stantially, it is not certain whether the best solution (statistically) without the consideration of inelastic effects remains the best when such effects are taken into account, and vice versa.

If, as has been assumed here, the nucleon is indeed a bound state in the πN system, its mass and the πN coupling constant could be calculated by solving Eqs. (4) and (6) together with (1), (3), and (5) to obtain $N(W)$ and $D(W)$, and then finding M and γ from $D(-M)=0$ and Eq. (9). The nucleon pole would not have to be inserted at the beginning of the calculation, because, as we have seen, it is absent from $N(W)$. It is true that the pole parameters in the crossed channel would indeed contribute to these equations in the complete problem. But, we could treat their values as variables, and then find those values for which the calculated parameters equal the assumed ones. Inelastic effects may be quite important in such a calculation, however. A similar procedure could be followed for any bound state.

ACKNOWLEDGMENTS

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Low-Energy Pion-Pion Scattering*

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A general method for solving the problem of low-energy pion-pion scattering is presented. Within each partial wave, the nearby unphysical singularities are calculated by the usual Chew-Mandelstam approach, while the more distant ones are treated by a generalization of the Ball-Wong technique. These two techniques are then combined with the requirement of self-consistency. A rough calculation is given in which only a narrow P-wave resonance is retained, and the only free parameter is the pion mass. The resulting resonance has a mass of 585 MeV and a half-width of 125 MeV. A method is also given for calculating Regge poles at low energies.

I. INTRODUCTION

THERE have been several attempts at calculating the low-energy pion-pion amplitude from the requirements of analyticity, elastic unitarity, crossing symmetry, and self-consistency.[1–5] Of these, only the method of Zachariasen[4] does not involve any arbitrary parameters in the physically interesting P-dominant case, at least if only the lowest order term is retained. However, when this method is extended to higher orders, one has to introduce cutoffs,[6] and so one once again introduces such parameters. The main difficulty in all of these calculations arises from the strong, incalculable, distant, unphysical singularities within each partial wave.

In the method presented here, which does not contain any arbitrary parameters, the nearby unphysical singularities are treated by the usual polynomial method of CM-I (reference 1). An effective-range formula is then set up to represent the remaining unphysical singularities. The parameters of this formula can be determined by requiring that the resulting partial-wave amplitude have the correct value and derivatives at some point between these singularities and the physical region. The amplitude at this point can, in turn, be calculated from the absorptive part in the crossed channel, through a fixed momentum-transfer dispersion relation. This absorptive part is always expandable in Legendre polynomials in the region of interest. Thus we have a self-consistency problem in which we must find partial-wave amplitudes such that the assumed forms equal the calculated ones.

Finally, the same type of procedure can be followed for complex values of the angular momentum l, once the physical problem has been solved at low energies. The main usefulness of such a calculation is that it gives the trajectories of the poles in the complex-l plane, as they move with energy. These Regge poles, in turn, dominate high-energy scattering in the crossed channel.[7,8]

* This work was done under the auspices of the U. S. Atomic Energy Commission.

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); hereafter referred to as CM-I.

[2] G. F. Chew and S. Mandelstam, Nuovo cimento **19**, 752 (1961); hereafter referred to as CM-II.

[3] V. V. Serebryakov and D. V. Shirkov, J. Exptl. Theoret. Phys. (U.S.S.R.) (to be published).

[4] F. Zachariasen, Phys. Rev. Letters **7**, 112, 268 (1961).

[5] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **7**, 390 (1961), to which the reader is also referred for additional references on the π-π problem.

[6] F. Zachariasen and C. Zemach (private communication).

[7] G. F. Chew, S. C. Frautschi and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

[8] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962), to which the reader is referred for additional references on Regge poles.

