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Two-Component First-Order Wave Equations

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It is shown that a suitable definition of the dotted spinors in terms of the undotted spinors allows the linearization of the two-component second order Feynman Gell-Mann wave equations. The linearized two-component first-order wave equations are similar to the original Jehle wave equations. They are covariant with respect to the restricted Lorentz group and are transformed into their complex conjugate wave equations by a reflection. Then it is shown that the current vector derived from them is a world vector.

1. INTRODUCTION

FEYNMAN and Gell-Mann[1,2] have postulated two-component second-order wave equations for the fermions and obtained the $V-A$ form for the β interaction, which has been experimentally verified. We should like to investigate the possibility of linearizing the Feynman–Gell-Mann second-order wave equations in order to incorporate for their two-component wave functions the advantages of a Dirac-like first-order wave equation also when the spin and magnetic moments obtain intrinsic expressions.

The definition that the dotted spinor is the complex conjugate of the undotted spinor leads invariably to a four-component first-order Dirac wave equation whose wave functions are covariant for the Lorentz group.[3] But the dotted and undotted spinors are arbitrary and unrelated spinors (B.J. Sec. III) and the above definition is made in order to correlate the Dirac wave functions to spinors. A redefinition of the dotted spinors in terms of the undotted spinors, which allows the construction of two-component first-order wave equations, and the consequent minor alterations in the usual spinor calculus are stated in Sec. 2.

These two-component wave equations are covariant for the restricted Lorentz group and only certain products of the dotted and undotted two-component spinors are covariant for the Lorentz group. This is sufficient to define a current vector which is covariant for the Lorentz group, and in Sec. 4 it is shown that the current vector derived from the first-order wave equations obtained in Sec. 3 is covariant for the Lorentz group. The equivalence of charge conjugation and complex conjugation for two-component wave functions is also suggested here.

Two-component first-order wave equations have been proposed by Jehle.[3–5] In Sec. 3, first the Jehle wave equations for free electrons are written in the momentum representation so that the conservation of energy and momentum is explicitly exhibited. The definition of the dotted spinor given in Sec. 2 is found to be necessary for this statement. Then, it is shown that the electromagnetic interaction can be introduced into the free-field wave equations exactly as for the Dirac wave equation and that the Jehle wave equations are correlated to the Feynman–Gell-Mann wave equation just as the Dirac wave equation is correlated to the Klein-Gordon wave equation.

2. SPINOR CALCULUS

Instead of correlating the arbitrary dotted and undotted spinors by defining the dotted spinor to be the complex conjugate of the undotted spinor (B.J. Sec. III), let the dotted spinor be defined by

$$\psi^{\dot\alpha}=i(\psi^*)^{\alpha},\quad \psi_{\dot\alpha}=-i(\psi^*)_{\alpha}, \tag{1}$$

or

$$(\psi^*)^{\dot\alpha}=i\psi^{\alpha},\quad (\psi^*)_{\dot\alpha}=-i\psi_{\alpha}, \tag{2}$$

where $(\psi^*)^{\alpha}\equiv(\psi^{\alpha})^*$ denotes the complex conjugate of the spinor ψ^{α}. The higher rank dotted and mixed spinors are accordingly defined by

$$\psi^{\dot\alpha\dot\beta}=-(\psi^*)^{\alpha\beta},\quad \psi^{\dot\alpha}{}_{\dot\beta}=(\psi^*)^{\alpha}{}_{\beta}, \tag{3}$$

$$\psi^{\dot\alpha\beta}=(\psi^*)^{\alpha\dot\beta},\qquad \psi^{\dot\alpha}{}_{\beta}=-(\psi^*)^{\alpha}{}_{\dot\beta}. \tag{4}$$

We note that

$$(\psi^{\dot\alpha})^*=-(\psi^*)^{\dot\alpha}, \tag{5}$$

$$(\psi^{\alpha\dot\beta})^*=-(\psi^*)^{\alpha\dot\beta},\quad (\psi^{\dot\alpha}{}_{\beta})^*=-(\psi^*)^{\dot\alpha}{}_{\beta}, \tag{6}$$

where in Eq. (6) the property (5) of the dotted spinors has been utilized so that the mixed spinor $\psi^{\dot\alpha\beta}$ is defined in terms of $\psi^{\alpha\dot\beta}$ by the combined use of (4) and (6) and the spinor $(\psi^*)^{\dot\alpha}$ is defined for obtaining convenient notation.

