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Prediction of Pion Phase Shifts*

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A method of deducing the pion-pion phase shifts from the pion-nucleon phase shifts is developed. A predominantly attractive S-wave interaction is found. A resonant P wave and an appreciable D wave are also deduced.

INTRODUCTION

THE equations of dispersion theory may be capable of predicting the phase shifts of pion-pion, and of pion-nucleon scattering in terms of a few parameters. However, the labor involved in the numerical solution of the self-consistent integral equations, and the problems posed by the multiplicity of solutions caused by nonlinearity, enhance the worth of a less ambitious undertaking. The pion-pion phase shifts can be calculated from the pion-nucleon phase shifts without the introduction of any unknown parameters, notwithstanding a subtraction, which reduces the incalculable high-energy effects.

The scattering amplitudes which are dispersed staisfy analyticity, crossing, and unitarity conditions. Most dispersion relations use analyticity to the full, while the approximations are at the expense of the other two requirements. Partial wave relations satisfy unitarity exactly, and in practice approximate the crossing. They allow the input of pion-nucleon information in a detailed manner[1]; but they suffer from the necessity of cutoffs, arising because of an analytic continuation into unphysical regions of angle, with a consequent divergence of the Legendre expansion in terms of which the continuation is made. Since it is here the intention to force the input information to yield the pion-pion phases, with no arbitrary parameters, it has been decided to use total amplitudes in the backward direction.[2] The resulting equations involve no continuation in angle. Further, they are linear in the unknown pion-pion amplitude, constituting Stieltjes equations, with unique solutions.

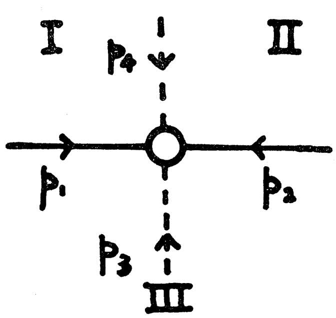

FIG. 1. The three channels of the $\pi n \to \pi n$ reaction.

1. KINEMATICS

The scattering is depicted in Fig. 1, where p_1, p_3 are the pion momenta, p_2, p_4 the nucleon momenta. Conservation of four momentum is expressed by

$$\sum_{i=1}^{4} p_i = 0. \tag{1.1}$$

The "invariant energies" are defined by

$$s_i = (p_i + p_4)^2, \quad i = 1, 2, 3, \tag{1.2}$$

but only two are independent, because of the mass-shell restriction:

$$\sum_{i=1}^{3} s_i = 2(M^2+1). \tag{1.3}$$

M is the nucleon mass, in units of the pion mass.

If

$$p_i = (\mathbf{p}_i, \epsilon_i), \quad i = 1, 2, 3, 4. \tag{1.4}$$

then the scattering matrix element between initial and final states is

$$S_{fi} = \delta_{fi} - (2\pi)^4 i \delta(\textstyle\sum p_i)[M/2(\Pi\epsilon_i)^{1/2}]\bar{u}_2 T u_1. \tag{1.5}$$

Here u_1 and u_2 are the nucleon spinors, and the T matrix has the form:

$$T = -A + i\gamma\cdot QB \quad \text{with} \quad Q = \tfrac{1}{2}(p_2 - p_1). \tag{1.6}$$

The amplitudes A and B can be decomposed in isospin space:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \delta_{\alpha\beta} \begin{bmatrix} A^+ \\ B^+ \end{bmatrix} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha] \begin{bmatrix} A^- \\ B^- \end{bmatrix}. \tag{1.7}$$

In this expression, α and β are the isospin indices of the pions. The four amplitudes $A^\pm$, $B^\pm$ are scalar functions of two independent variables.

The invariant amplitudes $A^\pm$, $B^\pm$ serve to describe three separate reactions, depending on the range of the arguments s_1, s_2, s_3, of which only two need be specified because of Eq. (1.3). If

$$s_1 > (M+1)^2, \quad s_3 < 0,$$

the amplitudes $A^\pm$, $B^\pm$ describe the reaction $\pi N \to \pi N$ [channel I of Fig. (1)]. An alternative pair of variables

* This work was supported by the Department of Scientific and Industrial Research, and by the U. S. Air Force, European Office (Aerospace Research).

[1] J. Hamilton and T. D. Spearman, Ann. Phys. (New York) 12, 172 (1961).

[2] H. Y. Tzu (to be published).

consist in ν, the square of the three momentum of the pion, and θ, the scattering angle between initial and final states, in the center of momentum system. The relations between the variables are

$$\begin{aligned} s_1&=[(\nu+M^2)^{1/2}+(\nu+1)^{1/2}]^2, \\ s_3&=-2\nu(1-\cos\theta). \end{aligned} \tag{1.8}$$

For the range

$$s_2>(M+1)^2, \quad s_3<0,$$

the particles 2 and 4 of Fig. 1 are incoming (channel II). This is the "crossed" $\pi N\to\pi N$ reaction; and if it is described by the corresponding momentum squared ν' and angle θ' the relations are

$$\begin{aligned} s_2&=[(\nu'+M^2)^{1/2}+(\nu'+1)^{1/2}]^2, \\ s_3&=-2\nu'(1-\cos\theta'). \end{aligned} \tag{1.9}$$

However, when

$$s_1<0, \quad s_3>4,$$

the invariant amplitudes describe the reaction $\pi\pi\to N\bar{N}$ (channel III), although the nucleons are virtual if $s_3<4M^2$. The natural variables to use in this channel are the three momenta of the pions, q, and of the nucleons, p, together with the scattering angle between the pion and the nucleons, φ. The relations are

$$\begin{aligned} s_1&=-p^2-q^2+2pq\cos\varphi, \\ s_3&=4(M^2+p^2)=4(1+q^2). \end{aligned} \tag{1.10}$$

The invariant amplitudes can be continued from these three "physical" regions into the complex planes of the s_i, and hence connections are implied between the three channels. This analytic continuation is considered in the following sections, where it is shown that if the amplitudes are known in the physical regions of channels I and II, then they can be decuced in that of channel III.