II. THE EFFECTIVE-RANGE FORMULA

If the amplitude for a given isotopic spin state I is $A_I(\nu, \cos\theta)$, the partial-wave amplitude is

$$A_{(l)I}(\nu)=\frac{1}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)A_I(\nu,\cos\theta), \quad (1)$$

where $\nu=(s/4)-1$, s is the square of the total energy with pion mass $\mu=1$, and θ is the scattering angle in the barycentric system. When this is solved by the N/D method, we have from CM-I,

$$A_{(l)I}(\nu)=N_l{}^I(\nu)/D_l{}^I(\nu), \quad (2)$$

with

$$N_l{}^I(\nu)=A_{(l)I}(\nu_0)+\frac{\nu-\nu_0}{\pi}\times\int_{-\infty}^{-1} d\nu' \frac{\mathrm{Im}A_{(l)I}(\nu')D_l{}^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}, \quad (3)$$

and

$$D_l{}^I(\nu)=1-\frac{\nu-\nu_0}{\pi}\int_0^{\infty} d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{R_l{}^I(\nu')N_l{}^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}, \quad (4)$$

where we have made subtractions at some point ν_0. The function $R_l{}^I(\nu)$ is the ratio of total to elastic partial-wave cross section. In the elastic approximation we have $R_l{}^I(\nu)=1$, a relation which is exact for $0<\nu<3$.

Now for any range $\nu_L<\nu<-1$, with $\nu_L>-9$, the function $\mathrm{Im}A_{(l)I}(\nu)$ can be calculated from the partial-wave cross sections in the crossed channel through Eq. (IV.7) of CM-I, according to which

$$\mathrm{Im}A_{(l)I}(\nu)=\frac{2}{\nu}\int_0^{-\nu-1} d\nu' P_l\left(1+2\frac{\nu'+1}{\nu}\right)\times\mathrm{Im}\tilde{A}_I\left(\nu', 1+2\frac{\nu+1}{\nu'}\right), \quad (5)$$

where

$$\mathrm{Im}\tilde{A}_I\left(\nu', 1+2\frac{\nu+1}{\nu'}\right)=\sum_{l'=0}^{\infty}(2l'+1)\sum_{I'=0}^{2}\beta_{II'}\,\mathrm{Im}A_{(l')I'}(\nu')\times P_{l'}\left(1+2\frac{\nu+1}{\nu'}\right), \quad (6)$$

with

$$\beta_{II'}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}.$$

To treat the region $-\infty<\nu<\nu_L$, we use an approach first applied to nucleon-nucleon[9] and pion-nucleon[10] scattering for setting up effective-range formulas. Putting $\nu'=-x^{-1}$ for $\nu'<\nu_L$, Eq. (3) becomes

$$N_l{}^I(\nu)=A_{(l)I}(\nu_0)+\frac{\nu-\nu_0}{\pi}\int_{\nu_L}^{-1} d\nu' \frac{\mathrm{Im}A_{(l)I}(\nu')D_l{}^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}+\frac{\nu-\nu_0}{\pi}\int_0^{x_L}\frac{dx}{x}\frac{\mathrm{Im}A_{(l)I}(-x^{-1})D_l{}^I(-x^{-1})}{(\nu_0+x^{-1})(1+x\nu)}, \quad (7)$$

where $x_L=-\nu_L{}^{-1}$. We can now approximate the kernel in the second integral in the same way as in S (reference 9). Using, say, an interpolation formula as in Eq. (6) of S, we have, for a given range of ν and for $0<x<x_L$,

$$\frac{1}{1+x\nu}\approx\sum_{i=1}^{n}\frac{G_i(x)}{1+x_i\nu}, \quad (8)$$

which inserted into Eq. (7) gives

$$N_l{}^I(\nu)=A_{(l)I}(\nu_0)+\frac{\nu-\nu_0}{\pi}\int_{\nu_L}^{-1} d\nu' \frac{\mathrm{Im}A_{(l)I}(\nu')D_l{}^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}+(\nu-\nu_0)\sum_{i=1}^{n}\frac{F_{(l)I}{}^i}{\omega_i+\nu}, \quad (9)$$

where $\omega_i=x_i{}^{-1}$ and the $F_{(l)I}{}^i$ are constant parameters. The last term has the same form as an n-pole formula. In this case, of course, the positions of these poles are not free parameters. The accuracy of this term can be easily estimated, being of the same order as the approximation (8). That possible oscillations, even growing ones in $\mathrm{Im}A_{(l)I}(\nu)$ should cause no difficulty was shown in Appendix A of S.

In states with $l>0$, l of the $F_{(l)I}{}^i$ can be eliminated by the condition that $A_{(l)I}(\nu)$ go to zero as ν^l for small ν. [This is automatically guaranteed if we consider $\nu^{-l}A_{(l)I}(\nu)$ as is done in CM-I, in which case a subtraction in $N_l{}^I(\nu)$ would likewise be unnecessary. However, this method is somewhat more inconvenient in other respects.]