Before showing how this redefinition of the dotted spinors in terms of the undotted spinors enables the construction of suitable two-component first-order wave equations we shall consider its effect on the B.J. spinor calculus and note that it remains essentially unchanged wherever only dotted or undotted spinors occur in a relation (B.J. Secs. III–VI); for example [B.J. (IV, 5),

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); L. M. Brown, *ibid.* **111**, 957 (1958); L. C. Biedenharn, Nuovo cimento **22**, 1097 (1961).

[2] Four-component fermion theories of $V-A$ β interaction have been proposed by E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109**, 1860 (1958); J. J. Sakurai, Nuovo cimento **7**, 714 (1958).

[3] W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953) which is hereafter called B.J. Its notation is followed but the dotted spinor is now defined as Eq. (1) of Sec. 2.

[4] H. Jehle, Phys. Rev. **75**, 1609 (1949).

[5] See also C. W. Kilmister, Phys. Rev. **75**, 568 (1949); J. Serpe, *ibid.* **76**, 1538 (1949), Physica **18**, 295 (1952); K. M. Case, Phys. Rev. **107**, 307 (1957); V. Heine, *ibid.* **107**, 620 (1957).

(IV, 8)],

$$\psi_\nu=\psi^\mu\gamma_{\mu\nu}=-\gamma_{\nu\mu}\psi^\mu, \tag{7}$$

$$\psi^\nu=-\psi_\mu\gamma^{\mu\nu}=\gamma^{\nu\mu}\psi_\mu. \tag{8}$$

However, on account of different rules for obtaining the dotted spinors from the undotted contravariant and covariant undotted spinors given in (1), we now obtain

$$\psi^\nu=(\gamma^*)^{\nu\mu}\psi_\mu=\gamma^{\nu\mu}\psi_\mu, \quad (\gamma^*)^{\mu\nu}=\gamma^{\mu\nu}, \tag{9}$$

so that [B.J. (IV, 10), (VI, 3)]

$$\gamma^{\mu\nu}=\gamma_{\mu\nu}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=-\gamma^{\dot{\mu}\dot{\nu}}=-\gamma_{\dot{\mu}\dot{\nu}}, \tag{10}$$

$$\gamma=\gamma_{12}\gamma_{\dot{1}\dot{2}}=-1. \tag{11}$$

For a Hermitian second-rank spinor $A_{\dot{\lambda}\mu}$ [B.J. (VII, 1)] we have again by means of (4) and (6)

$$A_{\dot{\lambda}\mu}=(A_{\dot{\mu}\lambda})^*=-A_{\mu\dot{\lambda}}, \tag{12}$$

so that if [B.J. (VII, 2)]

$$A_{\dot{\lambda}\mu}=\sigma^k{}_{\dot{\lambda}\mu}A_k, \tag{13}$$

then [B.J. (VII, 3)]

$$\sigma^k{}_{\dot{\lambda}\mu}=-\sigma^k{}_{\mu\dot{\lambda}}. \tag{14}$$

The connection between the world tensors and spinors is obtained by equating the invariant quadratic form $g^{kl}A_kA_l$ to the invariant [B.J. (VII, 4)]

$$-A^{\dot{\lambda}\mu}A_{\dot{\lambda}\mu}=-\gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\nu}A_{\dot{\rho}\nu}A_{\dot{\lambda}\mu}=-(2/\gamma)|A_{\dot{\lambda}\mu}|, \tag{15}$$

on account of (11), so that in place of B.J. (VII, 5) we now have

$$\sigma^{k\dot{\lambda}\mu}\sigma^l{}_{\dot{\lambda}\mu}=\gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\nu}\sigma^k{}_{\dot{\rho}\nu}\sigma^l{}_{\dot{\lambda}\mu}=-g^{kl}. \tag{16}$$

