Quantum Noise in Linear Amplifiers

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The classical definition of noise figure, based on signal-to-noise ratio, is adapted to the case when quantum noise is predominant. The noise figure is normalized to "uncertainty noise." General quantum mechanical equations for linear amplifiers are set up using the condition of linearity and the requirement that the commutator brackets of the pertinent operators are conserved in the amplification. These equations include as special cases the maser, the parametric amplifier, and the parametric up-converter. Using these equations the noise figure of a general amplifier is derived. The minimum value of this noise figure is equal to 2. The significance of the result with regard to a simultaneous phase and amplitude measurement is explored.

INTRODUCTION

THE availability of coherent signals at optical frequencies has stimulated research in their use for communication purposes. Ways of processing optical frequencies are considered that are similar to those of the low end of the coherent frequency spectrum. With the use of classical communication techniques, classical performance criteria will be applied. One purpose of this paper is to extend classical noise performance criteria to linear quantum amplifiers in which *the predominant noise is quantum mechanical* in nature. These criteria will be applied to a wide class of linear quantum mechanical amplifiers.

The purpose of a sensitive linear amplifier is to increase the power, or photon flux, of an incoming signal with as small a noise contamination as possible so that the signal may be conveniently detected at high power levels. The incoming signal, if used for communication purposes, carries amplitude modulation, phase modulation, frequency modulation, or some other type of modulation. Here we shall discuss noise problems mainly in the context of amplifiers processing sinusoidal carriers with narrow band amplitude and/or phase modulation. In this connection it must be noted that the presence of a modulation of bandwidth B calls for a minimum rate of detection. The received signal must be detected within a succession of observation times each of duration τ, where $\tau=\frac{1}{2}B$, in order to utilize the information contained in the modulation.

Noise in masers, including sponteneous emission noise, has been analyzed in many papers including the classical papers by Shimoda, Takahasi, and Townes,[1] and Serber and Townes.[2] A quantum mechanical treatment of the parametric amplifier and up-converter has been presented in a paper by Louisell *et al.*[3] We shall develop a unified set of equations for all "linear" amplifiers, special cases of which are the maser, the parametric amplifier, and the parametric up-converter. On the basis of these equations and the criteria of noise performance, it will be possible to present a proof on the limiting noise performance achievable by any one of these amplifiers used singly or in combination with other linear amplifiers. The connection of the fundamental noise of these amplifiers with the uncertainty principle will be studied.

I. NOISE FIGURE

In the noise theory of classical amplifiers (i.e., amplifiers operating with a very large number of quanta) the deterioration of the signal-to-noise ratio as the signal passes the amplifier is used as a measure of amplifier noise performance. The signal-to-noise ratio (SNR) is defined in the classical limit as the ratio of signal power to noise power. Mathematically one may describe a phase and amplitude modulated signal in the presence of noise by

$$A(t)=A_0(t)\cos[\omega_0 t+\phi_0(t)]+\delta A_p\cos[\omega_0 t+\phi_0(t)]+\delta A_q\sin[\omega_0 t+\phi_0(t)]. \quad (1.1)$$

The first term in this equation represents the signal in the absence of noise. The remaining two terms are the inphase and quadrature perturbations of the amplitude due to the noise. These are slowly varying with time if the noise is narrowband. We envisage an ensemble of identical signal waveforms with accompanying noise. The signal part may be extracted from the waveform by taking an ensemble average indicated by the brackets $\langle\ \rangle$

$$\langle A(t)\rangle=A_0(t)\cos[\omega_0 t+\phi_0(t)]. \quad (1.2)$$

[1] K. Shimoda, H. Takahasi, and C. H. Townes, J. Phys. Soc. Japan **12**, 686 (1957).

[2] R. Serber and C. H. Townes, in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960), pp. 233–255.

[3] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1646 (1961).

