Motion of Electron with Radiation Damping and Boundary Conditions

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The classical equation of motion of an electron with radiation damping is investigated. It is shown that a physically reasonable solution satisfying the given initial conditions can be obtained (i.e., the Cauchy problem formulated) only in the absence of radiation damping. On the other hand, once the radiation damping is taken into account, a boundary value problem arises and initial and final conditions must be prescribed. In conclusion, a comparison of our results and those of Schwartz is given.

1. BASIC EQUATION

THE problem of solving the classical equation of motion of the point electron, with radiation damping taken into account, has been rather widely discussed in recent years.[1–5] This equation, in the case of the harmonic oscillator in nonrelativistic approximation, has the form

$$D\left(\frac{d}{dt}\right)x=\left(\frac{d^2}{dt^2}+\omega_0^2-\gamma\frac{d^3}{dt^3}\right)x=f(t). \quad (1)$$

Here $\gamma=\frac{2}{3}r_0/c$, $r_0=e^2/m_0c^2$ is the electron radius, ω_0 is the oscillator frequency, and $f(t)$ is the external force depending upon the time t. As Eq. (1) contains the third time derivative, in order to obtain a unique solution it is not sufficient to give only the initial coordinate and velocity values, e.g., in the form

$$x=0,\quad \dot{x}=dx/dt=0 \quad \text{at} \quad t=-\infty \quad (2)$$

but it is, in addition, necessary to require the vanishing of the acceleration after the action of all forces disappears,[2,3]

$$d^2x/dt^2=0 \quad \text{at} \quad t=\infty. \quad (3)$$

Only if this condition is fulfilled are there no self-accelerating solutions after the external forces disappear. Thus, strictly speaking, the Cauchy problem (i.e., the initial value problem) cannot be formulated correctly if radiation damping is present.

The Green's function for Eq. (1) satisfying the relation

$$D(d/dt)G(t-t')=\delta(t-t'), \quad (4)$$

and the boundary conditions (2) and (3), have the form[3,5]

$$G(t-t')=\frac{1}{\gamma[\omega_2^2+(\omega_1+\omega_3)^2]}\times\begin{cases}\dfrac{\omega_2-i(\omega_1+\omega_3)}{2\omega_2}e^{-(\omega_1-i\omega_2)(t-t')}+\text{c.c.}, & t'<t\\ e^{\omega_3(t-t')}, & t'>t.\end{cases} \quad (5)$$

Here ω_3 and $-\omega_1\pm i\omega_2$ are the roots of the equation

$$D(\omega)=\omega^2+\omega_0^2-\gamma\omega^3=0,$$

ω_1, ω_2, and ω_3 being real positive quantities. Suppose that $\gamma\omega_0\ll 1$; then one obtains

$$\begin{aligned}&\omega_3\simeq(1/\gamma)(1+\gamma^2\omega_0^2);\quad \omega_2\simeq\omega_0(1-\tfrac{5}{8}\gamma^2\omega_0^2),\\ &\omega_1\simeq\tfrac{1}{2}\gamma\omega_0^2(1-2\gamma^2\omega_0^2).\end{aligned} \quad (6)$$

2. INVESTIGATION OF THE SOLUTIONS

At $t'<t$ the Green's function describes the retarded action of the force on the coordinates, and at $t'>t$ it describes the advanced action.

If $\omega_0\gamma\ll 1$, the advanced part of the Green's function, being proportional to $e^{\omega_3(t-t')}$, will assume nonvanishing values effectively only during a time interval $\Delta t\simeq\gamma$.