The alterations in the remaining B.J. Sec. VII are straightforward and for convenience we note that the B.J. Sec. VII Eqs. (6)–(12) now become

$$A^l=-\sigma^{l\dot{\lambda}\mu}A_{\dot{\lambda}\mu}, \tag{17}$$

$$T^{\dot{\mu}\nu\dot{\rho}\sigma}=\sigma^{k\dot{\mu}\nu}\sigma^{l\dot{\rho}\sigma}T_{kl}, \tag{18}$$

$$T^{kl}=\sigma^{k\dot{\mu}\nu}\sigma^{l\dot{\rho}\sigma}T_{\dot{\mu}\nu\dot{\rho}\sigma}, \tag{19}$$

$$-g_{kl}\sigma^k{}_{\dot{\lambda}\mu}\sigma^{l\dot{\rho}\sigma}=\delta_{\dot{\lambda}}{}^{\dot{\rho}}\delta_\mu{}^\sigma=\gamma_{\dot{\lambda}}{}^{\dot{\rho}}\gamma_\mu{}^\sigma, \tag{20}$$

$$-g_{kl}\sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\rho}\sigma}=\gamma^{\dot{\lambda}\dot{\rho}}\gamma^{\mu\sigma}, \tag{21}$$

$$-A_{\dot{\lambda}\mu}B^{\dot{\lambda}\mu}=A_kB^k, \tag{22}$$

$$\sigma^{l\mu\dot{\lambda}}\sigma^k{}_{\dot{\lambda}\sigma}+\sigma^{k\mu\dot{\lambda}}\sigma^l{}_{\dot{\lambda}\sigma}=-g^{kl}\delta^\mu{}_\sigma, \tag{23}$$

respectively. Since $\sigma^{k\dot{\lambda}\mu}$ also is Hermitian, Eq. (23) can be written in the form

$$\sigma^l{}_{\dot{\sigma}\lambda}\sigma^{k\lambda\dot{\mu}}+\sigma^k{}_{\dot{\sigma}\lambda}\sigma^{l\lambda\dot{\mu}}=-g^{kl}\delta_{\dot{\sigma}}{}^{\dot{\mu}}, \tag{24}$$

and by raising and lowering the indices of σ's in (23) and (24) we finally obtain

$$\sigma^l{}_\mu{}^{\dot{\lambda}}\sigma^k{}_{\dot{\lambda}}{}^\sigma+\sigma^k{}_\mu{}^{\dot{\lambda}}\sigma^l{}_{\dot{\lambda}}{}^\sigma=-g^{kl}\delta_\mu{}^\sigma, \tag{25}$$

$$\sigma^{l\dot{\sigma}}{}_\lambda\sigma^{k\lambda}{}_{\dot{\mu}}+\sigma^{k\dot{\sigma}}{}_\lambda\sigma^{l\lambda}{}_{\dot{\mu}}=-g^{kl}\delta^{\dot{\sigma}}{}_{\dot{\mu}}, \tag{26}$$

in place of B.J. (I, 8).

An explicit representation of the σ's is obtained by noting that the Eqs. B.J. (VIII, 1) and (VIII, 2) remain unaltered here and that one side of B.J. (VIII, 3) should now be multiplied by -1.

3. JEHLE WAVE EQUATIONS

We should now like to show that the Jehle[3,4] two-component wave equations,

$$\sqrt{2}\sigma^{k\dot{\nu}}{}_\rho(\partial_k+i\epsilon\varphi_k)\psi^\rho=\mu\psi^{\dot{\nu}}, \tag{27}$$

$$\sqrt{2}\sigma^{k\nu}{}_{\dot{\rho}}(\partial_k+i\epsilon\varphi_k)\psi^{\dot{\rho}}=\mu\psi^\nu, \tag{28}$$

describe the electron field. Then it is sufficient to show that after suitable operations have been performed they lead to the Feynman–Gell-Mann second-order wave equations. Equations (27) and (28) differ from the original Jehle wave equations because of the different significance of the dotted spinor here but, as for the Jehle wave equations, Eq. (28) is the complex conjugate of Eq. (27). This becomes evident by noting that on account of Eqs. (1)–(6)

$$(\psi^\rho)^*=-i\psi^{\dot{\rho}}, \quad (\psi^{\dot{\nu}})^*=-i\psi^\nu,$$
$$(\sigma^{k\dot{\nu}}{}_\rho)^*=-(\sigma^{k*})^{\dot{\nu}}{}_\rho=\sigma^{k\nu}{}_{\dot{\rho}}.$$

In momentum representation[6] Eqs. (27) and (28) become

$$\sqrt{2}\sigma^{k\dot{\nu}}{}_\rho(P_k-\lambda\varphi_k)\psi^\rho=imc\psi^{\dot{\nu}}, \tag{29}$$

$$\sqrt{2}\sigma^{k\nu}{}_{\dot{\rho}}(P_k-\lambda\varphi_k)\psi^{\dot{\rho}}=-imc\psi^\nu, \tag{30}$$

where $\lambda=-\epsilon\hbar$.

