$I_0\mathcal{A}_{ij}$:

$$I_0\mathcal{A}_{nn}=2\,\mathrm{Re}(ma^*)+2|c|^2+2|h|^2-2|g|^2,$$

$$I_0\mathcal{A}_{mm}=2\,\mathrm{Re}[a(g^*-h^*)-m(g^*+h^*)]+4\,\mathrm{Im}[cC^*(22)],$$

$$I_0\mathcal{A}_{ll}=2\,\mathrm{Re}[a(g^*+h^*)-m(g^*-h^*)]-4\,\mathrm{Im}[cC^*(22)],$$

$$I_0\mathcal{A}_{nl}=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]$$
$$+2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]+2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],$$

$$I_0\mathcal{A}_{ln}=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]$$
$$-2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]-2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)], \quad \text{(A3.8)}$$

$$I_0\mathcal{A}_{lm}=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)+(a-m)C^*(23)]+4\,\mathrm{Im}[gC^*(16)],$$

$$I_0\mathcal{A}_{ml}=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)-(a-m)C^*(23)]-4\,\mathrm{Im}[gC^*(16)],$$

$$I_0\mathcal{A}_{mn}=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]$$
$$+2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]+2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)],$$

$$I_0\mathcal{A}_{nm}=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]$$
$$-2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]-2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)].$$

$I_0\mathcal{K}_{ij}^{(1)}$:

$$I_0\mathcal{K}_{nn}^{(1)}=2\,\mathrm{Re}(am^*)+2|c|^2+2|g|^2-2|h|^2,$$

$$I_0\mathcal{K}_{mm}^{(1)}=2\,\mathrm{Re}[(a+m)g^*+(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],$$

$$I_0\mathcal{K}_{ll}^{(1)}=2\,\mathrm{Re}[(a+m)g^*-(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],$$

$$I_0\mathcal{K}_{nl}^{(1)}=2\,\mathrm{Re}[(a-g+h)C^*(25)+(a+g-h)C^*(29)-cC^*(13)+cC^*(11)]$$
$$+2\,\mathrm{Im}[(g+h-m)C^*(14)+(m+g+h)C^*(12)+cC^*(26)+cC^*(28)],$$

$$I_0\mathcal{K}_{ln}^{(1)}=2\,\mathrm{Re}[(a-g+h)C^*(25)-(a+g-h)C^*(29)+cC^*(13)+cC^*(11)]$$
$$+2\,\mathrm{Im}[(g+h-m)C^*(14)-(m+g+h)C^*(12)-cC^*(26)-cC^*(28)], \quad \text{(A3.9)}$$

$$I_0\mathcal{K}_{nm}^{(1)}=2\,\mathrm{Re}[(a-g-h)C^*(26)-(a+g+h)C^*(28)+cC^*(12)-cC^*(14)]$$
$$+2\,\mathrm{Im}[(m+h-g)C^*(13)-(m+g-h)C^*(11)-cC^*(29)-cC^*(25)],$$

$$I_0\mathcal{K}_{mn}^{(1)}=2\,\mathrm{Re}[(a-g-h)C^*(26)+(a+g+h)C^*(28)+cC^*(12)+cC^*(14)]$$
$$+2\,\mathrm{Im}[(m+h-g)C^*(13)+(m+g-h)C^*(11)-cC^*(29)+cC^*(25)],$$

$$I_0\mathcal{K}_{lm}^{(1)}=2\,\mathrm{Re}[(a-m)C^*(22)+(a+m)C^*(23)]+4\,\mathrm{Im}[gc^*-hC^*(16)],$$

$$I_0\mathcal{K}_{ml}^{(1)}=2\,\mathrm{Re}[(a-m)C^*(22)-(a+m)C^*(23)]-4\,\mathrm{Im}[gc^*+hC^*(16)].$$

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Width of Three-Pion Resonances*

WILLIAM R. FRAZER† AND DAVID Y. WONG
University of California, La Jolla, California

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A model of three-pion resonances is constructed taking into account only π-ρ intermediate states. The π-ρ interaction is replaced by a simple pole. The calculated ω-resonance width is approximately 10–20 MeV, depending on the range of the π-ρ interaction. These values of the width are consistent with present experimental data.