III. EVALUATION OF THE EFFECTIVE-RANGE PARAMETERS

To evaluate the remaining parameters in Eq. (9), we use the fixed momentum-transfer dispersion relation (CM-I)

$$A_I\left(\nu(s), 1+\frac{t}{2\nu(s)}\right)=\frac{1}{\pi}\int_4^{\infty} dt'\,\mathrm{Im}\tilde{A}_I\left(\nu(t'), 1+\frac{s}{2\nu(t')}\right)\times\left[\frac{1}{t'-t}+\frac{(-1)^I}{t'-(4-s-t)}\right], \quad (10)$$

where $t=-2\nu(s)(1-\cos\theta)$, and $\nu(s)=(s/4)-1$. From Fig. 1 we see that the integration is along some line $s=$constant. But, from the Regge pole analysis of Chew,

[9] L. A. P. Balázs, Phys. Rev. **125**, 2179 (1962); hereafter referred to as S.

[10] L. A. P. Balázs, preceding paper [Phys. Rev. **128**, 1935 (1962)].

Frautschi, and Mandelstam,[7] we have along such a line

$$\mathrm{Im}\tilde{A}_I\left(\nu(t), 1+\frac{s}{2\nu(t)}\right)\propto t^{\alpha_I(s)}, \quad \text{as} \quad t\to\infty, \qquad (11)$$

where the $\alpha_I(s)$ are shown in Fig. 2.[8] From this diagram, and from Eq. (10), we see that in the region of convergence of the polynomial expansion (6), i.e., for $4>s>-32$, the integral in Eq. (10) converges for $I=1$ and 2, while an S-wave subtraction has to be made for $I=0$.

If we substitute Eq. (10) into Eq. (1), we have

$$A_{(l)I}(\nu)=\frac{4}{\pi\nu}\int_0^\infty d\nu'\,\mathrm{Im}\tilde{A}_I\left(\nu', 1+2\frac{\nu+1}{\nu'}\right)\times Q_l\left(1+2\frac{\nu'+1}{\nu}\right), \qquad (12)$$

where we have used the fact that Legendre functions of the second kind $Q_l(z)$ obey the relation $Q_l(z)=(-1)^{l+1}Q_l(-z)$, and that in the pion-pion problem, $(-1)^l=(-1)^I$. We can now determine $A_{(l)I}(\nu_0)$ and $(n-l)$ of the $F_{(l)I}{}^i$ by requiring that $A_{(l)I}(\nu)$ as given by Eqs. (2), (4), and (9) and $(n-l)$ of its derivatives be equal to the corresponding quantities as given by Eq. (12) at $\nu=\nu_F$. The point ν_F may be taken anywhere in the region $\nu_L<\nu<0$, except at the branch points -1 and -4, where the higher derivatives are infinite. The most appropriate value of ν_F is one not too close to ν_L, but which at the same time avoids the strong peaking of the higher waves in the crossed channel that occurs in the neighborhood of $s=0$. A natural choice would, therefore, be the lowest point of the central triangle in Fig. 1, i.e., at $s=-4$ or $\nu_F=-2$. It is also usually convenient to put $\nu_0=\nu_F$, although it may be more appropriate to make ν_0 coincide with one of the $-\omega_i$ in an accurate calculation, since this reduces the number of parameters.

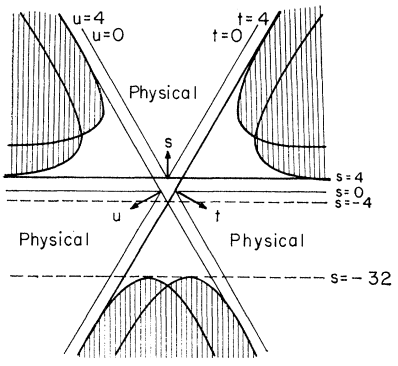

FIG. 1. The Mandelstam diagram for π-π scattering, with $u=4-s-t$. The double spectral functions are nonvanishing in the shaded regions. The expansion (6) converges between $s=4$ and $s=-32$.

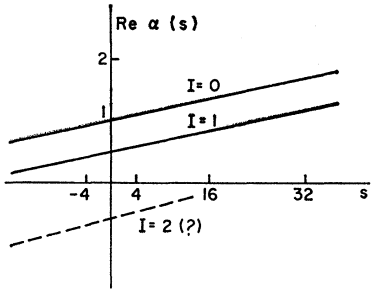

FIG. 2. Schematic plots of the real parts of the complex angular momentum $\alpha_I(s)$ as functions of s for the topmost Regge poles. These functions are real for $s<4$.

