that varies in space and time as $\exp(i\omega t - i\mathbf{q}\cdot\mathbf{r})$, where ω is the angular frequency and $\mathbf{q}$ the wave vector, can be conveniently expressed in terms of the longitudinal and transverse dielectric constants as done, for example, by Lindhard.[2] An equivalent formulation consists in expressing the results in terms of the electrical conductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$ appropriate to the wave vector $\mathbf{q}$ and the frequency ω. An example of a special case of particular interest is the discussion of Reuter and Sondheimer[3] in connection with the study of the anomalous skin effect in metals. Pippard[4] obtained the same results as Lindhard in his work on ultrasonic attenuation in metals by using a kinetic method. General formulations for the calculation of the conductivity have been given by Lax[5] and by Kubo.[6]

The present paper is concerned with the electrical conductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$ of an electron gas in the presence of a uniform magnetic field of induction $\mathbf{B}_0$. This problem has been discussed recently in the situation in which $\mathbf{q}=0$ by several authors.[7,8] In the situation in which $q\neq 0$ the conductivity tensor has been obtained[9–13] by solving the Boltzmann transport equation under conditions applicable to several cases of interest. Quantum-mechanical discussions of this problem have been given by Mattis and Dresselhaus,[14] by Zyryanov,[15] and by Zyryanov and Kalashnikov.[16]

The purpose of this paper is to discuss the response of an electron gas in a magnetic field $\mathbf{B}_0$ to the most general electromagnetic disturbance. The procedure used is that of the self-consistent-field method as described, for example, by Ehrenreich and Cohen.[17] In Sec. II we show how the calculation is carried out and we exhibit explicitly the matrix elements that are of interest. Some of their mathematical properties, including those that insure the gauge invariance of the theory, are given in the Appendix. Two geometries of particular interest are discussed, namely, those in which $\mathbf{q}$ is either perpendicular to or parallel to $\mathbf{B}_0$. Finally, we show that in the semiclassical limit we obtain the same results as Cohen *et al.*[12]

In Sec. III, after making some general remarks about the attenuation and velocity of sound in metals in the presence of a magnetic field, we discuss the application of the results of the present work to longitudinal acoustic waves propagating in the direction of $\mathbf{B}_0$. We find that both the velocity and absorption coefficient of acoustic waves exhibit an oscillatory variation as a function of the magnetic field. These oscillations have a period proportional to B_0^{-1} and have the same physical origin as the oscillations in the magnetic susceptibility of metals (de-Haas–van-Alphen effect). The effects described above are observable in rather pure metallic samples and at sufficiently low temperatures. A more precise description of the conditions for observability will be given at the appropriate place.

We now state explicitly the assumptions and approximations made in this work. A metal is assumed to consist of a free electron gas in the presence of a uniform background of positive ions such that the system is electrically neutral. No "real metal" effects, such as those arising from the actual energy-band structure and from a finite collision time τ for the electrons, are considered (here we assume $\tau=\infty$). The conductivity tensor is obtained by solving the equation of motion of the density matrix to first order in the electromagnetic

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field intensities, and we assume that the response to a source field that varies as $\exp(i\omega t - i\mathbf{q}\cdot\mathbf{r})$ varies in exactly the same fashion (this approximation is often called the random phase approximation).

The following is a list of some of the symbols used in this work and their meaning:

m = mass of the electron.
$\mathbf{p}$ = canonical momentum of a particle.
$\omega_0 = |e|B_0/mc$ = cyclotron frequency of an electron in a magnetic field of induction B_0.
$\omega_p = (4\pi Ne^2/\Omega m)^{1/2}$ = plasma frequency of the electrons.
$\Omega_p = (4\pi Nze^2/\Omega M)^{1/2}$ = plasma frequency of the positive ions of the metal.
M = mass of the atom in the metal.
z = number of conduction electrons per atom.
N = number of electrons in the sample under study.
Ω = volume of the sample.
$\zeta_0 = (\hbar^2/2m)(3\pi^2 N/\Omega)^{2/3}$ = Fermi energy at 0°K of the electron gas in the absence of the magnetic field.
E_0 = Fermi energy of the electron gas in the field B_0.

II. DERIVATION OF THE CONDUCTIVITY TENSOR

A. General Theory

Let us consider an electron gas consisting of N electrons confined within a cubic box of side L_0 and volume $\Omega = L_0^3$ in the presence of a magnetic field of induction $\mathbf{B}_0$, and of an electromagnetic disturbance that varies as $\exp(i\omega t - i\mathbf{q}\cdot\mathbf{r})$. We define $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ as the scalar and vector potentials for the self-consistent field produced by the disturbance and $\mathbf{A}_0 = (0, B_0 x, 0)$ as the vector potential of the dc magnetic field $\mathbf{B}_0$. We have taken a Cartesian coordinate system with the z axis parallel to $\mathbf{B}_0$. Without loss of generality the x axis can be chosen so that the wave vector $\mathbf{q}$ lies in the y-z plane.

