$I=\frac{1}{2}$, $J=\frac{1}{2}$ State in πN Scattering with the Nucleon as a Bound State

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An approximate formula for the $I=\frac{1}{2}$, $J=\frac{1}{2}$ phase shift is derived. Assuming that the nucleon is a bound state, this formula depends only on the pion and nucleon masses, and the πN coupling constant (f^2). No purely phenomenological parameters are involved. The formula is roughly consistent with experiment in the 0- to 220-MeV range.

I. INTRODUCTION

IN the S-matrix theory recently proposed by Chew,[1] it is postulated that none of the strongly interacting particles is elementary. Thus, all stable particles must be bound states, and their masses and coupling constants can, in principle, be determined dynamically. Although a complete calculation would be beyond present techniques, it should be possible to test such a postulate by an experimental study of various scattering processes. In particular, one may investigate the consequences of the assumption that the nucleon is a bound state on the low-energy behavior of the P wave $I=\frac{1}{2}$, $J=\frac{1}{2}$ state in πN scattering, since the nucleon pole is present in this state.[2]

Therefore, making this assumption, a two-parameter formula is first set up to approximate the effect of the interaction singularities. The parameters are then adjusted so as to give a bound-state pole with just the position and residue of the nucleon pole. This removes all arbitrary constants and leaves a formula which depends only on the nucleon and pion masses, and the πN coupling constant. Such a procedure is similar to the one followed in the ^3S state in n-p scattering, where the scattering length and effective range can be calculated from the position and residue of the deuteron pole; i.e., from the binding energy and square of the asymptotic normalization coefficient for the bound-state wave function of the deuteron.[1] This procedure can also be compared with that followed by Chew and Low,[3] who correlated the parameters of the interaction singularities with the position and width of the 3-3 resonance (which can be considered as an unstable bound state). The only important difference in the present case is that part of the interaction singularities coincides with the nucleon pole, a difficulty which does not arise for the deuteron and the 3-3 resonance.

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[2] For a possible test at high energies, see G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

II. THE $I=\frac{1}{2}$, $J=\frac{1}{2}$ PARTIAL-WAVE AMPLITUDE

Consider the partial-wave amplitude[4]

$$g(W)=\frac{W^2}{(W+M)^2-1}\,\frac{\eta_1(W)\exp[2i\delta_1(W)]-1}{2iq(W)}, \tag{1a}$$

$$=\frac{-W^2}{(W+M)^2-1}\times\frac{\eta_{11}(-W)\exp[2i\delta_{11}(-W)]-1}{2iq(-W)}, \tag{1b}$$

where we have used the MacDowell reflection property,[5] and where, taking the pion mass to be unity, W= total energy in the barycentric system; M= nucleon mass; δ_1, $\eta_1=S$ wave $I=\frac{1}{2}$, $J=\frac{1}{2}$ phase shift and inelastic parameter; δ_{11}, $\eta_{11}=P$ wave $I=\frac{1}{2}$, $J=\frac{1}{2}$ phase shift and inelastic parameter; and

$$q(W)=\tfrac{1}{2}W^{-1}\{[(W+M)^2-1][(W-M)^2-1]\}^{1/2}. \tag{2}$$

Since we are concerned only with the amplitude on the real axis, we shall replace the correct singularity structure[4] of $g(W)$, as shown in Fig. 1(a), by the equivalent structure of Fig. 1(b). That this is always possible to any order of approximation can be seen quite readily from the familiar Coulomb-law analogy (see Chapter I of reference 1). Furthermore, we shall only consider the δ_{11} phase shift, because the corresponding physical cut, running from $-M-1$ to $-\infty$, is quite close to the nucleon pole, which is at $W=-M$.

To solve the resulting problem, we use the N/D method of Chew and Mandelstam.[6] We write

$$g(W)=N(W)/D(W), \tag{3}$$

where we define $D(W)$ to be a function with a cut in the left-hand physical region, and obeying the subtracted

[4] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960). See also W. R. Frazer and J. R. Fulco, *ibid.* **119**, 1420 (1960).