According to (5) the solution of Eq. (1) may be expressed in the form[6]

$$x(t)=\int_{-\infty}^{\infty}G(t-t')f(t')dt'. \quad (7)$$

If the function $f(t)$ can be represented as a Fourier integral,

$$f(t)=\int_{-\infty}^{\infty}\varphi(\omega)e^{i\omega t}d\omega, \quad (8)$$

then on substituting (8) into (7) and integrating over t', one obtains:

$$x(t)=\int_{-\infty}^{\infty}\frac{\varphi(\omega)e^{i\omega t}}{(i\omega)^2+\omega_0^2-\gamma(i\omega)^3}d\omega. \quad (9)$$

In our case, the change of the order of integrations is permissible for all functions possessing Fourier transforms since the function $D(i\omega)=(i\omega)^2+\omega_0^2-\gamma(i\omega)^3$ has

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
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[3] D. Ivanenko and A. Sokolov, *Klassische Feldtheorie* (German translation) (Akademie Verlag, Berlin, 1953).
[4] F. Rohrlich, Ann. Phys. (New York) **13**, 93 (1961).
[5] M. Schwartz, Phys. Rev. **123**, 1903 (1961).

[6] In general the solution of the homogeneous equation (eigenvibrations),

$$x_0(t)=C_1e^{-(\omega_1-i\omega_2)t}+C_2e^{-(\omega_1+i\omega_2)t}+C_3e^{\omega_3t}, \quad (7a)$$

is to be added to the solution (7), describing the forced oscillations. To satisfy the boundary condition (3) we must put $C_3=0$ (no self-acceleration solutions). Although the constants C_1 and C_2 may be nonzero in the case when the initial coordinate and velocity are given at a certain moment $t_1\neq\infty$, the terms containing them tend to zero at $t\to\infty$ due to the influence of damping.

no zeros on the real ω axis. Thus the solution (9) is equivalent to (7).[7]

An attempt was made[8] to interpret the quantum character of the motion of a charged particle as the result of the influence of fluctuation forces due to the second quantized transversal photon field on the classical model of a damped harmonic oscillator. In this case, the force $f(t)$ is to be put equal to

$$f(t)=-(e/m_0c)[\partial A(t)/\partial t]. \tag{10}$$

Here $A(t)$ is the x component of the transversal part of the vector potential at the origin. These quantities do not commute at different times[9]:

$$[A(t),A(t')]=-(4\hbar/3ic)(\partial/\partial t)\delta(t-t'). \tag{11}$$

Let us perform a Fourier transformation of $A(t)$ and use the method of dividing by an operator (see reference 8) to obtain x and p. Then one can show that the coordinate in the dipole approximation

$$x=\frac{e\hbar^{1/2}}{2\pi m_0}\int d^3k\omega^{1/2}\left[\frac{ia(\mathbf{k})e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega^3}+\text{c.c.}\right], \tag{12}$$

and the momentum

$$\begin{aligned}p&=m_0\frac{dx}{dt}-\gamma m_0\frac{d^2x}{dt^2}+\frac{e}{c}A(t)\\&=\frac{e\omega_0^2\hbar^{1/2}}{2\pi}\int\frac{d^3k}{\omega^{1/2}}\left[\frac{a^\dagger(\mathbf{k})e^{i\omega t}}{\omega_0^2-\omega^2+i\gamma\omega^3}+\text{c.c.}\right],\end{aligned} \tag{13}$$

become noncommuting operators[10]:

$$[p,x]=(\hbar/i)\gamma\omega_3/(3\gamma\omega_3-2). \tag{14}$$

To obtain these results the commutation relations

$$[a(\mathbf{k}),a^\dagger(\mathbf{k}')]=\tfrac{2}{3}\delta(\mathbf{k}-\mathbf{k}') \tag{15}$$

were taken into account. Here $ck=c|\mathbf{k}|=\omega$, and the factor $\frac{2}{3}$ arises from averaging over the angles; the integrals of Eqs. (12) and (13) possess spherical symmetry.