The Hamiltonian for a single electron in the presence of the self-consistent electromagnetic field and of $\mathbf{B}_0$ is

$$H = (1/2m)[\mathbf{p} - (e/c)\mathbf{A}_0 - (e/c)\mathbf{A}]^2 + e\phi. \tag{1}$$

To first order in $\mathbf{A}$

$$H = H_0 + H_1, \tag{2}$$

where

$$H_0 = (1/2m)[\mathbf{p} - (e/c)\mathbf{A}_0]^2, \tag{3}$$

and

$$H_1 = -(e/2c)(\mathbf{v}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{v}) + e\phi. \tag{4}$$

The operators H_0 and

$$\mathbf{v} = (1/m)[\mathbf{p} - (e/c)\mathbf{A}_0]$$

are the Hamiltonian and velocity operators for an electron in the field $\mathbf{B}_0$. The stationary states of H_0 are characterized by the wave functions[18]

$$|\nu\rangle = |nk_yk_z\rangle = L_0^{-1}\exp(ik_y y + ik_z z) \times u_n(x + \hbar k_y/m\omega_0), \tag{5}$$

and by the eigenvalues

$$E_\nu = E_{nk_z} = \hbar\omega_0(n + \tfrac{1}{2}) + \hbar^2 k_z^2/2m. \tag{6}$$

The allowed values of the wave numbers k_y and k_z are obtained by imposing periodic boundary conditions on $|\nu\rangle$ with period L_0. The quantum number n can take any non-negative integral value and $u_n(x)$ is a normalized wave function for a simple harmonic oscillator[19] corresponding to a particle of mass m and characteristic frequency ω_0.

The electron current density induced by the self-consistent field is obtained by taking the trace of the product of the current density operator and the single-particle density matrix.[20] The density matrix f is calculated to first order in the fields ϕ and $\mathbf{A}$ as follows. The operator f must satisfy the equation of motion

$$i\hbar\partial f/\partial t = [H, f]. \tag{7}$$

In the absence of the perturbation, H_1, f reduces to its equilibrium value

$$f_0(H_0) = \{\exp[(H_0 - E_0)/kT] + 1\}^{-1},$$

which is diagonal in the Landau representation defined by the functions (5) and, therefore, satisfies the condition

$$f_0(H_0)|\nu\rangle = f_0(E_\nu)|\nu\rangle.$$

We now set $f = f_0 + f_1$, where f_1 is a small change in f from its equilibrium value f_0 caused by the self-consistent field. Furthermore, we assume that f_1 varies in time as $\exp(i\omega t)$ so that the equation of motion reads

$$-\hbar\omega f_1 = [H_0, f_1] + [H_1, f_0]. \tag{8}$$

In Eq. (8) terms of higher order than the first in ϕ and $\mathbf{A}$ have been neglected. By taking off-diagonal matrix elements of Eq. (8) in the Landau representation we find

$$\langle\nu|f_1|\nu'\rangle = [f_0(E_{\nu'}) - f_0(E_\nu)] \times (E_{\nu'} - E_\nu - \hbar\omega)^{-1}\langle\nu|H_1|\nu'\rangle. \tag{9}$$

The induced current density at position $\mathbf{x}$ and time t is

$$\mathbf{j}^{(1)}(\mathbf{x},t) = \mathrm{Tr}\{f j_{\mathrm{op}}^{(1)}\} = \sum_\nu \left\langle \nu \left| \frac{1}{2} e\left(\mathbf{v} - \frac{e}{mc}\mathbf{A}\right)\delta(\mathbf{x}-\mathbf{r})f + \mathrm{H.c.}\right| \nu \right\rangle, \tag{10}$$

where H.c. designates the Hermitian conjugate of the preceding operator, and $j_{\mathrm{op}}^{(1)}$ is defined implicitly by the second equality in Eq. (10). The induced charge density $\rho^{(1)}(\mathbf{x},t)$ is obtained from a similar relation. The Fourier components $\mathbf{j}^{(1)}(\mathbf{q},\omega)$ and $\rho^{(1)}(\mathbf{q},\omega)$ of the induced current and charge densities are given by the relations[21]

[18] L. D. Landau, Z. Physik **64**, 629 (1930). The representation defined by the wave functions (5) shall henceforth be called the Landau representation.

[19] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw Hill Book Company, Inc., New York, 1955), 2nd ed., pp. 64–65.

[20] See, C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1958), pp. 107–113.

[21] In this discussion, since we assumed $\tau = \infty$, $\mathbf{K}$ and $\mathbf{K}^*$ are each other's complex conjugates. However, if we wish to include relaxation effects in our treatment we can do so by adding a phenomenological term $i\hbar f_1/\tau$ to the left-hand side of Eq. (7). All equations are formally the same as those presented here except that ω must be replaced by $\omega - i/\tau$. In this case we must substitute $\mathbf{K}'$ for $\mathbf{K}^*$ in Eq. (12) where $\mathbf{K}'(\mathbf{q},\omega) = \mathbf{K}^*(\mathbf{q},\omega^*)$.