The above procedure is actually a generalization of a technique used by Ball and Wong in the $\pi\pi\to N\bar{N}$ problem.[11] These authors, however, chose the branch point -1 for ν_F, which enabled them to evaluate only the value and first derivative at $\nu=\nu_F$. An alternative generalization would be to take the value, and perhaps derivatives, at several points in the region $\nu_L<\nu<0$.

In Eq. (12), $\mathrm{Im}\tilde{A}_I$ can always be evaluated from the partial-wave cross sections in the crossed channel through Eq. (6). This expansion always converges at $\nu=\nu_F$, as is obvious from Fig. 1. At large values of ν' it may be more convenient to approximate this function by retaining the contribution of only a small number of Regge poles.[7,8] This contribution is quite small, however.

The method of this section can be carried to any order of accuracy if $R_l{}^I(\nu)$ is known, or, equivalently, if inelastic effects are inserted. This is because the accuracy of the approximation (8) can be increased indefinitely as n increases, and any number of derivatives can be taken at $\nu=\nu_F$. It is, in fact, essentially an analytic continuation from $\nu=\nu_F$ into the physical region of the function

$$N_l{}^I(\nu)-A_{(l)I}(\nu_0)-\frac{\nu-\nu_0}{\pi}\int_{\nu_L}^{-1}d\nu'\frac{\mathrm{Im}A_{(l)I}(\nu')D_l{}^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)},$$

which is free of singularities in the region $\nu_L<\nu<\infty$. Such a function can be regarded in some sense as the partial-wave amplitude from which the physical and nearby unphysical singularities have been removed by unitarity and crossing, respectively.

Since we have to make an S-wave subtraction in the $I=0$ state, the above procedure cannot be applied to the $I=0$, $l=0$ state. But, once the other waves have been calculated, the constants $A_{(0)0}(\nu_0)$ and $F_{(0)0}{}^i$ can be computed by using the exact crossing conditions of CM-II (reference 2). As there is an infinite number of such conditions, this state can also be determined to any order of accuracy.

IV. THE P-WAVE APPROXIMATION

We now illustrate the preceding techniques in the approximation of consistently neglecting everything except the P wave. This is a valid approximation because, although the S wave in the crossed channel distorts the P-wave amplitude at $\nu=\nu_F$, it can be shown that the

[11] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **6**, 29 (1961).

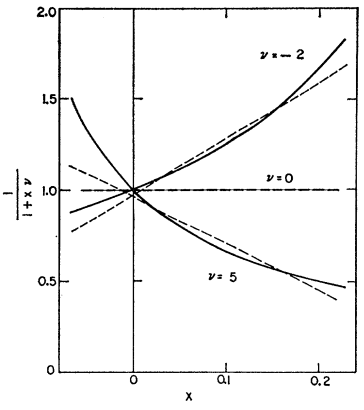

FIG. 3. Plots of $(1+x\nu)^{-1}$ (solid lines) and the approximate form given by Eqs. (8) and (13) (dashed lines) for various values of ν.

amplitude at the resonance will not be affected much by a consistent neglect of the S wave. We shall also use a zero-width resonance approximation. From Eqs. (5) and (6) this means that if the resonance is at $\nu=\nu_R$, the left-hand cut starts at $\nu=-\nu_R-1$. If we take $\nu_L=-\nu_R-1$, this in turn means that the first integral in Eqs. (7) and (9) vanishes. In addition, we take $R_l{}^I(\nu)=1$. This is a reasonable approximation for $0<\nu\lesssim 10$ (CM-I).

In the range $0<\nu\lesssim 5$, which we would expect to be the dominant region, we can approximate the kernel $(1+x\nu)^{-1}$ by a straight line for $0<x\lesssim 0.21$, which is the appropriate interval if $\nu_R\gtrsim 3.5$. In other words, in Eq. (8) we set $n=2$, and put

$$G_{1,2}(x)=(x-x_{2,1})/(x_{1,2}-x_{2,1}), \tag{13}$$

with $x_1=0.16$ and $x_2=0.02$, which corresponds to $\omega_1=6.25$ and $\omega_2=50$ in Eq. (9). Within this range of ν, the accuracy is of the order of several percent, as can be seen from Fig. 3. Of course the accuracy continues to be reasonable for much higher values of ν.