Now the second-order free electron wave equation can be stated in the equivalent forms:

$$[\Box(x)+\mu^2]\psi=0, \tag{31}$$

or

$$(P^2-m^2c^2)\psi=0, \tag{32}$$

$$P_k\psi=i\hbar\partial_k\psi, \quad P_k\psi^*=-i\hbar\partial_k\psi^*. \tag{33}$$

The original Jehle free-electron wave equation reproduces (31) but not (32) on account of the changed sign of the mass term there, while Eq. (27) for $\epsilon=0$ does not reproduce (31) but Eq. (29) reproduces Eq. (32). So that the definition of dotted spinor as the complex conjugate of the undotted spinor leads to second-order wave equations of the form (31), the definition of the dotted spinor by means of Eq. (1) leads to second-order wave equations of the form (32) and a physical significance of the definition of the dotted spinor in terms of the undotted spinor is obtained. It also becomes

[6] By momentum representation we mean the application of Eq. (33) to Eqs. (27) and (28) and not the Fourier transformation of Eqs. (27) and (28).

necessary to choose either Eq. (31) or Eq. (32) to be the fundamental second-order free electron wave equation and then to perform the transformations (33) to obtain the corresponding wave equation in the other representation.

Now multiplying Eq. (29) [(30)] by $\sqrt{2}\sigma^{l\alpha}{}_{\dot\nu}(P_l-\lambda\varphi_l)$ [$\sqrt{2}\sigma^{l\alpha}{}_{\dot\nu}(P_l-\lambda\varphi_l)$], we obtain the required second-order two-component wave equations for the electron

$$[(P-\lambda\varphi)^2-m^2c^2]\psi^{\dot\alpha}=\sigma^{k\dot\alpha}{}_{\nu}\sigma^{l\nu}{}_{\dot\rho}F_{lk}\psi^{\dot\rho}, \tag{34}$$

$$[(P-\lambda\varphi)^2-m^2c^2]\psi^{\alpha}=\sigma^{k\alpha}{}_{\dot\nu}\sigma^{l\dot\nu}{}_{\rho}F_{lk}\psi^{\rho}. \tag{35}$$

By operating in the momentum representation and, therefore, in accordance with the acceptance of (32) we can now multiply the left-hand side of (27) by $\sqrt{2}\sigma^{l\alpha}{}_{\dot\nu}(\partial_l+i\epsilon\varphi_l)$ and its right-hand side by $-\sqrt{2}\sigma^{l\alpha}{}_{\dot\nu}\times(\partial_l-i\epsilon\varphi_l)$, which is essential for obtaining the correct form of the electromagnetic interaction term in the second-order wave equations as shown in Eqs. (34) and (35).

We conclude that the Jehle two-component wave equations (27) and (28) or (29) and (30) are the linearized forms of the Feynman–Gell-Mann second-order wave equations (34) and (35) and that in the momentum representation they are correlated to each other as the Dirac equation is correlated to the Klein-Gordon wave equation.

4. LORENTZ COVARIANCE AND CHARGE CONJUGATION

Equation (29) or Eq. (30) is covariant with respect to the restricted Lorentz group (B.J. Sec. VIII) and we should now like to show that the combined wave equations (29) and (30) are covariant for the Lorentz group. It is known (B.J. Sec. VIII, footnote 15) that the combined original Jehle wave equations and their charge conjugate wave equations are covariant for the Lorentz group. Thus, in the present formulation charge conjugation is shown to be equivalent to complex conjugation. This simplification results in an economy in the number of wave functions needed to describe the electron and positron fields and is supported by field theories where the operator wave functions ψ^α [$\psi^{\dot\alpha}$] are assigned the dual role of the emission of an electron and absorption of a positron (absorption of an electron and emission of a positron).