The noise part is extracted as follows:

$$\langle A(t)^2\rangle-\langle A(t)\rangle^2$$
$$=\langle\delta A_p{}^2\rangle\cos^2[\omega_0 t+\phi_0(t)]+\langle\delta A_q{}^2\rangle\sin^2[\omega_0 t+\phi_0(t)]$$
$$+\langle\delta A_p\delta A_q\rangle\sin 2[\omega_0 t+\phi_0(t)]. \quad (1.3)$$

Additive stationary noise is characterized by

$$\langle\delta A_p{}^2\rangle=\langle\delta A_q{}^2\rangle \quad \text{and} \quad \langle\delta A_p\delta A_q\rangle=0. \quad (1.4)$$

The noise as defined here measures the mean square deviation from the signal of the measured ensemble of waveforms all containing the same signal. If the noise is stationary, this mean square deviation becomes

$$\langle A(t)^2\rangle-\langle A(t)\rangle^2=\langle\delta A_p{}^2\rangle. \quad (1.5)$$

A signal-to-noise ratio may be defined, based on a time average, over an observation time T, long compared to the inverse bandwidth of signal and noise. The signal power is

$$S=\frac{1}{T}\int_0^T A_0{}^2(t)\cos^2(\omega_0 t+\phi_0)dt=\tfrac{1}{2}[A_0{}^2(t)]_{\text{av}}, \quad (1.6)$$

where the square bracket indicates a time average.

The noise is defined correspondingly as

$$N=\frac{1}{T}\int_0^T \{\langle A^2(t)\rangle-\langle A(t)\rangle^2\}dt$$
$$=\tfrac{1}{2}\{\langle\delta A_p{}^2\rangle+\langle\delta A_q{}^2\rangle\}_{\text{av}}=\langle\delta A_p{}^2\rangle. \quad (1.7)$$

The last equality holds for stationary noise. The noise figure of an amplifier is defined as the signal-to-noise ratio at the input of the amplifier divided by the signal-to-noise ratio at the output[4]

$$F=(S_i/N_i)/(S_0/N_0). \quad (1.8)$$

In defining S_i/N_i one envisages measurements of the signal and noise at the amplifier input and output by a noise-free measurement apparatus. The noise figure is usually defined by choosing a standard input noise corresponding to thermal noise of the input source of 290°K. If the amplifier is linear,

$$S_0=GS_i, \quad (1.9)$$

and the amplifier noise is additive,

$$N_0=GN_i+N_a, \quad (1.10)$$

then, one has for the "excess noise figure," $F-1$,

$$F-1=N_a/GN_i. \quad (1.11)$$

The excess noise figure measures the increase of the mean square deviation of the normalized signal as caused by the amplifier noise.

When adapting the noise figure of linear amplifiers for the quantum case, one faces two problems. First of all, one must establish that linear quantum mechanical amplifiers, like linear classical amplifiers, introduce additive noise. Secondly, in the limit when quantum effects are of importance, physical measurements, in general, introduce uncertainties, i.e., noise, and it is not clear that the concept of a noise-free measurement as envisaged in the classical noise figure definition can be applied. Thus, for example, simultaneous measurements of amplitude and phase of an electric field cannot be carried out with arbitrarily high precision but must obey the principle of complementarity. It is clear, therefore, that special measures have to be taken if the uncertainty introduced by the measurement is to be negligible compared to the noise in the system.

In quantum theory, a physical quantity is described by an operator. The expectation value of the operator represents the result of a set of measurements on an ensemble of identically prepared systems. In the evaluation of quantum noise we shall consider field amplitude measurements on an ensemble of amplifiers all of which are fed by a transmitter signal that is nonthermally attenuated (such as the attenuation of a radiation field by the inverse square law). At the transmitter the signal has a classical power level and, thus, can be accurately phase and amplitude controlled. If a measurement of the electric field $E(t)$ at the receiver inputs is performed at the time t, the signal is defined by $\langle E(t)\rangle$, and the noise by[5] $\langle E(t)^2\rangle-\langle E(t)\rangle^2$. By performing many such measurements at random time instants delayed with respect to each other by times long compared to the inverse bandwidth of the receivers, one finds the signal power

$$S=[\langle E(t)\rangle^2]_{\text{av}}, \quad (1.12)$$

and the noise power

$$N=[\langle E(t)^2\rangle-\langle E(t)\rangle^2]_{\text{av}}. \quad (1.13)$$

Such measurements do not mutually intefere, and, therefore, are not accompanied by a fundamental uncertainty. These measurements are analogous to the "noise-free" measurements implied in the classical signal-to-noise ratio definition. The signal-to-noise ratio is thus

$$S/N=[\langle E(t)\rangle^2]_{\text{av}}/[\langle E(t)^2\rangle-\langle E(t)\rangle^2]_{\text{av}}. \quad (1.14)$$

With the aid of definition (1.14), Eq. (1.8) may be used as the definition for the quantum noise figure. In the case of linear quantum amplifiers, as discussed in this paper, Eqs. (1.9), (1.10), and (1.11) are also made valid. It is not expedient to normalize the quantum noise figure to thermal noise. It is more appropriate to normalize it to the minimum noise within the observation time τ of $h\nu/2$ that is associated with attenuated signals that before attenuation were classically phase and amplitude controlled (see Appendix). Then, if we express the amplifier noise power multiplied by the observation time τ in terms of a photon number n_a at

[4] H. T. Friis, Proc. Inst. Radio Engrs. **33**, 458 (1945).