Now the above approximations obviously distort $R_1{}^1(\nu)N_1{}^1(\nu)$ at large values of ν. This distortion arises from the inadequacy of both the elastic approximation and the effective-range formula at high energies. However, as was shown in reference 10, Eqs. (3) and (4) are still valid provided that we add to the right side of Eq. (3) the additional term

$$\Delta N_1{}^1(\nu)=\frac{\nu-\nu_0}{\pi}\int_{\nu_I}^{\infty}d\nu'\,\frac{\mathrm{Im}[A_{(1)1}(\nu')D_1{}^1(\nu')]}{(\nu'-\nu_0)(\nu'-\nu)}$$
$$=\frac{\nu-\nu_0}{\pi}\int_{x_I}^{0}\frac{dx}{x}\,\frac{\mathrm{Im}[A_{(1)1}(-x^{-1})D_1{}^1(-x^{-1})]}{(\nu_0+x^{-1})(1+x\nu)}, \tag{14}$$

where $\nu_I=-x_I{}^{-1}$ is the lowest value of ν at which this distortion begins to become large. But the approximation given by Eq. (8) holds also for $x_I<x<0$, as can be seen from Fig. 3. Thus, the form of Eq. (9) is unchanged by the addition of $\Delta N_1{}^1(\nu)$ if we insert the approximation (8) into Eq. (14).

To set up the zero-width resonance approximation we first use the fact that, for a dynamical resonance, since

$$\mathrm{Re}D_1{}^1(\nu_R)=0 \tag{15}$$

we may to a good approximation[12] put $\mathrm{Re}D_1{}^1(\nu)\approx-(\nu-\nu_R)/(\nu_R-\nu_0)$, and $N_1{}^1(\nu)\approx(\nu/\nu_R)N_1{}^1(\nu_R)$, as can be seen *a posteriori* from Fig. 4. From Eq. (2), since $\mathrm{Im}D_1{}^1(\nu)=-[\nu/(\nu+1)]^{1/2}N_1{}^1(\nu)$, this leads to the Breit-Wigner form for $\nu>0$, namely,

$$\mathrm{Im}A_{(1)1}(\nu)=\frac{\nu(\Gamma_1{}^1)^2[\nu^3/(\nu+1)]^{1/2}}{(\nu-\nu_R)^2+(\Gamma_1{}^1)^2[\nu^3/(\nu+1)]}, \tag{16}$$

where

$$\nu_R\Gamma_1{}^1=(\nu_R-\nu_0)N_1{}^1(\nu_R). \tag{17}$$

In the zero-width approximation, Eq. (16) becomes

$$\mathrm{Im}A_{(1)1}(\nu)=\pi\nu_R\Gamma_1{}^1\delta(\nu-\nu_R). \tag{18}$$

Taking $\nu_0=\nu_F=-2$, the constants $A_{(1)1}(\nu_0)$, $F_{(1)1}{}^1$, and $F_{(1)1}{}^2$ can now be calculated from the requirements that $A_{(1)1}(0)=0$ and that Eqs. (9) and (4) give the same values of $A_{(1)1}(\nu_0)$ and $A_{(1)1}{}'(\nu_0)$ as would be obtained from Eqs. (12), (6), and (18). We can then calculate ν_R and $\Gamma_1{}^1$ through Eqs. (9), (4), (15), and (17). Thus we have to find ν_R and $\Gamma_1{}^1$ such that these calculated values equal the original assumed ones. This is a straightforward, if somewhat tedious calculation that can be carried out by hand and does not entail any numerical integration. It leads to $\nu_R=3.4$ and $\nu_R\Gamma_1{}^1=2.6$. The corresponding values of $A_{(1)1}(-2)$, $F_{(1)1}{}^1$, and $F_{(1)1}{}^2$ are -0.0979, -1.326, and 13.05, respectively, which can be used to calculate the phase shift $\delta_1{}^1$ through Eq. (V.20) of CM-I, according to which

$$[\nu/(\nu+1)]^{1/2}\cot\delta_1{}^1=\mathrm{Re}D_1{}^1(\nu)/N_1{}^1(\nu). \tag{19}$$

The corresponding cross section is plotted in Fig. 5. The half-width of the distribution is 125 MeV, while the position of the resonance, at which $\delta_1{}^1=90$ deg, is 585 MeV.