$$\mathbf{j}^{(1)}(\mathbf{q},\omega)=(\omega_p^2/4\pi c)\times[-\mathbf{A}(\mathbf{q},\omega)-\mathbf{I}\cdot\mathbf{A}(\mathbf{q},\omega)+\mathbf{K}\phi(\mathbf{q},\omega)], \quad (11)$$

and

$$\rho^{(1)}(\mathbf{q},\omega)=(\omega_p^2/4\pi c^2)[-\mathbf{K}^*\cdot\mathbf{A}(\mathbf{q},\omega)+L\phi(\mathbf{q},\omega)]. \quad (12)$$

In these equations $\mathbf{A}(\mathbf{q},\omega)$ and $\phi(\mathbf{q},\omega)$ are the Fourier components of the electromagnetic potentials and the symbols $\mathbf{I}$, $\mathbf{K}$, and L stand for (the tensor $\mathbf{I}$ is expressed in dyadic notation)

$$\mathbf{I}(\mathbf{q},\omega)=(m/N)\sum_{\nu\nu'}[f_0(E_{\nu'})-f_0(E_\nu)](E_{\nu'}-E_\nu-\hbar\omega)^{-1}\times\langle\nu'|\mathbf{V}(\mathbf{q})|\nu\rangle\langle\nu'|\mathbf{V}(\mathbf{q})|\nu\rangle^*, \quad (13)$$

$$\mathbf{K}(\mathbf{q},\omega)=(mc/N)\sum_{\nu\nu'}[f_0(E_{\nu'})-f_0(E_\nu)](E_{\nu'}-E_\nu-\hbar\omega)^{-1}\times\langle\nu'|\mathbf{V}(\mathbf{q})|\nu\rangle\langle\nu'|\exp(i\mathbf{q}\cdot\mathbf{r})|\nu\rangle^*, \quad (14)$$

and

$$L(\mathbf{q},\omega)=(mc^2/N)\sum_{\nu\nu'}[f_0(E_{\nu'})-f_0(E_\nu)]\times(E_{\nu'}-E_\nu-\hbar\omega)^{-1}|\langle\nu'|\exp(i\mathbf{q}\cdot\mathbf{r})|\nu\rangle|^2. \quad (15)$$

The operator $\mathbf{V}(\mathbf{q})$ which appears in Eqs. (13) and (14) is defined by

$$\mathbf{V}(\mathbf{q})=\tfrac{1}{2}\exp(i\mathbf{q}\cdot\mathbf{r})\mathbf{v}+\tfrac{1}{2}\mathbf{v}\exp(i\mathbf{q}\cdot\mathbf{r}). \quad (16)$$

The matrix elements of $\exp(i\mathbf{q}\cdot\mathbf{r})$ and of $\mathbf{V}(\mathbf{q})$ are given by the following equations

$$\langle n'k_y'k_z'|\exp(i\mathbf{q}\cdot\mathbf{r})|nk_yk_z\rangle=\delta(k_y',k_y+q_y)\delta(k_z',k_z+q_z)f_{n'n}(q_y), \quad (17)$$

$$\langle n'k_y'k_z'|V_x(\mathbf{q})|nk_yk_z\rangle=i(\hbar\omega_0/2m)^{\frac{1}{2}}\delta(k_y',k_y+q_y)\times\delta(k_z',k_z+q_z)X_{n'n}^{(-)}(q_y), \quad (18)$$

$$\langle n'k_y'k_z'|V_y(\mathbf{q})|nk_yk_z\rangle=\delta(k_y',k_y+q_y)\delta(k_z',k_z+q_z)\times[(\hbar q_y/2m)f_{n'n}(q_y)+(\hbar\omega_0/2m)^{\frac{1}{2}}X_{n'n}^{(+)}(q_y)], \quad (19)$$

and

$$\langle n'k_y'k_z'|V_z(\mathbf{q})|nk_yk_z\rangle=\delta(k_y',k_y+q_y)\times\delta(k_z',k_z+q_z)(\hbar/m)(k_z+\tfrac{1}{2}q_z)f_{n'n}(q_y). \quad (20)$$

The symbol $\delta(k',k)$ is zero unless $k'=k$ in which case it is equal to unity, $f_{n'n}(q)$ is the two-center integral of harmonic oscillator wave functions defined by

$$f_{n'n}(q)=\int_{-\infty}^{\infty}dx\,u_{n'}(x+\hbar q/m\omega_0)u_n(x), \quad (21)$$

and

$$X_{n'n}^{(\pm)}(q)=(n+1)^{1/2}f_{n',n+1}(q)\pm n^{1/2}f_{n',n-1}(q). \quad (22)$$

A few useful mathematical properties of these matrix elements are displayed in the Appendix.

Equations (11) and (12) are not ostensibly gauge invariant. If $\mathbf{j}^{(1)}(\mathbf{q},\omega)$ and $\rho^{(1)}(\mathbf{q},\omega)$ are to be independent of the choice of gauge $(\mathbf{A},\phi)$, then the conditions

$$\omega\mathbf{K}-c\mathbf{q}-c\mathbf{I}\cdot\mathbf{q}=0 \quad (23)$$

and

$$\omega L-c\mathbf{K}^*\cdot\mathbf{q}=0 \quad (24)$$

must be satisfied. These relations can be easily verified using the results given in the Appendix.