[5] R. Senitzky, Phys. Rev. **111**, 3 (1958).

the amplifier output,

$$N_a = h\nu n_a, \tag{1.15}$$

one has

$$F-1=2n_a/G. \tag{1.16}$$

Since the basic noise energy to which we compare the amplifier noise corresponds to half a photon within the observation time, it is natural to measure the output noise content, referred to the input by division through G, in terms of the energy of half a photon *at the output frequency*. In this sense, Eq. (1.16) can be applied to the general case for different input and output frequencies if G is interpreted as the *photon number gain*.

II. THE EQUATIONS OF LINEAR AMPLIFIERS

Consider a system of weakly interacting particles that, in the absence of radiation, has N available energy levels. If the particles obey Bose-Einstein statistics, the entry of a particle into a particular level (i) and its exit from this level may be described by creation and annihilation operators, $b_i^\dagger$ and b_i that obey commutator relations.

$$[b_i, b_j^\dagger] = \delta_{ij}, \tag{2.1}$$

$$[b_i, b_j] = 0. \tag{2.2}$$

The operation of b_i on a wave function of the system with n_i particles in level i produces a wave function with $n_i - 1$ particles in that level, the operation of $b_i^\dagger$ produces one with $n_i + 1$ particles in the level i. The number of particles in level i is given by

$$n_i = b_i^\dagger b_i. \tag{2.3}$$

If the particles obey Fermi-Dirac statistics, a Bose-Einstein description is possible,[2] provided that the number of available states in a particular level is much greater than the number of particles occupying it. If the states within a particular energy level are all equivalent for the physical processes envisaged, wave functions may be constructed that correspond to a discrete number of particles in a particular energy level with no distinction made between different states within that level.[6] Creation operators $b_i^\dagger$ and annihilation operators b_i that increase or decrease the number of particles in a particular energy level upon operation on a wave function can be shown to obey the commutation relations (2.1) and (2.2), provided the number of particles in the energy level is much less than the number of states of the level.[7] Linear quantum amplifiers must of necessity operate by means of transitions from and to weakly occupied levels since nonlinear effects occur as soon as the deviation from Bose-Einstein behavior of these levels becomes appreciable.

The electromagnetic field is represented by the creation and annihilation operators $a_i^\dagger$ and a_i for photons in the modes i, j, etc., of the electromagnetic system. These obey the usual commutation relations

$$a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}, \tag{2.4}$$

$$a_i a_j - a_j a_i = 0. \tag{2.5}$$

When the fields are made to interact with matter, coupling results among the various operators of the system. In the interaction representation, differential equations in time are obtained relating the time rate of change of every operator to all the other operators of energy levels and electromagnetic modes partaking in the interaction. These equations are, in general, nonlinear. However, under proper "biasing" or "pumping" conditions linear interactions may result among some of the energy-level and photon operators of the system. Thus, consider a system like the three-level maser in which a pumping excitation establishes a steady state in the occupation of two energy levels. If a small signal is applied with a frequency corresponding to the energy difference between one of the pumped levels and an intermediate level, the small perturbations produced in the steady state of the strongly excited levels may be disregarded, and linear equations are obtained for the operators of the weakly excited level and the photons of the small applied signal. In more complicated devices, more level and photon operators may be particpating in the interaction initiated by the applied small signal. In any case, if the device is to act as a linear amplifier for some photons it is a necessary requirement that the equations of motion permit a linearization of the equations for the signal photon operators and the operators directly involved in the small signal interaction. If the amplifier is to be time independent, a further requirement is that the coefficients of the differential equations be time independent. These two requirements strongly restrict the form of the equations. Integration of the differential equations in time leads to linear relations among the operators at an initial time t_0, and final time t_1.