In the limit $\gamma\to 0$, i.e., $\gamma\omega_3\to 1$, Eq. (14) takes the form of Heisenberg uncertainty relation for the operators p and x:

$$[p,x]=\hbar/i. \tag{16}$$

We do not intend to discuss here the physical aspect of the problem. But it is necessary to return once again to mathematical reasoning leading to Eq. (16), as Schwartz in his paper[5] expressed some doubts on the validity of using the Fourier transform [see (9)], believing that dividing by an operator leads to "misleading solutions." In particular, Schwartz believes that starting from the exact solution when, according to (7),

$$\begin{aligned}x&=-\frac{e}{m_0c}\int_{-\infty}^{\infty}\frac{\partial G(t-t')}{\partial t}A(t')dt',\\p&=\frac{e\omega_0^2}{c}\int_{-\infty}^{\infty}G(t-t'')A(t'')dt'',\end{aligned} \tag{17}$$

we are led to the following commutation relations[11]:

$$\begin{aligned}&[m_0(dx/dt),x]=[\gamma m_0(d^2x/dt^2),x]=0,\\&(e/c)[A(t),x]=[p,x],\end{aligned} \tag{18}$$

whereas our solution in the form (9) is stated to give other results.

In this connection we must emphasize that, as was shown by our direct calculations, both methods yield completely identical results:

$$\begin{aligned}&\left[m_0\frac{dx}{dt},x\right]=\frac{2\hbar}{i(3\gamma\omega_3-2)},\\&\left[\gamma m_0\frac{d^2x}{dt^2},x\right]=0,\\&\frac{e}{c}[A(t),x]=-\frac{\hbar}{i}\frac{2-\gamma\omega_3}{3\gamma\omega_3-2}\neq[p,x].\end{aligned} \tag{19}$$

These results do not coincide with corresponding relations (18) of Schwartz.

3. SOME PARTICULAR CASES

Starting from the general solution (5), one may obtain the Green's function corresponding to various particular cases.

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[7] To obtain Eq. (9) from Eq. (7) one may use the following relation that holds also in the limiting case $t\to\infty$:

$$\int_t^\infty e^{\omega_3(t-t')}f(t')dt'=\frac{1}{\omega_3}\left(f(t)+\frac{1}{\omega_3}\dot f(t)+\frac{1}{\omega_3^2}\ddot f(t)+\cdots\right).$$

It is seen from Eq. (9) that when the constant force $\varphi(\omega)=A\delta(\omega)$ acts upon the damped oscillator, one obtains in the limit $t\to\infty$:

$$x(t)\to A/\omega_0^2;\quad \dot x(t)\to 0;\quad \ddot x(t)\to 0.$$

If, on the other hand, the force is harmonic, $\varphi(\omega)=(A/2i)\times[\delta(\omega-\omega_1)-\delta(\omega+\omega_1)]$, then not only the coordinate but the velocity and acceleration as well perform undamped oscillations in the limit $t\to\infty$.

[8] A. A. Sokolov and V. S. Tumanov, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 802 (1956) [translation: Soviet Phys.—JETP **3**, 958 (1957)]; see also A. A. Sokolov, *Introduction to Quantum Electrodynamics* (Moscow, 1958, in Russian), p. 190.

[9] The Eq. (11) may be obtained from the commutation relations for the x component of the transverse electromagnetic vector potential: $[A(\mathbf{r}_1t),A(\mathbf{r}',t')]=F(t,t',R)$. Here $R=|\mathbf{r}'-\mathbf{r}|$, and

$$F(t,t',R)=\frac{1}{\pi^2}\frac{c\hbar}{2i}\int\left(1-\frac{k_x^2}{k^2}\right)e^{i\mathbf{k}\mathbf{R}}\sin[ck(t-t')]\frac{d^3k}{k}.$$

Going to the limit $R\to 0$ (dipole approximation) one obtains Eq. (11).

[10] Note by the way that the factor 2 instead of 4 must be used in the right-hand side of the corresponding formula of Schwartz [see Eq. (39) of reference 5]. Only in this case will it go over into our Eq. (14). Besides that, the relations $\omega_1^2+\omega_2^2=\omega_3(\omega_3-1/\gamma)$ and $\omega_1=\frac{1}{2}(\omega_3-1/\gamma)$ must be taken into account.