It is convenient to consider separately the two independent geometrical arrangements in which $\mathbf{q}$ is either parallel to or perpendicular to $\mathbf{B}_0$. The general case in which $\mathbf{q}$ makes an arbitrary angle with $\mathbf{B}_0$ can be analyzed in a similar fashion, however, the expressions one obtains are more complicated than those found in these simple geometries.

B. Propagation in a Transverse Magnetic Field at 0°K (q perpendicular to $\mathbf{B}_0$)

In this situation we find that the components of the induced current density can be conveniently expressed in the form

$$j_x^{(1)}=\sigma_{xx}E_x+\sigma_{xy}E_y, \quad (25)$$

$$j_y^{(1)}=\sigma_{yx}E_x+\sigma_{yy}E_y, \quad (26)$$

and

$$j_z^{(1)}=\sigma_{zz}E_z. \quad (27)$$

The components of the conductivity tensor defined by Eqs. (25)–(27) can be shown to be

$$\sigma_{xx}=\frac{\omega_p^2}{4\pi i\omega}\left[1-\frac{2m\omega_0}{\hbar}\frac{1}{N}\sum_{nk_yk_z\alpha}{}'\left(\frac{\partial f_{n+\alpha,n}}{\partial q}\right)^2\frac{\alpha}{\alpha^2-(\omega/\omega_0)^2}\right], \quad (28)$$

$$\sigma_{yy}=\frac{im\omega_p^2\omega}{4\pi\hbar\omega_0q^2}\frac{2}{N}\sum_{nk_yk_z\alpha}{}'f_{n+\alpha,n}^2\frac{\alpha}{\alpha^2-(\omega/\omega_0)^2}, \quad (29)$$

$$\sigma_{xy}=-\sigma_{yx}=(i\omega_0/2\omega q)(\partial/\partial q)(q^2\sigma_{yy}), \quad (30)$$

and

$$\sigma_{zz}=\frac{\omega_p^2}{4\pi i\omega}\left[1-\frac{2\hbar}{m\omega_0}\frac{1}{N}\sum_{nk_yk_z\alpha}{}'k_z^2f_{n+\alpha,n}^2\frac{\alpha}{\alpha^2-(\omega/\omega_0)^2}\right]. \quad (31)$$

In Eqs. (28)–(31) the summations extend over all values of the quantum numbers for which $E_{nk_z}\leq E_0<E_{n+\alpha,k_z}$. This restriction is indicated by a prime following the summation sign. One can also show that it is possible to keep only the restriction $E_{nk_z}\leq E_0$ provided the sum over α is performed from $-n$ to infinity. The induced charge density can be found from Eqs. (25)–(27) by using the equation of continuity.

It is interesting to notice that in the limit in which $E_0/\hbar\omega_0$ is very large compared to unity the components of the conductivity tensor [Eqs. (28)–(31)] reduce to those obtained by Cohen *et al.*[12] in the semiclassical limit. This result is accomplished by replacing the sum over n in Eqs. (28)–(31) by an integration and making the substitution $n=n_0\sin^2\theta$, where n_0 is the largest integer that does not exceed $(E_0/\hbar\omega_0)-\frac{1}{2}$. The summation over n can, therefore be replaced by an integration over θ from 0 to $\pi/2$. It is easy to convince oneself that the largest contributions to the components of the conductivity tensor arise from large values of n (when, as in this case, $n_0 \gg 1$). We can thus replace $f_{n+\alpha,n}$ by its asymptotic behavior for large[22] n

$$f_{n+\alpha,n} \approx J_\alpha[\{(4n+2\alpha+2)\xi\}^{1/2}] \approx J_\alpha(w\sin\theta), \quad (32)$$

where J_α is the Bessel function of order α, $\xi = \hbar q^2/2m\omega_0$, and $w = qv_0/\omega_0$, v_0 being the velocity of an electron on the surface of the Fermi sphere. We find, after performing the sums over k_y and k_z, the following approximate expressions:

$$\sigma_{xx} = \frac{3\omega_p^2}{4\pi i\omega} \sum_{\alpha=-\infty}^{\infty} \frac{s_\alpha(w)}{1+\alpha(\omega_0/\omega)}, \quad (33)$$

$$\sigma_{yy} = -\frac{3\omega_p^2}{4\pi i\omega}\left(\frac{\omega}{qv_0}\right)^2 \left[1 - \sum_{\alpha=-\infty}^{\infty} \frac{g_\alpha(w)}{1+\alpha(\omega_0/\omega)}\right], \quad (34)$$

$$\sigma_{xy} = \frac{3\omega_p^2}{4\pi qv_0} \sum_{\alpha=-\infty}^{\infty} \frac{g_\alpha'(w)}{1+\alpha(\omega_0/\omega)}, \quad (35)$$

and

$$\sigma_{zz} = \frac{3\omega_p^2}{4\pi i\omega} \sum_{\alpha=-\infty}^{\infty} \frac{r_\alpha(w)}{1+\alpha(\omega_0/\omega)}. \quad (36)$$