[5] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

[6] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

dispersion relation

$$D(W)=1+\frac{W+M+1}{\pi}\times\int_{-\infty}^{-M-1} dW'\frac{\operatorname{Im}D(W')}{(W'-W)(W'+M+1)},\quad(4)$$

with

$$\operatorname{Im}D(W)=-W^{-2}q(-W)[(W+M)^2-1]\operatorname{Re}N(W),\quad\text{for}\quad W<-M-1,\quad(5)$$

where the function is approached from above the cut. This corresponds to $N(W)$ being real for $-M-1>W>-W_I$, as can be seen from Eqs. (1b) and (3). Here W_I is the threshold of the inelastic region.

We can now find the singularities of $N(W)=g(W)D(W)$. It is obvious that this function has no singularities in addition to those possessed by $g(W)$ and $D(W)$. Since, as we saw, $\operatorname{Im}N(W)=0$ for $-M-1>W>-W_I$, the dispersion relation for $N(W)$ becomes

$$N(W)=\frac{1}{\pi}\int_{-(M^2+2)^{1/2}}^{\infty} dW'\frac{\operatorname{Im}g(W')D(W')}{W'-W}+\frac{1}{\pi}\int_{-\infty}^{-W_I} dW'\frac{\operatorname{Im}[g(W')D(W')]}{W'-W}.\quad(6)$$

The second integral comes from inelastic effects. In practice, the integrand is small until W' corresponds to 600 MeV, and so we may effectively take $W_I\approx M+4$, even though the correct value is smaller.

Equations (4) and (6)—together with (1), (3), and (5)—can be solved if we know $\operatorname{Im}g(W)$ for $W>-M-1$ and $\eta_{11}(-W)$ for $W<-W_I$.[7] If, as we are assuming here, the nucleon is a bound state, we then have $D(-M)=0$[8]; i.e., there is a pole in $D^{-1}(W)$ at $W=-M$. Hence there is no pole in $N(W)$, since the corresponding delta function in $\operatorname{Im}g(W)$ is canceled by $D(W)$, which is zero at the same point in Eq. (6). To obtain the residue of this pole, we expand

$$\frac{1}{g}=\frac{D(-M)}{N(-M)}+(W+M)\left[\frac{\partial}{\partial W}\left(\frac{D}{N}\right)\right]_{W=-M}+\cdots.\quad(7)$$

Since $D(-M)=0$, this means that, near $W=-M$,

$$g(W)=\frac{-1}{-M-W}\left[\frac{\partial}{\partial W}\left(\frac{D}{N}\right)\right]_{W=-M}^{-1}=\frac{-[N(-M)/D'(-M)]}{-M-W};\quad(8)$$

i.e., we have a pole in $g(W)$ at $W=-M$, with residue

$$\gamma=-[N(-M)/D'(-M)].\quad(9)$$

It is useful to note that we may always redefine $\operatorname{Im}D(W)$ by distorting $\operatorname{Re}N(W)$ in Eq. (5) into any other function $F(W)$ for $W<-W_I$ without changing the form of Eq. (6), provided only that the integrals (4) and (6) still exist. If $F(W)$ is such that on solving the resulting equations we are still able to obtain a bound state, then that bound state must be the nucleon, since there is none other in this problem.

III. THE APPROXIMATION

We shall first neglect all unphysical singularities for $W<W_R\approx-M+1$ except for the short cut at $W=-M$ arising from the nucleon pole in the crossed channel, which we shall call the S cut. This is equivalent to neglecting the nearby portions of the circular cut in Fig. 1(a), and should not be unreasonable, because this cut depends on $\pi\pi$ scattering and the $\pi\pi$ resonance has a large mass, which corresponds mostly to the more distant portions of the cut. Nevertheless, this approximation can be expected to introduce some error into the function $N(W)$ at $W\approx-M$, in any quantitative calculation. For $W<-M-1$, however, this error is much smaller because, in addition to $D(W')$ being small in the neighborhood of $W'=-M$, $(W'-W)$ is much larger in Eq. (6).