Taking as an example a system within which a photon operator interacts with an energy level operator, one has two possible cases:

$$\begin{aligned} a(t_1) &= M_{aa}a(t_0) + M_{ab}b^\dagger(t_0), \\ b(t_1) &= M_{ba}a(t_0) + M_{bb}b^\dagger(t_0), \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} a(t_1) &= M_{aa}a(t_0) + M_{ab}b(t_0), \\ b(t_1) &= M_{ba}a(t_0) + M_{bb}b(t_0). \end{aligned} \tag{2.7}$$

The first system of equations is of the type obtained by Serber and Townes[2] for the ideal maser. In this system, the creation operator $b^\dagger$ of the energy level couples to the annihilation operator a of the photons. In the second system the annihilation operators of the energy level b couples to a. We shall later see the significance of this

[6] For example, if one is interested only in the interaction with an electromagnetic field of the spin of weakly interacting particles, all states of the particles that have the same spin are equivalent.

[7] H. A. Haus, Internal Memorandum, Massachusetts Institute of Technology Research Laboratory of Electronics, 1962 (unpublished).

latter type of coupling. An analysis of the parametric amplifier leads to equations of the form (2.6) where $b^\dagger$ is replaced by a creation operator representing the photons at the idler frequency.[3]

The general form of the linear equations is immediately apparent for a linear system of arbitrary degree of complexity. If one comprises all operators representing "output" quantities at $t=t_1$ in a column matrix $\mathbf{v}$, those representing "input" quantities at $t=t_0$ in a column matrix $\mathbf{u}$, one may write the general linear relations in matrix form as

$$\mathbf{v}=\mathbf{M}\mathbf{u}. \tag{2.8}$$

In the example of Eq. (2.6)

$$\mathbf{u}=\begin{bmatrix} a(t_0) \\ b^\dagger(t_0) \end{bmatrix}, \quad \text{and} \quad \mathbf{v}=\begin{bmatrix} a(t_1) \\ b^\dagger(t_1) \end{bmatrix}.$$

The requirements of linearity and time independence have led to an equation of the form of (2.8). There is another important condition that has to be met by Eq. (2.8) which restricts the matrix $\mathbf{M}$: The commutator brackets of the operators must be conserved. Indeed, considering a system with photons and Bose-Einstein particles, the commutator brackets are invariants of the complete (nonlinear) equations of motion. The linearized equations that describe the small signal interactions, must conserve commutator brackets insofar as they are good approximations to the behavior of the same set of operators in the complete equations. Consider next a system with photons and Fermi-Dirac particles. One notes that the complete equations of motion preserve the commutator brackets of the photon operators and anticommutator brackets of the particle operators. As mentioned before, one may construct effectively Bose-Einstein operators from the input Fermi-Dirac operators. Corresponding operators may be constructed for the output. These will also be effectively Bose-Einstein provided the number of particles in the level of interest has remained small compared to the number of states in the level (as needs be if the amplification is to be linear). Therefore, the commutator brackets of the constructed operators are also conserved in the amplification. Let us study the matrix $\mathbf{C}$ consisting of the commutator brackets of the operators contained in the input matrix $\mathbf{u}$. It is easily seen that this matrix is constructed by

$$\mathbf{C}=\mathbf{u}\mathbf{u}^\dagger-(\mathbf{u}_t{}^\dagger\mathbf{u}_t)_t. \tag{2.9}$$

Here the subscripts t indicate the transpose. The operators obey initially the commutation relations (2.1), (2.2), (2.4), and (2.5). Photon operators and level operators commute. The commutator relations can be conveniently summarized by defining a matrix $\mathbf{P}$ of the same order as $\mathbf{u}$ by

$$\mathbf{P}=\text{diag}(\pm1,\pm1,\pm1,\cdots\pm1), \tag{2.10}$$

where for the ith diagonal element, the plus sign is chosen when an annihilation operator appears in the ith row of the matrix $\mathbf{u}$, and the minus sign if a creation operator appears. Using this definition for $\mathbf{P}$, one has from Eq. (2.9)

$$\mathbf{C}=\mathbf{u}\mathbf{u}^\dagger-(\mathbf{u}_t{}^\dagger\mathbf{u}_t)_t=\mathbf{P} \tag{2.11}$$

at $t=t_0$. Because

$$\mathbf{C}(t_1)=\mathbf{C}(t_0), \tag{2.12}$$

one has from Eqs. (2.8) and (2.12)

$$\begin{aligned}\mathbf{v}\mathbf{v}^\dagger-(\mathbf{v}_t{}^\dagger\mathbf{v}_t)_t=\mathbf{P}=\mathbf{M}\mathbf{u}\mathbf{u}^\dagger\mathbf{M}^\dagger-[(\mathbf{u}^\dagger\mathbf{M}^\dagger)_t(\mathbf{M}\mathbf{u})_t]_t \\ =\mathbf{M}(\mathbf{u}\mathbf{u}^\dagger-\mathbf{u}_t{}^\dagger\mathbf{u}_t)\mathbf{M}^\dagger=\mathbf{M}\mathbf{P}\mathbf{M}^\dagger.\end{aligned}$$