The functions $s_\alpha(w)$, $g_\alpha(w)$, and $r_\alpha(w)$ are those defined by Cohen *et al.*[12] Here we reproduce their defining expressions for the sake of convenience:

$$s_\alpha(w) = \int_0^{\pi/2} d\theta \sin^3\theta \{J_\alpha'(w\sin\theta)\}^2, \quad (37)$$

$$g_\alpha(w) = \int_0^{\pi/2} d\theta \sin\theta \{J_\alpha(w\sin\theta)\}^2, \quad (38)$$

and

$$r_\alpha(w) = \int_0^{\pi/2} d\theta \sin\theta \cos^2\theta \{J_\alpha(w\sin\theta)\}^2. \quad (39)$$

C. Propagation in a Longitudinal Magnetic Field (q parallel to $\mathbf{B}_0$)

In this case the induced current density is again given by relations such as those of Eqs. (25)–(27). However, from symmetry considerations (which can also be verified directly) it follows that

$$\sigma_{xx} = \sigma_{yy}, \quad (40)$$

and

$$\sigma_{xy} = -\sigma_{yx}. \quad (41)$$

These conditions allow us to write the components of the induced current density in the form

$$j_x^{(1)} \pm i j_y^{(1)} = (\sigma_{xx} \mp i\sigma_{xy})(E_x \pm iE_y), \quad (42)$$

and

$$j_z^{(1)} = \sigma_{zz} E_z. \quad (43)$$

[22] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 199.

In these equations $\sigma_{xx} \mp i\sigma_{xy}$ and σ_{zz} are given by

$$\sigma_{xx} \mp i\sigma_{xy} = \frac{\omega_p^2}{4\pi i\omega}\left[1 - \frac{\hbar\omega_0}{N} \sum_{nk_yk_z}{}' \left\{ \frac{n+1}{\epsilon(k_z+q) - \epsilon(k_z) + \hbar(\omega_0 \mp \omega)} + \frac{n}{\epsilon(k_z+q) - \epsilon(k_z) - \hbar(\omega_0 \mp \omega)} \right\}\right], \quad (44)$$

and

$$\sigma_{zz} = -\frac{\omega_p^2}{4\pi i\omega}\left(\frac{\omega}{q}\right)^2 \frac{m}{N} \sum_{nk_yk_z}{}' \left[\frac{1}{\epsilon(k_z+q) - \epsilon(k_z) - \hbar\omega} + \frac{1}{\epsilon(k_z+q) - \epsilon(k_z) + \hbar\omega}\right]. \quad (45)$$

Here

$$\epsilon(k_z) = \hbar^2 k_z^2/2m. \quad (46)$$

The sums in Eqs. (44) and (45) are extended over all sets of quantum numbers nk_yk_z for which $E_{nk_z} \leq E_0$.

The expressions in Eqs. (44) and (45) can be evaluated in a rather simple way. As an example we give the result of the calculation of σ_{zz}. The real and imaginary parts of σ_{zz} are obtained making use of

$$(z \pm i0^+)^{-1} = \mathcal{P}(1/z) \mp i\pi\delta(z). \quad (47)$$

Here 0^+ is an infinitesimal positive quantity, $\mathcal{P}(1/z)$ indicates that in any integration on z of the right-hand side of Eq. (47) it is understood that one must take the principal value of the integral at the singularity $z=0$, and $\delta(z)$ is the Dirac delta function of argument z.

The imaginary part of σ_{zz} turns out to be

$$\mathrm{Im}(\sigma_{zz}) = \frac{3\omega_p^2\omega\omega_0}{8\pi q^3 v_0^3} \sum_{n=0}^{n_0} \left\{ \ln\left|\frac{K_n - (m\omega/\hbar q) + \frac{1}{2}q}{K_n - (m\omega/\hbar q) - \frac{1}{2}q}\right| + \ln\left|\frac{K_n + (m\omega/\hbar q) + \frac{1}{2}q}{K_n + (m\omega/\hbar q) - \frac{1}{2}q}\right| \right\}, \quad (48)$$

while the real part is

$$\mathrm{Re}(\sigma_{zz}) = (3\omega_p^2\omega_0\omega/8q^3v_0^3)\Lambda. \quad (49)$$

In these equations

$$K_n = (2m/\hbar^2)^{1/2}[E_0 - (n+\tfrac{1}{2})\hbar\omega_0]^{\frac{1}{2}}, \quad (50)$$

and Λ is the number of integers in the interval (Λ_+, Λ_-), where

$$\Lambda_\pm = \frac{E_0}{\hbar\omega_0} - \frac{1}{2} - \frac{m\omega^2}{2\hbar\omega_0 q^2}\left(1 \pm \frac{\hbar q^2}{2m\omega}\right)^2. \quad (51)$$

The limiting behavior of $\mathrm{Im}(\sigma_{zz})$ for long wavelengths is of particular interest. In this case we can rewrite Eq. (48) in the form

$$\mathrm{Im}(\sigma_{zz})=\frac{3\omega_p{}^2\omega\omega_0}{8\pi q^2v_0{}^3}\sum_{n=0}^{n_0}\left[\left(K_n-\frac{m\omega}{\hbar q}\right)^{-1}+\left(K_n+\frac{m\omega}{\hbar q}\right)^{-1}\right]. \quad (52)$$

In the present section we have given a rather formal derivation of the conductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$ for an electron gas in two geometrical arrangements of special interest. In the next section we discuss some applications of this work.