V. CALCULATION OF REGGE POLE TRAJECTORIES

It has been proposed by Froissart and shown by Squires that Eq. (12) can be used to uniquely continue

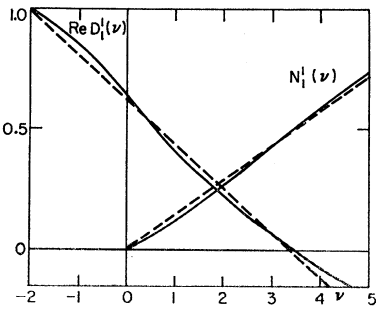

FIG. 4. The functions $N_1{}^1(\nu)$ and $\mathrm{Re}D_1{}^1(\nu)$ calculated from Eqs. (9) and (4) (solid lines) compared with the approximate forms $(\nu/\nu_R)N_1{}^1(\nu_R)$ and $-[(\nu-\nu_R)/(\nu_R-\nu_0)]$ (dashed lines), with the parameters of Sec. IV.

[12] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), p. 78.

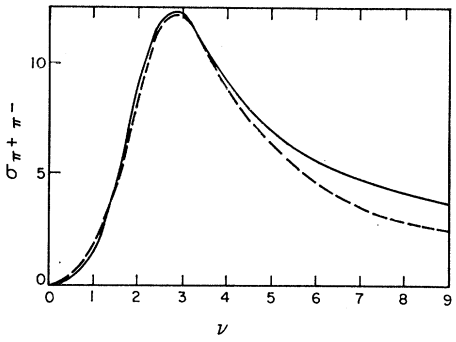

FIG. 5. The $I=1$, P-wave total cross section $=(12\pi/\nu)\sin^2\delta_1{}^1$ $=(12\pi/\nu)[\nu/(\nu+1)]^{1/2}\,\mathrm{Im}A_{(l)1}(\nu)$ calculated from Eqs. (9), (4), and (19) (solid lines), and from Eq. (16) (dashed line), with the parameters of Sec. IV.

the function $A_{(l)I}(\nu)$ to complex values of l.[13] Actually, in this case it is more appropriate to consider the function $B_I(\nu,l)=\nu^{-l}A_{(l)I}(\nu)$, since $A_{(l)I}(\nu)$ is singular for $-1<\nu<0$, whereas $B_I(\nu,l)$ is not.[14] Now for physical l, the N/D solution in the elastic approximation,[1] except for possible subtractions, has the form

$$\nu^{-l}A_{(l)I}(\nu)=B_I(\nu,l)=N_I(\nu,l)/D_I(\nu,l), \tag{20}$$

where

$$N_I(\nu,l)=\frac{1}{\pi}\int_{-\infty}^{-1}d\nu'\,\frac{\mathrm{Im}B_I(\nu',l)D_I(\nu',l)}{\nu'-\nu}, \tag{21}$$

and

$$D_I(\nu,l)=1-\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'^{2l+1}}{\nu'+1}\right)^{1/2}\times\frac{N_I(\nu',l)}{(\nu'-\nu_0)(\nu'-\nu)}. \tag{22}$$

It has been proposed by Prosperi that equations of this type, together with Eq. (12), can now be continued to complex l.[15]

The treatment of the above equations can now be carried out exactly as in Secs. II and III. In general, however, because the effective-range approximation distorts the high-energy behavior of $\nu^l N_I(\nu,l)$, the integral in Eq. (22) will not converge. Such a difficulty can be removed by actually imposing the requirement that $\nu^l N_I(\nu,l)$ have, at least approximately, the correct asymptotic behavior. This provides an additional condition for determining the effective-range parameters. Inelastic effects can be taken into account by the method of the third paragraph of the preceding section for small ν. For complex l, it is important, not so much to solve for the partial-wave amplitude as it is to find the poles of $B_I(\nu,l)$, i.e., to find those values of $l=\alpha(\nu_P)$ for which at some value $\nu=\nu_P$,

$$D_I(\nu_P,\alpha(\nu_P))=0. \tag{23}$$

Equation (23) is thus the equation of motion of the topmost Regge trajectory for a given value of I, i.e., the relation between the position $\alpha(\nu_P)$ in the complex-l plane and the energy. The residue of this pole in $B_I(\nu,l)$ as a function of ν for a given l is

$$\Gamma_I(\nu_P,\alpha(\nu_P))=-N_I(\nu_P,\alpha(\nu_P))\times[\partial D_I(\nu,\alpha(\nu_P))/\partial\nu]_{\nu=\nu_P}{}^{-1}. \tag{24}$$