It follows that

$$\mathbf{M}\mathbf{P}\mathbf{M}^\dagger=\mathbf{P}. \tag{2.13}$$

One may easily check that in the special cases of an ideal maser and parametric amplifier mentioned previously, the expressions as derived in references 2 and 3 obey the relation (2.13). Here this relation has been derived in general as a consequence of the requirement of conservation of commutator brackets. It is this relation that imposes a fundamental limit on the noise performance of all linear amplifiers. It is of interest to show that in an amplifier with photon gain greater than unity the signal creation (annihilation) operator must couple to at least one annihilation (creation) operator of a molecular state or of a photon of frequency different from the signal frequency. Suppose the output photons pertain to the operator a_j, the input photons to a_k. Let these, in turn, correspond to v_m or $v_m{}^\dagger$ and u_n or $u_n{}^\dagger$. If the signal level is sufficiently large, one may disregard the uncertainty contained in the commutator relations, i.e., one may set

$$\langle v_m{}^\dagger v_m\rangle=n_m\cong\langle v_m v_m{}^\dagger\rangle, \tag{2.14}$$

where n_m is the number of output photons,

$$\langle u_n{}^\dagger u_n\rangle=n_n\cong\langle u_n u_n{}^\dagger\rangle, \tag{2.15}$$

with n_n, the number of input photons. All the other input quantities are assumed unexcited

$$\langle u_i{}^\dagger u_i\rangle=0,\quad \langle u_i u_i{}^\dagger\rangle=1,\quad i\neq n. \tag{2.16}$$

We thus have from Eqs. (2.8), (2.14)–(2.16),

$$n_m\cong|M_{mn}|^2 n_n. \tag{2.17}$$

$|M_{mn}|^2$ is the photon gain of the amplifier. But, from Eq. (2.13)

$$\textstyle\sum_i P_{ii}|M_{mi}|^2=P_{mm}. \tag{2.18}$$

We recall that $P_{ii}=\pm1$, depending upon whether u_i stands for an annihilation operator or creation operator. Thus, this equation shows that $|M_{mn}|^2$ is necessarily less than unity, unless at least one of the P_{ii}'s is of opposite sign to P_{mm}. A simple situation exists when u_m interacts with only one other operator u_l,

$$|M_{mm}|^2+(P_{ll}/P_{mm})|M_{ml}|^2=1. \tag{2.19}$$

The ideal maser, Eq. (2.6), uses the same input and output frequency, the photons of which couple to a single level operator $b^\dagger$ ($=u_l$). The gain $|M_{mm}|^2$ is greater than unity, because P_{ll}/P_{mm} is equal to -1.

The parametric amplifier using the same input and output frequency couples the signal photon operator $a_1=u_m$ to the idler photon operator $a_2^\dagger=u_l$. Again $P_{ll}/P_{mm}=-1$ and the gain $|M_{mm}|^2>1$.

The ideal parametric up-converter couples the input signal frequency operator $a_1=u_m$ to the output frequency operator $a_2=u_l$. $P_{ll}/P_{mm}=1$ and the gain $|M_{ml}|^2\lessgtr 1$.

A system within which a photon operator $a=u_m$ couples to a level operator $b=u_l$ exhibits gain less than, or at most equal to, unity since $P_{ll}/P_{mm}=1$. [Compare Eq. (2.7).]

An interesting special case is the "lossless scatterer," with the P_{ii}'s all of the same sign. The coupling between a transmitting antenna and a receiver may be represented in this way. Here all u's represent annihilation operators a_i pertaining to the same frequency, but different spatial modes. The relation (2.13) assures conservation of the total photon number, but only a fraction $|M_{jk}|^2$ of the input (transmitter) photons represented by, say $\langle u_k^\dagger u_k\rangle$ reach the receiver as output photons $\langle v_j^\dagger v_j\rangle$.

III. AMPLIFIER NOISE

It should be recalled that the u's contain the a's and b's, or their Hermitian conjugates. In the study of signal and noise we are interested in averages of expectation values of electric field components, whose operators are given by

$$E_j(t)=k_j i(a_j^\dagger e^{i\omega_j t}-a_j e^{-i\omega_j t}), \tag{3.1}$$

where k_j is some real constant dependent upon the geometry of the system.