III. ULTRASONIC ATTENUATION AND DISPERSION

A. General Considerations

In this section we discuss a few applications of the expressions for the conductivity tensor to the propagation of acoustic waves in metals. The motion of the ions of a metal acts as the driving disturbance responsible for the establishment of the self-consistent electromagnetic field which couples the electronic and ionic motions. An acoustic wave traveling in a metal can induce both real and virtual excitations of the electron gas because it is accompanied by an electromagnetic field. In real transitions the electron gas gains energy irreversibly at the expense of the energy of the sound wave. This energy loss is the source of the attenuation of the sound wave. The virtual transitions give rise to dielectric screening of the motion of the ions and change the characteristic frequencies of the normal modes of vibration of the lattice from their values when the electrons are held fixed. In particular, the frequency of a longitudinal acoustic wave can be simply evaluated within the framework of the plasma model considered in this paper. In the absence of a magnetic field, one obtains the well-known result $\omega=(zm/3M)^{1/2}v_0q$ of Bohm and Staver.[23]

The coefficient of attenuation γ of the energy of a sound wave is defined as the ratio of Q, the power absorbed per unit volume, to the incident power per unit area normal to the direction of propagation. The quantity Q is given by the formula

$$Q=\tfrac{1}{2}\,\mathrm{Re}(\mathbf{j}^{(1)*}\cdot\mathbf{E}). \quad (53)$$

The electric field $\mathbf{E}$ and the electronic current density $\mathbf{j}^{(1)}=\boldsymbol{\sigma}\cdot\mathbf{E}$ must be obtained self-consistently by solving Maxwell's equations together with the expressions giving the total current density as the sum of contributions from the electrons and from the positive ions. The ionic current density is simply $(N/\Omega)|e|\mathbf{u}$, where $\mathbf{u}$ is the velocity field of the sound wave. The general expression for the attenuation coefficient turns out to be

$$\gamma=(\Omega_p{}^2/4\pi s)\,\mathrm{Re}(\hat{u}\cdot\mathbf{P}\cdot\hat{u}), \quad (54)$$

where s is the velocity of sound and $\hat{u}=\mathbf{u}/|\mathbf{u}|$ is a unit vector in the direction of polarization of the wave. The 3×3 tensor $\mathbf{P}$ is the inverse of the tensor $\mathbf{P}^{-1}$ whose components are

$$(\mathbf{P}^{-1})_{\mu\nu}=\sigma_{\mu\nu}-\Gamma_{\mu\nu}. \quad (55)$$

The tensor $\mathbf{\Gamma}$ is defined by

$$\Gamma_{\mu\nu}=\frac{ic^2q^2}{4\pi\omega}\left(1-\frac{\omega^2}{c^2q^2}\right)\delta_{\mu\nu}-\frac{ic^2q_\mu q_\nu}{4\pi\omega}. \quad (56)$$

The suffixes μ, ν designate projections on the axes (xyz) of the Cartesian coordinate system that has been adopted.

In the previous section we discussed separately the two independent geometries, $\mathbf{q}$ parallel to $\mathbf{B}_0$ and $\mathbf{q}$ perpendicular to $\mathbf{B}_0$. In each of these cases one can have either longitudinal or transverse waves (i.e., $\mathbf{u}$ parallel or perpendicular to $\mathbf{q}$). In the case of transverse waves in a transverse magnetic field one must distinguish the two possible situations in which $\mathbf{u}$ may be either parallel to or perpendicular to $\mathbf{B}_0$. For each particular situation expressions for the ultrasonic attenuation can be found easily from Eq. (54) and the appropriate values of the components of the conductivity tensor. The study of propagation of sound waves at an arbitrary angle to $\mathbf{B}_0$ can be analyzed in terms of the two simple geometries considered here. In the semiclassical limit we have already shown that the conductivity tensor reduces to the results of the treatment based on the Boltzmann transport equation. The attenuation coefficient has been studied, in this limit and for all geometries of interest, by Cohen *et al.*[12] and the use of Eqs. (33)–(36) together with the introduction of a phenomenological relaxation time τ would merely reproduce the results of reference 12. For this reason we need not discuss further the classical aspects of the problem. Thus, we shall only concern ourselves here with some simple quantum effects which are independent of a phenomenological relaxation time τ. A more thorough study of quantum effects in ultrasonic attenuation is given in the following paper.[24]