Thus, putting $W'=(x^{-1}-M)$ and combining the two integrals (except for the S-cut contribution) in Eq. (6), we have

$$N(W)=\frac{1}{\pi}\int_{-(M^2+2)^{1/2}}^{-M+(1/M)} dW'\frac{\operatorname{Im}g(W')D(W')}{W'-W}+\frac{1}{\pi}\int_{x_I}^{x_R} dx\frac{\operatorname{Im}[g(x^{-1}-M)D(x^{-1}-M)]}{x[1-(W_A+M)x]}\times\frac{1-(W_A+M)x}{1-(W+M)x},\quad(10)$$

where $x_R=(W_R+M)^{-1}\approx1$, $x_I=(M-W_I)^{-1}\approx-0.25$ (effectively), and W_A is some convenient value of W in the range of interest. We shall take $W_A=-1.5-M$.

Now within the first integral in Eq. (10)—since the interval is small, and since $D(-M)=0$—we may put

$$D(W)\approx D'(-M)(W+M).\quad(11)$$

To treat the second integral in Eq. (10), we shall use an approach first applied to nucleon-nucleon scattering for obtaining approximate formulas depending on only a small number of parameters.[9] From Fig. 2, we see that in the range $0>(W+M)>-2.25$ we may make the

[7] This is not the most convenient method in an actual calculation, but is more appropriate for setting up effective-range formulas. For an alternative method see, for instance, M. Froissart, Nuovo cimento **22**, 191 (1961).

[8] G. F. Chew and. S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

[9] L. A. P. Balázs, Phys. Rev. **125**, 2179 (1962).

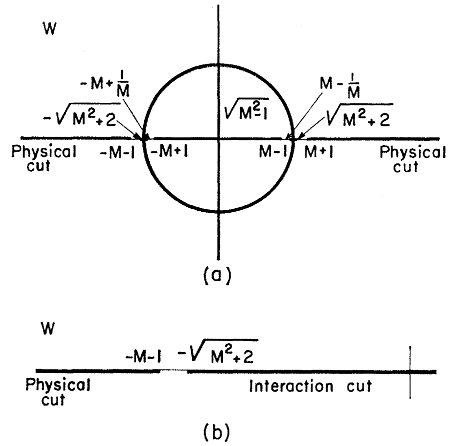

FIG. 1. (a) The singularity structure of $g(W)$ in the complex W plane. (b) The singularity structure of $g(W)$ with the unphysical singularities off the real axis replaced by an equivalent cut, which has its discontinuity adjusted so as to give the correct amplitude on the real axis.

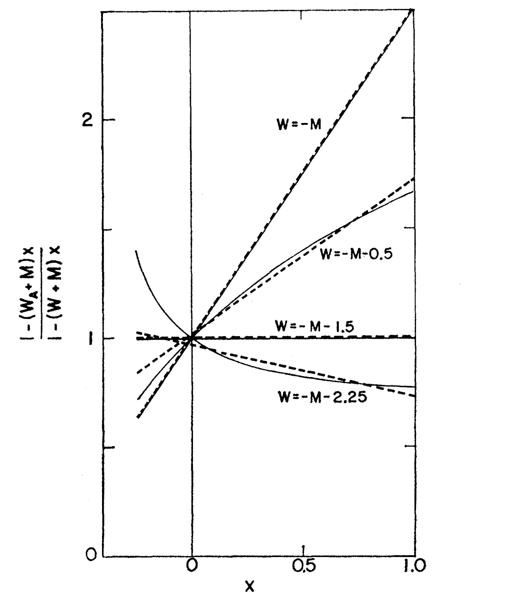

FIG. 2. Comparison of $[1-(W_A+M)x]/[1-(W+M)x]$ (solid lines) with the corresponding approximate forms given by Eq. (12) (dashed lines), as functions of x.

straight-line approximation

$$\frac{1-(W_A+M)x}{1-(W+M)x}\approx\frac{1-(W_A+M)x_1}{1-(W+M)x_1}\left(\frac{x-x_2}{x_1-x_2}\right)+\frac{1-(W_A+M)x_2}{1-(W+M)x_2}\left(\frac{x-x_1}{x_2-x_1}\right),\qquad(12)$$

where $x_1=0.081$ and $x_2=0.735$.