WE have estimated the width of $I=0$ three-pion resonances on the basis of a disperion-theoretic calculation of the three-pion scattering amplitude, finding for the ω meson a full width of 10–20 MeV. We have employed an angular-momentum expansion of the three-pion state in order to reduce the calculation to manageable dimensions. The picture of an $I=0$, three-pion resonance as a quasi-bound state of a pion and a ρ meson then arises from the approximation of consider-

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† Alfred P. Sloan Foundation Fellow.

ing only states of low angular momentum. Such a picture leads to a three-pion resonance whose width is much smaller than the ρ-meson width. However, our calculation gives an ω width of an order of magnitude broader than the estimation of Gell-Mann, Sharp, and Wagner,[1] and several orders of magnitude broader than that of Feinberg.[2]

Let us now consider our quasi-bound-state model in more detail. The angular momentum expansion for three relativistic particles has been worked out by Wick,[3] who generalized the scheme introduced by Dalitz in the study of τ-meson decay.[4] First, two of the three pions, having four-momenta q_1 and q_2, are characterized by their angular momentum l, invariant mass-squared σ, and helicity λ; then the two pions are combined with the third (four-momentum q_3) to form a state of total angular momentum. Cook and Lee[5] have shown that the generalized unitarity condition[6,7] then simplifies to the form, for $(3m_\pi)^2 \leq s \leq (5m_\pi)^2$,

$$\begin{aligned}\mathrm{disc}\langle\lambda'l'\sigma'|T^J(s)|\lambda l\sigma\rangle &= \frac{2i}{\pi}\sum_{l'',\lambda''}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} d\sigma'' \frac{q(s,\sigma'')}{s^{1/2}}\left(\frac{\sigma''-4}{\sigma''}\right)^{1/2} \\ &\quad\times\langle\lambda'l'\sigma'|T^J(s_+)|\lambda''l''\sigma_+''\rangle \\ &\qquad\times\langle\lambda''l''\sigma_-''|T^J(s_-)|\lambda l\sigma\rangle, \quad (1)\end{aligned}$$

where s is the square of the invariant mass of the three-pion system, and where $q(s,\sigma)$ is the momentum of the third pion in the over-all c.m. system. We use the subscripts $\pm$ as an abbreviation of $\pm i\epsilon$, and use the symbol "disc" to indicate the discontinuity across the branch cut in s. Since an $I=0$ state of three pions is totally antisymmetric, only odd values of l occur. We consider only $l=1$, thus neglecting F and higher waves in the pion-pion system. The amplitude must, of course, be symmetrized, but let us defer this complication for a moment. Equation (1) can be simplified still further by considering states of given parity. Specifically, we consider the two quantum-number assignments which have been discussed in connection with three-pion resonances of negative G parity, namely, 1^- and 0^-. Since both these assignments have negative parity, both require $L=1$, where L is the angular momentum of the third pion in the over-all c.m. system. In both cases the scattering amplitude satisfies a unitarity relation of the form of Eq. (1), but as long as states with $l\geq 3$ are neglected, the sum over helicities is not present as long as one considers amplitudes corresponding to 1^- and 0^-. This equation is very similar to the ordinary two-body unitarity condition, except for the additional dependence on σ.

To facilitate consideration of the dependence on σ and σ', we factor out the initial and final-state interactions and define a new amplitude M as follows:

$$T^J(s,\sigma',\sigma)=f(\sigma')M^J(s,\sigma',\sigma)f(\sigma), \quad (2)$$

where $f(\sigma)$ is a function having the phase of pion-pion P-wave scattering and incorporating the threshold factors appropriate to the present problem. We use a Breit-Wigner type formula,

$$f(\sigma)=\frac{[\gamma(\sigma-4)q^2(s,\sigma)]^{1/2}}{m_\rho^2-\sigma-i\gamma[(\sigma-4)^3/\sigma]^{1/2}}. \quad (3)$$

From the dispersion-theoretic viewpoint the definition of the function $M(s,\sigma',\sigma)$ has been chosen such that the branch points at $\sigma=4m_\pi^2$ and $\sigma'=4m_\pi^2$ associated with pion-pion scattering have been removed. Therefore, we conjecture that $M(s,\sigma',\sigma)$ will not vary rapidly with σ' and σ. In particular, we use the sharply peaked nature of $f(\sigma)$ to simplify Eq. (1) to the form

$$(1/2i)\ \mathrm{disc}M^J(s,\sigma',\sigma) \approx \kappa(s)M^J(s_+,\sigma',m_\rho^2)M^J(s_-,\sigma,m_\rho^2), \quad (4)$$

where

$$\kappa(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2} d\sigma\frac{q^3(s,\sigma)}{s^{1/2}}\left[\frac{(\sigma-4)^3}{\sigma}\right]^{1/2} \times\frac{\gamma}{(m_\rho^2-\sigma)^2+\gamma^2(\sigma-4)^3/\sigma}. \quad (5)$$

Our procedure up to this point has been parallel to that used by Ball, Frazer, and Nauenberg in treating the state $\pi+\pi+N$.[7] The approximation made in deriving Eq. (4) should be quite reasonable for values of s sufficiently large that the phase-space integration runs over the region of the ρ-meson peak. Then we need only assume that M does not vary significantly as a function of σ in the region of this peak. In using Eq. (4) for values of s less than $(m_\rho+m_\pi)^2$, we are neglecting the σ dependence of M over a wider region.