The velocity of sound as a function of an applied magnetic field can also be obtained from the knowledge of the components $\sigma_{\mu\nu}(\mathbf{q},\omega)$ of the conductivity tensor. If $S_{\mu\nu}(\mathbf{q},\omega)$ are the components of the electrical conductivity tensor for the positive ions, then the total current density is

$$J_\mu=\textstyle\sum_\nu(\sigma_{\mu\nu}+S_{\mu\nu})E_\nu. \quad (57)$$

There is another relation connecting $\mathbf{J}$ and $\mathbf{E}$ which is a consequence of Maxwell's equations, namely,

$$J_\mu=\textstyle\sum_\nu\Gamma_{\mu\nu}E_\nu, \quad (58)$$

where $\mathbf{\Gamma}$ is defined by Eq. (56). The angular frequency ω of a longitudinal sound wave of wave vector $\mathbf{q}$ is a solution of the determinantal equation

$$\|\sigma_{\mu\nu}(\mathbf{q},\omega)+S_{\mu\nu}(\mathbf{q},\omega)-\Gamma_{\mu\nu}(\mathbf{q},\omega)\|=0. \quad (59)$$

[23] D. Bohm and T. Staver, Phys. Rev. **84**, 836 (1952).

[24] J. J. Quinn and S. Rodriguez, following paper [Phys. Rev. **128**, 2494 (1962)].

In general, only one solution $\omega=\omega(\mathbf{q})$ of (59) corresponds to the propagation of an acoustic wave. Furthermore $\omega=\omega(\mathbf{q})$ is complex. The real part ω_1 of the frequency ω gives the phase velocity of sound $s=\omega_1/q$ while the imaginary part ω_2 is related to the attenuation coefficient $\gamma=2\omega_2/s$. In the absence of a magnetic field (59) yields the result of Bohm and Staver[23] for the sound velocity and the result of Pippard[4] for the coefficient of ultrasonic attenuation.

B. Propagation of Longitudinal Acoustic Waves in a Longitudinal Magnetic Field

The simplest quantum effects in the dispersion and attenuation of sound waves in metals in a dc magnetic field occur when we have a longitudinal wave propagating in the direction of the field $\mathbf{B}_0$. In the semiclassical theory both the velocity and attenuation coefficient in this situation are independent of $\mathbf{B}_0$. This does not turn out to be the case in the quantum theory as stated in the introduction. The oscillatory behavior in s and γ predicted here is therefore a purely quantum-mechanical effect.

Applying Eqs. (59) and (56) to this particular case and using $S_{zz}=\Omega_p{}^2/4\pi i\omega$, we find

$$\omega^2=\Omega_p{}^2(1-4\pi i\sigma_{zz}/\omega)^{-1}. \tag{60}$$

Using the expressions (49) and (52) for the real and imaginary parts of σ_{zz} one can find the velocity of sound and the coefficient of ultrasonic attenuation. The velocity of sound is given by the implicit equation

$$s^2=\frac{2}{3}\left(\frac{\Omega_p}{\omega_p}\right)^2\frac{v_0{}^3}{\omega_0}\times\left\{\sum_{n=0}^{n_0}\left[(K_n-ms/\hbar)^{-1}+(K_n+ms/\hbar)^{-1}\right]\right\}^{-1}. \tag{61}$$

Inspection of the result (61) reveals the oscillatory character of s as a function of $\mathbf{B}_0$. More details of this work have been given elsewhere.[25]. The coefficient of ultrasonic attenuation is given approximately by

$$\gamma=(\pi\omega_0/2v_0)\Lambda. \tag{62}$$

The quantity Λ, which has been defined as the number of integers in the interval between Λ_+ and Λ_- [see Eq. (51)], is proportional to the number of electrons in Landau states that can be excited above the Fermi level by phonons of energy $\hbar\omega$. The dependence of Λ on B_0 can be easily visualized by considering the restrictions imposed upon the possible absorption of a phonon by the laws of conservation of energy and momentum and by the Pauli principle. In particular, these restrictions imply that the electrons that contribute to the attenuation have a component of velocity in the direction of $\mathbf{B}_0$ equal to the velocity of sound.

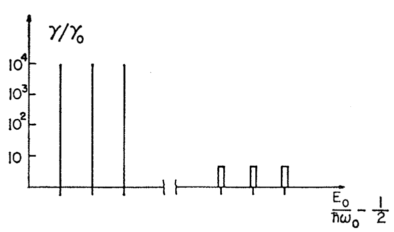

FIG. 1. Schematic plot of ultrasonic attenuation as a function of $E_0/\hbar\omega_0$. The marks on the horizontal axis occur at integral values of $(E_0/\hbar\omega_0)-\frac{1}{2}$ (see text for additional description).