Substituting the approximations (11) and (12) into Eq. (10), we get

$$N(W)=\frac{D'(-M)}{\pi}\int_{-(M^2+2)^{1/2}}^{-M+(1/M)}dW'\,\frac{\operatorname{Im}g(W')(W'+M)}{(W'-W)}+\sum_{i=1}^{2}\frac{\alpha_i}{W_i-W},\qquad(13)$$

where

$$W_i=x_i^{-1}-M$$

and

$$\alpha_{1,2}=\frac{1-(W_A+M)x_{1,2}}{\pi x_{1,2}}\times\int_{x_I}^{x_S}dx\,\frac{\operatorname{Im}[g(x^{-1}-M)D(x^{-1}-M)]}{x[1-(W_A+M)x]}\left(\frac{x-x_{2,1}}{x_{1,2}-x_{2,1}}\right).$$

The first term in Eq. (13) can be neglected for $W<-M-1$, because the numerator in the integrand is small compared with the denominator. The second term has the same form as a two-pole formula. In this case, of course, the positions of these poles are not free parameters. The accuracy of this term can be easily estimated, being of the same order as the approximation (12), which, as can be seen from Fig. 2, is correct to within several percent in the range of interest. That possible oscillations, even growing ones, in $\operatorname{Im}g(W)$ should cause no difficulty was shown in Appendix A of reference 9.

To obtain an expression for $D(W)$ valid in the same range, we substitute Eq. (13) into Eq. (5). Equation (4) then becomes

$$D(W)=1+(W+M+1)\times[\alpha_1H(W,W_1)+\alpha_2H(W,W_2)],\qquad(14)$$

where we have dropped the first term in Eq. (13), and where

$$H(W,W_i)=\frac{1}{\pi}\int_{-\infty}^{-M-1}dW'\times\frac{q(-W')[(W'+M)^2-1]}{W'^2(W'-W)(W'-W_i)(W'+M+1)},\qquad(15)$$

or, since for $W<-M-1$, Eq. (2) becomes $q(-W)\approx\frac{1}{2}W^{-1}(W-M)[(W+M)^2-1]^{1/2}$,

$$H(W,W_i)=-\frac{M+2\pi L(0)}{4\pi(M^2-1)}+\frac{C(W)-C(W_i)}{W-W_i},\qquad(16)$$

where

$$C(W)=\frac{(W-M)(W+M-1)}{W}\times\left(\frac{M+2\pi L(0)}{4\pi(M^2-1)}+\frac{1-2\pi ML(0)}{2\pi W}+\frac{[(W+M)^2-1]L(W)-[M^2-1]L(0)}{W^2}\right),$$

TABLE I. The δ_{11} phase shift calculated from Eq. (19) with $\alpha_1=68.5$ and $\alpha_2=3.83$ (which corresponds to $f^2=0.08$).

Lab energy (MeV)	Phase shift (deg) Calculated	Experimental[a]
30	-1.15	-0.11 ± 6.30
41.5	-1.76	-0.34 ± 2.12
98	-4.96	-1.20 ± 0.23
150	-7.76	$-3^{+1.3}_{-1.5}$
170	-8.90	$-3^{+2}_{-2.3}$
220	-11.39	-5.4^{+6}_{-4}
225	-11.59	-4.3 ± 2.5

[a] These values were taken from S. W. Barnes, H. Winick, K. Miyake, and K. Kinsey, Phys. Rev. **117**, 238 (1960) (30 to 98 MeV); from Solution *A* of H. Y. Chiu and E. L. Lomon, Ann. Phys. (New York) 6, 50 (1959) (150 to 220 MeV); and from the Fermi (ii) solution of J. Deahl, M. Derrick, J. Fetkovich, T. Fields, and G. B. Yodh, Phys. Rev. 124, 1987 (1961) (225 MeV).