Let the photon operator a_j, or $a_j^\dagger$, be represented by u_j, so that

$$\langle E_j(t)\rangle=\pm k_j i\langle u_j^\dagger e^{\pm i\omega_j t}-u_j e^{\mp i\omega_j t}\rangle$$
$$=\pm 2k_j|\langle u_j\rangle|\cos(\omega_j t+\Phi), \tag{3.2}$$

and

$$[\langle E_j(t)\rangle^2]_{\text{av}}=2k_j^2[|\langle u_j\rangle|^2]_{\text{av}}=2k_j^2[|\langle u_j^\dagger\rangle|^2]_{\text{av}}, \tag{3.3}$$

because the average indicated by the square bracket is equivalent to a time average. Similarly,

$$[\langle E_j(t)^2\rangle]_{\text{av}}=k_j^2[\langle u_j^\dagger u_j+u_ju_j^\dagger\rangle]_{\text{av}}. \tag{3.4}$$

Since the final expressions are symmetric in u and $u^\dagger$ and do not depend on time, both cases $u_j=a_j$ and $u_j=a_j^\dagger$ lead to the same final expressions for $[\langle E(t)\rangle^2]_{\text{av}}$ and $[\langle E(t)^2\rangle]_{\text{av}}$. Thus one may use the u's directly to obtain the required averages.

Suppose the matrix $\mathbf{M}$ of Eq. (2.8) represents an amplifier with its input described by the operator u_n and its output by v_m. Then the output signal-to-noise ratio is given by [compare Eqs. (1.14) and (3.2)–(3.4)]

$$\frac{S_0}{N_0}=\frac{[2|\langle v_m\rangle|^2]_{\text{av}}}{[\langle v_m^\dagger v_m\rangle+\langle v_mv_m^\dagger\rangle-2|\langle v_m\rangle|^2]_{\text{av}}}. \tag{3.5}$$

Now, from Eq. (2.8)

$$v_m=\textstyle\sum_i M_{mi}u_i. \tag{3.6}$$

Because all inputs but the signal input m are fed by incoherent noise, we have for i, $j\neq m$, n

$$\langle u_i\rangle=\langle u_i^\dagger\rangle=0, \tag{3.7}$$

$$\begin{aligned}\langle u_i^\dagger u_j\rangle&=0, \text{ if } u_i \text{ stands for an annihilation operator,}\\ &=\delta_{ij}, \text{ if } u_i \text{ stands for a creation operator,}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\langle u_iu_j^\dagger\rangle&=0, \text{ if } u_i \text{ stands for a creation operator,}\\ &=\delta_{ij}, \text{ if } u_i \text{ stands for an annihilation operator.}\end{aligned}$$

Thus

$$\langle v_m\rangle=M_{mn}\langle u_n\rangle, \tag{3.9}$$

and

$$\langle v_m^\dagger v_m\rangle+\langle v_mv_m^\dagger\rangle-2|\langle v_m\rangle|^2$$
$$=\sum_{i\neq n}|M_{mi}|^2+|M_{mn}|^2\{\langle u_n^\dagger u_n+u_nu_n^\dagger\rangle-2|\langle u_n\rangle|^2\}. \tag{3.10}$$

But the input noise is taken as the noise accompanying a signal after a large radiative attenuation. This noise expressed in terms of a photon number is equal to the uncertainty noise of $\frac{1}{2}$ photon as shown in the Appendix. The input signal-to-noise ratio to which the noise figure will be normalized has the same form as Eq. (3.5) except that u_n replaces v_m and is equal to

$$S_i/N_i=2[n_n]_{\text{av}}, \tag{3.11}$$

where $[n_n]_{\text{av}}$ is the average number of input photons. Introducing Eqs. (3.9)–(3.11) into Eq. (3.5) and using the noise figure definition (1.8), we have

$$F=(\textstyle\sum_i|M_{mi}|^2)/|M_{mn}|^2. \tag{3.12}$$

The input signal has dropped out as is characteristic of a linear amplifier. The noise figure F of (3.12) can be made unity. It is found, however, that the gain $|M_{mn}|^2$ of the "amplifier" is then also necessarily equal to unity. If one wants gain, F must be optimized for fixed gain. Another way of accomplishing this is to use the definition of "noise measure"[8] M,

$$M=(F-1)/(1-1/G). \tag{3.13}$$

Here G is the photon number gain, in the present case $|M_{mn}|^2$. The noise measure M has a nontrivial minimum as we now proceed to show. We obtain

$$M=(\textstyle\sum_{i\neq n}|M_{mi}|^2)/(|M_{mn}|^2-1). \tag{3.14}$$

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[8] H. A. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks* (Technology Press, Cambridge, Massachusetts, 1959).