The residue of the corresponding pole in $A_{(l)I}(\nu)$ in the complex-l plane is then

$$\beta(\nu_P)=\nu_P{}^{\alpha(\nu_P)}\Gamma_I(\nu_P,\alpha(\nu_P))[d\alpha(\nu)/d\nu]_{\nu=\nu_P}. \tag{25}$$

Now this method can be used only for $\mathrm{Re}l>\mathrm{Re}\alpha(\nu_F)$, since Eq. (12) diverges for smaller values of $\mathrm{Re}l$. To calculate high-energy cross sections,[7,8] however, we need $\alpha(\nu)$ and $\beta(\nu)$ for $\nu<0$, which can only partially be obtained by the above procedure. The remaining part can be calculated from the dispersion relations

$$\alpha(\nu)=\frac{1}{\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}\alpha(\nu')}{\nu'-\nu}, \tag{26}$$

and

$$\beta(\nu)=\frac{1}{\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}\beta(\nu')}{\nu'-\nu}, \tag{27}$$

where we have assumed that $\alpha(\nu)$ and $\beta(\nu)$ each has only the right-hand cut.[16] Suppose now that ν_B is the largest value of ν for which $\alpha(\nu)$ and $\beta(\nu)$ can be calculated. Then, should Eqs. (26) and (27) prove insufficient because of a large contribution from $\nu>\nu_B$, one may represent the singularities in that region by effective-range formulas as in Sec. II. The parameters of these formulas may then be determined by requiring that they give the correct $\alpha(\nu)$ and $\beta(\nu)$ and perhaps their derivatives at some point or points ν_A, where $\alpha(\nu_A)>\alpha(\nu_F)$, and $\nu_A<\nu_B$.

The above procedure only gives the topmost Regge pole for a particular isotopic spin state. To obtain the next pole, we must first subtract out the contribution of the topmost pole for $\mathrm{Im}\tilde{A}_I$ in Eq. (12), and then explicitly insert that pole into $B_I(\nu,l)$. This is to assure the necessary convergence of Eq. (12). We may now proceed as before in calculating this next pole. This procedure may be repeated any number of times.

VI. CONCLUSION

A general method for calculating the low-energy pion-pion amplitude has been given, both for physical and unphysical angular momenta. The method takes into account both long-range and short-range forces (nearby and distant unphysical singularities), the latter being calculated with the help of the fixed momentum-transfer dispersion relation (10). Incidentally, such a relation was also used by Chew and Mandelstam (CM-II). These authors, however, used it essentially to calculate the left-hand cut, a calculation leading to

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[14] A. O. Barut and D. Zwanziger (private communication).
[15] G. Prosperi (private communication).
[16] P. G. Burke (private communication).

divergences in the more distant singularities. In the present work, this relation is used to calculate the full amplitude at $\nu=\nu_F$, which does not lead to any divergences.

A rough calculation, in which only a sharp P-wave resonance is retained, gives a mass of 585 MeV and a half-width of 125 MeV. The mass is smaller and the width larger than the latest experimental values of 725 and 75 MeV, respectively.[17,18] This is not unreasonable, however, in view of the crude approximations that were made. Moreover, the experimental pion-pion cross sections are deduced from pion production experiments with the help of simple models which themselves may not be very reliable. The calculated width is also larger than that obtained by Serebryakov and Shirkov,[3] whose width is smaller than the experimental value. On the other hand, it is smaller than the width obtained by Chew and Mandelstam (CM-II) and Zachariasen.[4] The latter also calculated the mass of the resonance without arbitrary parameters and obtained 350 MeV, which is smaller than the value calculated here.

It is interesting to note that the calculation of Serebryakov and Shirkov depends quite sensitively on the assumed value of the pion-pion coupling constant, whereas all the other calculations do not. This appears to be a peculiarity of the particular approximation scheme used by these authors, rather than a feature of the general theory. Indeed, as was first pointed out by Chew,[19] a calculation of the $I=2$ state, where an S-wave subtraction is unnecessary, would automatically give this constant. Such a calculation is certainly possible with the present method.