[11] See reference 5, p. 1907.

(a) In the case of an undamped harmonic oscillator one must put $\gamma=0$ in Eq. (5). Then

$$G(t-t')=(1/\omega_0)\sin\omega_0(t-t'),\quad t'<t$$
$$=0,\quad t'>t. \tag{20}$$

Thus, in the absence of damping, only the retarded part of the solution will remain, i.e., we have the solution of the Cauchy problem and the final condition may be discarded.

Now let the force be given in the form of the expansion (8). Then the solution (7) leads to

$$x(t)=\int_{-\infty}^{\infty}\varphi(\omega)e^{i\omega t}\left(\frac{1}{\omega_0^2-\omega^2}-i\pi\frac{\omega}{\omega_0}\delta(\omega_0^2-\omega^2)\right)d\omega, \tag{21}$$

the principal value being taken of the integral of $(\omega_0^2-\omega^2)^{-1}$.

On the other hand, the direct dividing by an operator would lead to an analogous expression but without the term proportional to $\delta(\omega_0^2-\omega^2)$. Thus, Schwartz's assertion that the solutions (7) and (9) lead to different results applies only in cases when the function $D(i\omega)$ possesses zeros on the real axis (such as in the case of a harmonic oscillator), but it is quite irrelevant in the case of our Eq. (1), reference 8, since there the roots of the function $D(i\omega)$ lie in the complex plane.

(b) If the elastic force is absent but the damping is present, then taking the time derivative of the Green's function (5) and putting $\omega_0^2=0$, one obtains the following expression for the Green's function determining the velocity:

$$\dot{G}=1,\quad t'<t$$
$$=e^{(t-t')/\gamma},\quad t'>t. \tag{22}$$

Consider now the case when the force is nonvanishing in the time interval $t_1\leq t\leq t_2$. Solving the Cauchy problem, i.e., imposing the following initial values before the force sets in:

$$\dot{x}(t_1)=\text{const},\quad \ddot{x}(t_1)=0,$$

one obtains

$$\dot{x}(t)=\text{const},\quad t<t_1$$
$$=\int_{t_1}^{t}(1-e^{(t-t')/\gamma})f(t')dt'+\text{const},\quad t_1<t<t_2 \tag{23}$$
$$=\int_{t_1}^{t_2}(1-e^{(t-t')/\gamma})f(t')dt'+\text{const},\quad t>t_2.$$

It is seen that the mass point continues to move with acceleration after all the forces have stopped their action ($t>t_2$).

This self-accelerating solution is absent only if one imposes the boundary conditions (2) and (3). In this particular case they take the form

$$\dot{x}(t_1)=\text{const},\quad \ddot{x}(t_2)=0,$$

and one obtains

$$=\int_{t_1}^{t_2}e^{(t-t')/\gamma}f(t')dt'+\text{const},\quad t<t_1$$
$$\dot{x}(t)=\int_{t_1}^{t}f(t')dt'$$
$$+\int_{t}^{t_2}e^{(t-t')/\gamma}f(t')dt'+\text{const},\quad t_1<t<t_2$$
$$=\text{const},\quad t>t_2 \tag{24}$$

In the absence of damping, $\gamma\rightarrow 0$, Eq. (23) yields a diverging value, while Eq. (24) becomes the solution of the Cauchy problem.

Thus the problem of boundary conditions in the presence of damping (i.e., of the third derivative) is only partially investigated even in the simplest cases as yet considered. One of the most important possible alternatives has just been considered in our paper (see also Sokolov *et al.*[12])

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[12] A. A. Sokolov, B. A. Lysov, and M. M. Kolesnikova, Transactions of the Third All-Union Conference on Theoretical Physics, Uzhgorod, 1961 (in Russian).