According to Eq. (2.13)

$$\sum_i P_{ii}|M_{mi}|^2 = P_{mm}.$$

Thus

$$|M_{mn}|^2 - 1 = -\sum_{i\neq n}(P_{ii}/P_{nn})|M_{mi}|^2 + (P_{mm}/P_{nn}) - 1. \quad (3.15)$$

Since we are, in general, interested in photon gain, $|M_{mn}|^2 > 1$, it follows that the right-hand side of (3.15) must be positive. In such a case, since $P_{ii}/P_{nn} = \pm 1$, it follows immediately that

$$\sum_{i\neq n}|M_{mi}|^2 \geq -\sum_{i\neq n}(P_{ii}/P_{nn})|M_{mi}|^2 + (P_{mm}/P_{nn}) - 1. \quad (3.16)$$

Thus, from (3.14)–(3.16) we find

$$M > 1. \quad (3.17)$$

At high gain $M = F - 1$. It follows that in that case

$$F \geq 2. \quad (3.18)$$

The quantum noise introduced in an amplifier leads to a noise-to-signal ratio at its output double of that at its input, if the input signal-to-noise ratio is assumed to be that of uncertainty noise. Considering specific devices, we note that the ideal maser characterized by Eq. (2.6) has a noise measure of unity, because the equality sign in Eq. (3.18) applies, as one can easily see. The same is true for the parametric amplifier. In general, one can see that coupling of the signal photon annihilation operator to other annihilation operators tends to increase the noise figure by virtue of the fact that negative terms appear in the sum on the right-hand side of Eq. (3.16).

We have introduced the concept of noise measure because it possessed a nontrivial minimum, whereas the noise figure could be made equal to unity at a complete sacrifice of gain. There are other advantages in the use of "noise measure" that have been successfully employed in the analysis of noise in classical linear amplifiers.[8] Using the concept of noise measure one may state in simple terms the limit on the optimum noise performance of a passive interconnection of linear two-port amplifiers (amplifiers with one input terminal pair and one output terminal pair). In the classical circuit-theoretical application for which this theorem was originally derived it states[8] that any passive interconnection of two-port amplifiers resulting in an overall two-port amplifier (amplifier with one input terminal pair and one output terminal pair) leads to an optimum (minimum) noise measure that cannot be lower than that of the best amplifier, namely, the amplifier with the lowest value of optimum noise measure. This proof can now be easily extended to the case of an interconnection of linear quantum amplifiers. In the quantum case, two-port amplifiers are those that are described by one input signal photon operator and one output signal photon operator as discussed in this section. A passive interconnection is one with a net positive (or zero) internal loss of photons. Systems with linear frequency transformations are not ruled out. Thus the proof in the present section, extended along the lines of reference 8, in effect shows that the optimum noise measure of a linear quantum mechanical system with one input and one output cannot be better than unity ($F = 2$ at high gain).

IV. SIMULTANEOUS PHASE AND AMPLITUDE MEASUREMENTS

The technical purpose of amplifiers is to raise signal levels so that signal processing may be effected at conveniently high power levels. In the process, amplifiers must introduce as little noise as possible. Since we have found that all linear quantum amplifiers introduce noise, one may ask the question whether, in those cases in which ultimate sensitivity is desired, one should not dispense with linear amplification. We shall discuss this question as one of principle, although it is clear that technical requirements may call for linear amplification for reasons other than those considered here.

We have defined the quantum noise figure on the basis of a noise-free measurement. In this context it is immediately apparent that the use of an amplifier is not desired in those cases in which a "noise-free" measurement can be administered to the incoming signal, namely, an instantaneous amplitude measurement alone.