The above technique can, of course, be applied to other strong-interaction problems. In such problems one must also deal with energetically unavailable quasi-physical regions, for instance, the region $0<\nu<(m^2-1)$ in the process $\pi\pi\to N\bar{N}$, where $m=$nucleon mass. While such regions are strictly speaking unphysical, the techniques used to handle them are essentially the same as for the physical regions.

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APPENDIX. HIGH-ENERGY EFFECTS AT LOW ENERGIES

We shall now discuss the mechanism of the third paragraph of Sec. IV in greater detail. It essentially arises from the nonuniqueness of the N/D method. There is, in fact, an infinite number of N/D methods. Suppose that, instead of defining $D_l^I(\nu)$ by Eq. (4), we define it by

$$D_l^I(\nu)=1+\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\,\frac{\mathrm{Im}D_l^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}, \quad \text{(A1)}$$

with

$$\mathrm{Im}D_l^I(\nu)=-\left(\frac{\nu}{\nu+1}\right)^{1/2}\mathrm{Re}N_l^I(\nu) \quad \text{for} \quad \nu<\nu_I, \quad \text{(A2)}$$

$$=H_l^I(\nu) \quad \text{for} \quad \nu>\nu_I, \quad \text{(A3)}$$

where $H_l^I(\nu)$ is any function for which the integral of (A1) does not diverge, and ν_I any value of ν such that the relation $R_l^I(\nu)=1$ is valid, at least approximately, for $\nu<\nu_I$.

We can now find the singularities of $N_l^I(\nu)=A_{(l)I}(\nu)D_l^I(\nu)$. It is obvious that this function cannot have any singularities in addition to those possessed by $A_{(l)I}(\nu)$ and $D_l^I(\nu)$. Thus it has, at most, a right-hand cut running from 0 to $+\infty$ and a left-hand cut running from -1 to $-\infty$. But, from Eqs. (2) and (A2), combined with the elastic unitarity condition $\mathrm{Im}[A_{(l)I}^{-1}(\nu)]=-[\nu/(\nu+1)]^{1/2}$, it follows that $N_l^I(\nu)$ is real for $0<\nu<\nu_I$. In other words, the discontinuity across the right-hand cut in the interval $0<\nu<\nu_I$ is zero. This, in turn, means that $N_l^I(\nu)$ satisfies the dispersion relation

$$N_l^I(\nu)=A_{(l)I}(\nu_0)+\frac{\nu-\nu_0}{\pi}\int_{-\infty}^{-1}d\nu'\,\frac{\mathrm{Im}A_{(l)I}(\nu')D_l^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}$$
$$+\frac{\nu-\nu_0}{\pi}\int_{\nu_I}^\infty d\nu'\,\frac{\mathrm{Im}[A_{(l)I}(\nu')D_l^I(\nu')]}{(\nu'-\nu_0)(\nu'-\nu)}, \quad \text{(A4)}$$

since, according to Eq. (A1), $D_l^I(\nu)$ is real for $\nu<0$. Thus, Eq. (7) is unchanged except for the additional term

$$\Delta N_l^I(\nu)=\frac{\nu-\nu_0}{\pi}$$
$$\times\int_{x_I}^0\frac{dx}{x}\,\frac{\mathrm{Im}[A_{(l)I}(-x^{-1})D_l^I(-x^{-1})]}{(\nu_0+x^{-1})(1+x\nu)}. \quad \text{(A5)}$$

Suppose now that the approximation (8) is such that it also holds for $x_I<x<0$. Then, if we insert that approximation into Eq. (A5), we find that the form of Eq. (9) is unchanged by the addition of $\Delta N_l^I(\nu)$, although the $F_{(l)I}{}^i$ will, of course, be different. Now this is true no matter what $H_l^I(\nu)$ may be. In particular, $H_l^I(\nu)$ may be just the function obtained by using Eq. (A2) for $\nu>\nu_I$, but with $N_l^I(\nu)$ given by Eq. (9) instead of Eq. (A4). This justifies the use of Eqs. (4) and (9) for $\nu<\nu_I$, despite the failure of Eq. (9) and the elastic approximation at large values of ν' within the integral of Eq. (4).

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[19] G. F. Chew (private communication).