and

$$L(W)=\frac{\ln\{(W+M)-[(W+M)^2-1]^{1/2}\}}{2\pi[(W+M)^2-1]^{1/2}},\quad \text{for } |(W+M)|>1$$

$$=-\frac{1}{2\pi[1-(W+M)^2]^{1/2}}\times\left(\frac{\pi}{2}-\tan^{-1}\frac{W+M}{[1-(W+M)^2]^{1/2}}\right),\quad \text{for } |(W+M)|<1.$$

Now it is true that we only have a good approximation for $N(W)$ in a limited range of W, whereas the range in the integral of Eq. (4) is from $-M-1$ to $-\infty$. That this should not give rise to any difficulties, however, was shown in the last paragraph of the preceding section.

The two parameters α_1 and α_2 can now be calculated from the two conditions necessary for the nucleon pole to be a bound-state pole, namely,

$$D(-M)=0, \tag{17}$$

and from Eqs. (9) and (13),

$$\gamma+\frac{1}{\pi}\int_{-[M^2+2]^{1/2}}^{-M+(1/M)} dW'\,\mathrm{Im}g(W') =\gamma_{\mathrm{eq}}=\frac{-1}{D'(-M)}\sum_{i=1}^{2}\frac{\alpha_i}{W_i-W}, \tag{18}$$

with $D(W)$ given by Eq. (14) in each case. Of these two conditions, Eq. (17) is the more accurate, since it only depends on the value of $N(W)$ for $W<-M-1$. Thus it could be used even if we did not know the singularities at $W\approx-M$. In Eq. (18), γ_{eq} is just the residue of the equivalent pole in $g(W)$ that would be obtained by replacing the S cut by a pole at $W=-M$ and adding to this the nucleon pole. This is equal to the corresponding

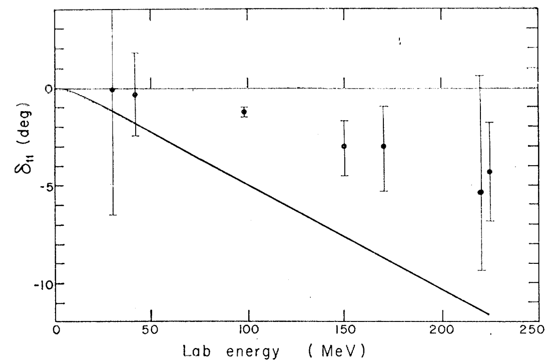

FIG. 3. Plot of the δ_{11} phase shift calculated with $f^2=0.08$, as a function of lab energy.

pole in the static theory,[3] i.e., $\gamma_{\mathrm{eq}}\approx(8/3)f^2M^2$, with f^2 equal to the renormalized pseudovector coupling constant. Taking $f^2=0.08$, we obtain $\alpha_1=68.5$ and $\alpha_2=3.83$.

Using Eqs. (1b), (3), (13), and (14)—and remembering that we may drop the integral in Eq. (13) for $W<-M-1$—we obtain for δ_{11},

$$\mathrm{Re}\frac{D}{N}=\frac{(W+M)^2-1}{-W^2}q(-W)\cot\delta_{11}(-W) =\Big[1+(W+M+1)\sum_{i=1}^{2}\alpha_i\,\mathrm{Re}H(W,W_i)\Big]\Big/\sum_{i=1}^{2}[\alpha_i/(W_i-W)], \tag{19}$$

which may be compared with the experimental phase shifts in the range of interest.[10] Such a comparison is given in Table I and Fig. 3, and is not unreasonable in view of the crude approximations that were made. Moreover, the experimental δ_{11} phase shifts are not very reliably known at present and the values given should probably not be taken too seriously.

IV. CONCLUSION

We have shown that the energy dependence of the δ_{11} phase shift in the 0- to 220-MeV range can be roughly predicted by assuming that the nucleon is a bound state.[11] We have not considered higher energies, since phase-shift analyses normally neglect inelastic effects at such energies, and it is not known how the results of such analyses would be modified by their inclusion. Even if they do not affect a particular solution sub-

[10] It is important to note that this is true only in a limited energy range. For instance, although Levinson's theorem is satisfied in this case, any considerations based on this theorem are probably meaningless.