Note that Eq. (4) is identical in form to the partial-wave unitarity condition for two-body scattering. The properties of the unstable ρ meson are contained in the generalized phase-space factor $\kappa(s)$. In the limit $\gamma\to 0$, $\kappa(s)$ reduces to the two-body P-wave phase space $q^3(s,m_\rho^2)/s^{1/2}$.

We shall show later that the effect of symmetrization is to modify the function $\kappa(s)$. Before taking up this question, let us show how the formalism we have developed can be used to estimate the width of three-pion resonances. In order to do this, one must somehow evaluate the interaction between the pions, then solve the N/D equations. The resonance should appear as a

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zero of the D function. As a first rough attempt in this direction we have represented the interaction by a pole (i.e., we have used an effective-range formula) whose residue is adjusted to fit the position of the resonance. We then find that

$$M(s)=\frac{1/\bar{\kappa}(s)}{s_r-s-i[\kappa(s)/\bar{\kappa}(s)]\theta(s-9m_\pi^2)}, \qquad (6)$$

where

$$\bar{\kappa}(s)\equiv\frac{s-s_0}{\pi}\mathcal{P}\int_{(3m_\pi)^2}^{\infty}ds'\frac{\kappa(s')}{(s'-s)(s'-s_0)(s'-s_r)}, \qquad (7)$$

where s_r is the position of the resonance and s_0 is the position of the interaction pole. For a narrow resonance we then have a width of $\Gamma\approx\kappa(s_r)/\bar{\kappa}(s_r)$. We shall see below that the width is not very sensitive to the position of s_0. In fact, for large negative s_0 the dependence is only logarithmic.

Since the width Γ is proportional to $\kappa(s_r)$, we can see from Eq. (5) why a three-pion $I=0$ resonance with energy well below $m_\rho+m_\pi$ should be narrow. The mass squared of the two-pion system cannot be large enough to lie in the region of the ρ-meson peak, and the decay occurs via the tail of the ρ-meson distribution. The existence of a second, lower-lying pion-pion resonance in the $J=I=1$ state would, of course, invalidate the present treatment.[8] It is interesting to note that $\kappa(s)$ is essentially equal to the decay probability corresponding to Fig. 1(a).

Finally, let us consider the effect of symmetrization on $\kappa(s)$. We used an expansion in states of the form $(12)3\rangle$, where pions 1 and 2 are combined to have angular momentum $l=1$. The following state will then have the proper symmetry: $(12)3\rangle+(23)1\rangle+(31)2\rangle$. The effect of introducing such a state is to change $\kappa(s)$ to be essentially equal to the decay matrix element calculated from the sum of the three diagrams of Fig. 1. This matrix element has been written down for both 0^- and 1^- by Shaw and Wong,[9] who pointed out that the symmetrization produces a tremendous suppression in $\kappa(s)$ for small s in the 0^- case and electromagnetic corrections must be included if the resonance is in the neighborhood of 550 MeV. The 1^- matrix element calculated from any one term in Fig. 1 is already totally antisymmetric except for the ρ-meson propagators [the denominators in Eq. (3)]. Nevertheless the symmetrization affects the calculation of Γ. For low values of s the three propagators are essentially constant and add coherently. For high values of s the result is approximately the sum of the squares, so that $\kappa(s)/\bar{\kappa}(s_r)$ is raised by a factor of about 3 for small s_r, as compared to the unsymmetrized calculation.

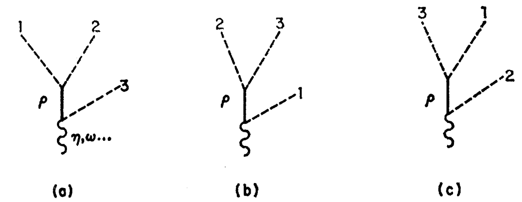

FIG. 1. Diagrams for the 3π decay of a quasi-bound-state.

The results we find for the ω (1^- assignment assumed) are summarized in the table below:

s_0	Full width of ω (MeV)
8	23.4
4	20.4
0	18.4
−10	14.7
−100	7.0

The disagreement between our result and that of Gell-Mann, Sharp, and Wagner[1] can probably be understood as a violation of unitary symmetry, which these authors assume in relating π^0 decay to the width of the ω. Current calculations of nucleon-nucleon scattering by Scotti and Wong[10] indicate that the ω-nucleon coupling is stronger by a factor of about 3 than the ρ-nucleon coupling; hence, the ω-gamma coupling may be three times weaker than the ρ-gamma coupling if the electromagnetic form factors of the nucleon are dominated by the ω and ρ states. This result is consistent with the calculation of the π^0 lifetime using our values of the ω width.

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