For weak magnetic fields when $\omega_0\ll\omega$ we have $\Lambda\approx\omega/\omega_0$ and obtain the well-known result[4] $\gamma=\pi\omega/2v_0$ for the coefficient of absorption of sound in the absence of a magnetic field. At higher magnetic fields when $\omega<\omega_0$, at most one Landau level contributes to the energy dissipation because under these conditions $\Lambda_--\Lambda_+=\omega/\omega_0<1$. In Fig. 1 we give a schematic plot of the ratio of the attenuation γ to the attenuation γ_0 in the absence of a magnetic field as a function of $(E_0/\hbar\omega_0)-\frac{1}{2}$. Two ranges of values of $(E_0/\hbar\omega_0)-\frac{1}{2}$ are shown. In the region to the left we have taken $(E_0/\hbar\omega_0)\approx 8\times10^4$ and $\omega/\omega_0=10^{-4}$. The attenuation exhibits spikes of width ω/ω_0 and height ω_0/ω (the width in Fig. 1 is grossly exaggerated), that occur at integral values of $(E_0/\hbar\omega_0)-\frac{1}{2}$ to a high degree of approximation. In the region to the right we show the case in which $E_0/\hbar\omega_0\approx1.6\times10^8$ and $\omega/\omega_0=0.2$. Here the spikes have a width of 0.2 and a height of 5. The average value of γ over a cycle (i.e., over an interval of unity in the variable $E_0/\hbar\omega_0$) is equal to γ_0. These giant oscillations which have been observed by Korolyuk and Prushchak[26] were predicted originally by Gurevich *et al.*[27] using a method which is rather different from the one used in this paper. However, the procedure used in the present work has the advantage that it can be used to obtain the temperature dependence of the amplitude of the giant oscillations; in fact, a straight forward (but lengthy) calculation gives

$$(\gamma/\gamma_0)-1=(4\pi kT/\hbar\omega)\sum_{n=1}^{\infty}(-1)^n\sin(\pi n\omega/\omega_0)\times\frac{\cos(2\pi n\zeta_0/\hbar\omega_0)}{\sinh(2n\pi^2kT/\hbar\omega_0)}. \tag{63}$$

In real metals the situation can become more complicated because of the influence of the collision mechanism and of the actual band structure of the solid. In particular, if the Fermi surface of the metal has sections in several energy bands, oscillations in γ can occur having widely different periods. The broadening of the Landau levels due to their finite lifetime can wash out the oscillations in the region where $\omega\approx\omega_0$. However, at high magnetic fields when $\omega\ll\omega_0$ the large fluctuations

[25] J. J. Quinn and S. Rodriguez, Phys. Rev. Letters **9**, 145 (1962).

[26] A. P. Korolyuk and T. A. Prushchak, Soviet Phys.—JETP **14**, 1201 (1962).

[27] V. L. Gurevich, V. G. Skobov, and Yu. A. Firsov, Soviet Phys.—JETP **13**, 552 (1961).

in γ/γ_0 should become observable in highly pure materials and low temperatures.

Most of the discussion in this paper has been centered on the calculation of transport coefficients for a degenerate electron gas at 0°K. At a temperature which is finite but much smaller than the Fermi degeneracy temperature our results hold to a good degree of approximation. The effects discussed here are probably observable if $kT \ll \hbar\omega_0 \ll E_0$ and in pure samples so that the collision broadening of the Landau levels is negligible as compared with their splitting $\hbar\omega_0$.

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APPENDIX

The purpose of this appendix is to give a few mathematical properties of the matrix elements $f_{n'n}(q)$. This quantity is defined by Eq. (21) and can also be expressed in the form

$$f_{n'n}(q)=i^{n-n'}\int_{-\infty}^{\infty}dx\,\exp(iqx)u_{n'}(x)u_n(x). \quad \text{(A1)}$$

Using the expressions for $u_n(x)$ given in reference (19) it is easy to prove that

$$f_{n'n}(q)=(n!/n'!)^{\frac{1}{2}}\xi^{\frac{1}{2}(n'-n)}\exp(-\tfrac{1}{2}\xi)L_n^{(n'-n)}(\xi). \quad \text{(A2)}$$

The formula (A2) holds only when $n' \geq n$. $L_n^{(\alpha)}(\xi)$ is an associated Laguerre polynomial[22] and $\xi=\hbar q^2/2m\omega_0$. An expression similar to (A2) can be found when $n'<n$ using (A2) together with

$$f_{n'n}(-q)=f_{nn'}(q)=(-1)^{n'-n}f_{n'n}(q). \quad \text{(A3)}$$

Using the property of completeness of the functions $u_n(x)$ and calculating diagonal matrix elements of the commutator

$$[[p_x^2, \exp(iqx)], \exp(-iqx)]=-2\hbar^2q^2, \quad \text{(A4)}$$

we obtain the sum rules

$$\sum_{n'=0}^{\infty} f_{n'n}^2(q)=1, \quad \text{(A5)}$$

and

$$\sum_{n'=0}^{\infty} (n'-n)f_{n'n}^2(q)=\xi. \quad \text{(A6)}$$

Taking derivatives with respect to q of (A5) and (A6) and using

$$\partial f_{n'n}/\partial q=(\hbar/2m\omega_0)^{\frac{1}{2}}X_{n'n}^{(-)}, \quad \text{(A7)}$$

we find other sum rules that turn out to be useful, together with (A5) and (A6) in establishing the gauge invariance of the theory. Another property of $f_{n'n}$ which is also necessary for this purpose is obtained by taking off diagonal matrix elements of the commutator

$$[H_0, \exp(i\mathbf{q}\cdot\mathbf{r})]=\hbar\mathbf{q}\cdot\mathbf{V}(\mathbf{q}). \quad \text{(A8)}$$

We find

$$(n'-n-\xi)f_{n'n}(q)=\xi^{1/2}X_{n'n}^{(+)}(q). \quad \text{(A9)}$$