[11] In a recent report by this author bearing the same title [University of California Radiation Laboratory Report UCRL-10026 (unpublished)] an error in sign in one of the formulas resulted in the prediction of a change of sign of the δ_{11} phase shift at about 180 MeV. Such a change of sign is favored in some phase-shift analyses [see, for instance, the third reference attached to Table I]. A conclusive experimental confirmation of such an effect would indicate the inadequacy of the model employed here.

stantially, it is not certain whether the best solution (statistically) without the consideration of inelastic effects remains the best when such effects are taken into account, and vice versa.

If, as has been assumed here, the nucleon is indeed a bound state in the πN system, its mass and the πN coupling constant could be calculated by solving Eqs. (4) and (6) together with (1), (3), and (5) to obtain $N(W)$ and $D(W)$, and then finding M and γ from $D(-M)=0$ and Eq. (9). The nucleon pole would not have to be inserted at the beginning of the calculation, because, as we have seen, it is absent from $N(W)$. It is true that the pole parameters in the crossed channel would indeed contribute to these equations in the complete problem. But, we could treat their values as variables, and then find those values for which the calculated parameters equal the assumed ones. Inelastic effects may be quite important in such a calculation, however. A similar procedure could be followed for any bound state.

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Low-Energy Pion-Pion Scattering*

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A general method for solving the problem of low-energy pion-pion scattering is presented. Within each partial wave, the nearby unphysical singularities are calculated by the usual Chew-Mandelstam approach, while the more distant ones are treated by a generalization of the Ball-Wong technique. These two techniques are then combined with the requirement of self-consistency. A rough calculation is given in which only a narrow P-wave resonance is retained, and the only free parameter is the pion mass. The resulting resonance has a mass of 585 MeV and a half-width of 125 MeV. A method is also given for calculating Regge poles at low energies.

I. INTRODUCTION

THERE have been several attempts at calculating the low-energy pion-pion amplitude from the requirements of analyticity, elastic unitarity, crossing symmetry, and self-consistency.[1–5] Of these, only the method of Zachariasen[4] does not involve any arbitrary parameters in the physically interesting P-dominant case, at least if only the lowest order term is retained. However, when this method is extended to higher orders, one has to introduce cutoffs,[6] and so one once again introduces such parameters. The main difficulty in all of these calculations arises from the strong, incalculable, distant, unphysical singularities within each partial wave.

In the method presented here, which does not contain any arbitrary parameters, the nearby unphysical singularities are treated by the usual polynomial method of CM-I (reference 1). An effective-range formula is then set up to represent the remaining unphysical singularities. The parameters of this formula can be determined by requiring that the resulting partial-wave amplitude have the correct value and derivatives at some point between these singularities and the physical region. The amplitude at this point can, in turn, be calculated from the absorptive part in the crossed channel, through a fixed momentum-transfer dispersion relation. This absorptive part is always expandable in Legendre polynomials in the region of interest. Thus we have a self-consistency problem in which we must find partial-wave amplitudes such that the assumed forms equal the calculated ones.

Finally, the same type of procedure can be followed for complex values of the angular momentum l, once the physical problem has been solved at low energies. The main usefulness of such a calculation is that it gives the trajectories of the poles in the complex-l plane, as they move with energy. These Regge poles, in turn, dominate high-energy scattering in the crossed channel.[7,8]

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* This work was done under the auspices of the U. S. Atomic Energy Commission.

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); hereafter referred to as CM-I.

[2] G. F. Chew and S. Mandelstam, Nuovo cimento **19**, 752 (1961); hereafter referred to as CM-II.

[3] V. V. Serebryakov and D. V. Shirkov, J. Exptl. Theoret. Phys. (U.S.S.R.) (to be published).

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[6] F. Zachariasen and C. Zemach (private communication).

[7] G. F. Chew, S. C. Frautschi and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

[8] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962), to which the reader is referred for additional references on Regge poles.