To represent inversions in the spin space, as in B.J. Sec. VIII, let the reflection $c^l{}_m=c_l{}^m$ [B.J. (VIII, 5)] be coordinated with the spinor charge conjugation transformation

$$\psi^{\mu}{}_{\nu\dot\sigma}\rightarrow C(\psi^{\mu}{}_{\nu\dot\sigma})=-(\psi^*)^{\mu}{}_{\nu\dot\sigma}=(\psi^{\mu}{}_{\nu\dot\sigma})^*, \tag{36}$$

$$C(\psi\eta\cdots)=(\psi\eta\cdots)C=C(\psi)C(\eta)\cdots, \tag{37}$$

then

$$C^2\psi=\psi,\quad C(x)=x^*, \tag{38}$$

$$C(c^l{}_m\sigma^{m\dot\rho}{}_\nu)=c^l{}_mC(\sigma^{m\dot\rho}{}_\nu)=\sigma^{l\dot\rho}{}_\nu, \tag{39}$$

where x is a scalar. When the transformations $c_m{}^l$ and C are applied simultaneously to Eqs. (29) and (30), they become

$$\sqrt{2}\sigma^{k\dot\nu}{}_\rho c_k{}^m(P_m-\lambda\varphi_m)C(\psi^\rho)=C(imc\psi^{\dot\nu}), \tag{40}$$

$$\sqrt{2}\sigma^{k\nu}{}_{\dot\rho}c_k{}^m(P_m-\lambda\varphi_m)C(\psi^{\dot\rho})=C(-imc\psi^{\nu}), \tag{41}$$

or

$$\sqrt{2}\sigma^{m\nu}{}_{\dot\rho}(P_m-\lambda\varphi_m)\psi^{\dot\rho}=-imc\psi^{\nu}, \tag{42}$$

$$\sqrt{2}\sigma^{m\dot\nu}{}_{\rho}(P_m-\lambda\varphi_m)\psi^{\rho}=imc\psi^{\dot\nu}, \tag{43}$$

respectively, and the covariance of the combined equations (29) and (30) for the Lorentz group is obtained.

We note that in the above procedure the C operation has been applied to the spinor $x\psi$, where x is a scalar quantity and not to the spinor ψ only as in B.J. Sec. VIII so that the transformed spinor of $x\psi$ is $C(x\psi)$ and not $xC(\psi)$. This definition introduces a change in sign in the right-hand sides of Eqs. (42) and (43) which is necessary if they are to be equated to Eqs. (29) and (30). On the other hand if the charge conjugation operator is defined by

$$\psi^{\mu\cdots}{}_{\nu\dot\sigma\cdots}\rightarrow C'(\psi^{\mu\cdots}{}_{\nu\dot\sigma\cdots})=i^{m-n}(\psi^{\mu\cdots}{}_{\nu\dot\sigma\cdots})^*, \tag{44}$$

where m and n are the numbers of the dotted and undotted indices, respectively, of the spinor $\psi^{\mu\cdots}{}_{\nu\dot\sigma\cdots}$ and it is restricted to apply to spinors ψ only in quantities of the form $x\psi$ we again obtain Eqs. (42) and (43) when the transformations $c^l{}_m$ and C' are simultaneously applied to the Eqs. (29) and (30). Thus, the operators C and C' give equivalent results and, perhaps, the only reasons to prefer C to C' are its simpler form and its equality to complex conjugation.

On lowering the indices, Eqs. (29) and (30) become

$$\sqrt{2}\sigma^k{}_{\dot\nu\rho}(P_k-\lambda\varphi_k)\psi^\rho=imc\psi_{\dot\nu}, \tag{45}$$

$$\sqrt{2}\sigma^k{}_{\nu\dot\rho}(P_k-\lambda\varphi_k)\psi^{\dot\rho}=-imc\psi_{\nu}. \tag{46}$$

A consequence of the covariance of the combined wave equations (45) and (46) with respect to the Lorentz group is that the current vector

$$j^k=q\psi^{\dot\nu}\sigma^k{}_{\dot\nu\rho}\psi^\rho, \tag{47}$$

whose definition follows directly from Eqs. (45) and (46) and which satisfies the continuity equation

$$\partial_kj^k=0, \tag{48}$$

is a world vector,

$$j'^k=qc^k{}_lC(\psi^{\dot\nu}\sigma^l{}_{\dot\nu\rho}\psi^\rho)=c^k{}_lj^l. \tag{49}$$