If one envisages simultaneous amplitude and phase measurement the situation is not that simple and deserves further study.[2,9] Suppose we intend to measure with an ideal measuring apparatus the phase ϕ_0 and amplitude A_0 of an incoming wave. The measurement introduces an rms uncertainty in phase $\Delta\phi$ and in photon number Δn such that at best

$$\Delta n \Delta\phi = \tfrac{1}{2}. \quad (4.1)$$

But if A_0^2 is measured in units of power, Δn is related to the inphase uncertainty of amplitude ΔA_p by (cf. Eq. (1.1)]

$$\tau A_0(\langle\delta A_p^2\rangle)^{1/2} = \tau A_0 \Delta A_p = h\nu\Delta n, \quad (4.2)$$

where ΔA_p is defined as $(\langle\delta A_p^2\rangle)^{1/2}$. Further

$$\Delta\phi = \Delta A_q / A_0. \quad (4.3)$$

We thus have

$$\tau\Delta A_p \Delta A_q = \tfrac{1}{2}h\nu \quad (4.4)$$

for an ideal measurement apparatus. If the measurement apparatus is preceded by an ideal linear amplifier of high gain then the measurement does not have to introduce an uncertainty beyond that introduced by the amplifier noise. The minimum amplifier noise at high gain referred to the input is equal to that caused by quantum attenuation and, when measured in terms of energy, is according to Eq. (3.18) equal to half a

[9] H. Heffner, Proc. Inst. Radio Engrs. 50, 1604 (1962).

photon within an observation time. Further, the noise is stationary. It follows from (1.4)

$$\langle\delta A_p{}^2\rangle=\langle\delta A_q{}^2\rangle. \tag{4.5}$$

Using Eq. (1.7),

$$\tau\langle\delta A_p{}^2\rangle=\tau(\Delta A_p)^2=\tfrac{1}{2}h\nu. \tag{4.6}$$

Using (4.5) and (4.6) we find that the uncertainty introduced into the final measurement by the amplifier noise is just equal to that introduced by an ideal detector not preceded by an amplifier, Eq. (4.4). It is characteristic of linear amplifiers that the final measurement results in equal inphase and quadrature uncertainties whereas a measurement performed without the use of a preamplifier may choose the relative magnitudes of each, subject only to Eq. (4.1).

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APPENDIX

Radiative Attenuation Noise

Using Eq. (2.8) to represent radiative attenuation, we interpret u_l as the annihilation operator of the input photons at the transmitter, v_k as the annihilation operator of the output photons at the receiver. All the other operators in $\mathbf{u}$ represent spatial modes not used for the transmission process and are unexcited, $\langle u_i\rangle=0$, $\langle u_i{}^\dagger u_i\rangle=0$, $\langle u_iu_i{}^\dagger\rangle=1$ for $i\neq l$. We have for the signal-to-noise ratio at the receiver, as in Eq. (3.5),

$$\left(\frac{S}{N}\right)_{\rm rec}=\frac{2[|\langle v_k\rangle|^2]_{\rm av}}{[\langle v_k{}^\dagger v_k+v_kv_k{}^\dagger\rangle-2|\langle v_k\rangle|^2]_{\rm av}}. \tag{A1}$$

Introducing Eq. (2.8), where according to Eq. (2.13), with $\mathbf{P}=\mathbf{1}$,

$$\mathbf{MM}^+=\mathbf{1} \tag{A2}$$

one has

$$\left(\frac{S}{N}\right)_{\rm rec}=\frac{2|M_{kl}|^2[|\langle u_l\rangle|^2]_{\rm av}}{(|M_{kl}|^2[u_l{}^\dagger u_l+u_lu_l{}^\dagger-2|\langle u_l\rangle|^2]_{\rm av}+\sum_{i\neq k}|M_{ki}|^2)}. \tag{A3}$$

The first term in the denominator represents the attenuated noise of the transmitted signal. The additive noise introduced in the transmission process is represented by $\sum_{i\neq k}|M_{kl}|^2$ and is stationary. If the attenuation is very large, $|M_{kl}|^2\ll 1$, and if the noise accompanying the signal at the input is not inordinately larger than that imposed by the uncertainty relation, then this term is negligible compared to the second one. Then the noise at the receiver is entirely determined by the zero-point fluctuations of the modes other than the one used for transmission regardless of the input noise. Further, using Eq. (A2)

$$\sum_{i\neq l}|M_{ki}|^2\cong\sum|M_{ki}|^2=1. \tag{A4}$$

Finally,

$$|M_{kl}|^2[|\langle u_l\rangle|^2]_{\rm av}\cong[n]_{\rm av},$$

the average number of received photons as long as this number is much larger than unity. We find

$$(S/N)_{\rm rec}\cong 2[n]_{\rm av}.$$