Dispersion relations will be written in the variable ν, the momentum squared in channel I, at fixed scattering angle, $\theta=\pi$. Equations (1.8) become

$$\begin{aligned} s_1&=[(\nu+M^2)^{1/2}+(\nu+1)^{1/2}]^2, \\ s_3&=-4\nu, \\ s_2&=2(M^2+1)-s_1-s_3=[(\nu+M^2)^{1/2}-(\nu+1)^{1/2}]^2. \end{aligned} \tag{1.11}$$

The amplitudes are defined in the s_1 plane, and this is mapped into a two sheeted ν plane. The two sheets are connected crosswise across a cut, $(-M^2<\nu<-1)$, distinguished by

$$\begin{aligned} s_1&=[(\nu+M^2)^{1/2}+(\nu+1)^{1/2}]^2 \quad \text{sheet I} \\ &=[(\nu+M^2)^{1/2}-(\nu+1)^{1/2}]^2 \quad \text{sheet II.} \end{aligned} \tag{1.12}$$

The square roots are defined to be positive for real $\nu>-1$. Thus the channel I physical region,

$$(M+1)^2<s_1<\infty$$

maps into the region

$$0<\nu<\infty$$

on sheet I.

The channel II physical region,

$$(M+1)^2<s_2<\infty$$

maps into the region

$$0<\nu<\infty$$

on sheet II, as may be seen by comparing the expressions for s_2 in terms of ν and ν' [Eq. (1.9)]. Further, by comparing the expressions for s_3, it is seen that $\theta'=\pi$. That is, the continuation of the backward amplitude from the channel I region into the channel II region gives the backward amplitude of the crossed $\pi N-\pi N$ reaction. For any fixed angle other than 0 or π this simple 1:1 mapping does not exist.

Similarly, comparing Eqs. (1.10) and (1.11), one adopts the definitions:

$$p=[-\nu-M^2]^{1/2}, \quad q=[-\nu-1]^{1/2},$$

where the product pq is defined to be positive for $\nu<-M^2$ on sheet I of ν, and negative on sheet II. Then it is readily seen that $\varphi=\pi$, or the continuation of the invariant amplitudes into the $\pi\pi$ region again refers to backward amplitudes, on both sheets of ν. Further, the $\pi\pi$ cut, namely,

$$4<s_3<\infty$$

maps into

$$-\infty<\nu<-1$$

on both sheets.

It is necessary now to relate the invariant amplitudes $A^\pm$, $B^\pm$, in the backward direction, to the pion-nucleon phase shifts $\delta_{L\pm}{}^T$.[3] Partial-wave amplitudes are defined in terms of these phase shifts:

$$f_{L\pm}{}^T=(1/\nu^{1/2})\exp i\delta_{L\pm}{}^T\sin\delta_{L\pm}{}^T, \quad T=\tfrac{1}{2},\tfrac{3}{2}, \quad L=0,1,2,\cdots. \tag{1.13}$$

Here T is the isospin of the system, and L the orbital angular momentum, while $\pm$ refers to the nucleon helicity. Spin amplitudes $f_1{}^T$, $f_2{}^T$ are given by

$$\begin{aligned} 2f_1{}^T&=\sum_{i=1}^{\infty}(-)^{L+1}L(L+1)[f_{L-1+}{}^T-f_{L+1-}{}^T], \\ 2f_2{}^T&=\sum_{i=1}^{\infty}(-)^{L+1}L(L+1)[f_{L-}{}^T-f_{L+}{}^T]. \end{aligned} \tag{1.14}$$

Certain $\pm$ combinations are defined:

$$\begin{aligned} 3f_s{}^+&=f_s{}^{1/2}+2f_s{}^{3/2}, \\ 3f_s{}^-&=f_s{}^{1/2}-f_s{}^{3/2}, \quad s=1,2, \end{aligned} \tag{1.15}$$

[3] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu. Phys. Rev. **106**, 1337 (1957).

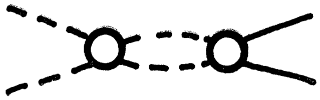

FIG. 2. Two-pion intermediate state in channel III.

and the invariant amplitudes are expressed

$$\begin{aligned} A^{\pm}/4\pi &= [(W+M)/(E+M)]f_1^{\pm} \\ &\quad -[(W-M)/(E-M)]f_2^{\pm}, \\ B^{\pm}/4\pi &= [1/(E+M)]f_1^{\pm}+[1/(E-M)]f_2^{\pm}. \end{aligned} \tag{1.16}$$

In a similar way, by considering the kinematics of $\pi\pi-N\bar{N}$ scattering, the invariant amplitudes in the channel III physical region can be related to the helicity amplitudes describing that reaction.[4] Here $\pm$ refers to the relative helicity of the nucleons in the final state. The expansion is

$$\begin{aligned} \frac{p^2}{4\pi}A^T &= \sum_{J=0}^{\infty}(-)^J(2J+1)(pq)^J \\ &\quad \times\{\tfrac{1}{2}M[J(J+1)]^{1/2}f_-^{TJ}-f_+^{TJ}\}, \\ \frac{1}{2\pi}B^T &= \sum_{J=1}^{\infty}(-)^{J+1}(2J+1)[J(J+1)]^{1/2}(pq)^{J-1}f_-^{TJ}, \\ &\quad T=0,\,1, \end{aligned} \tag{1.17}$$

where

$$\begin{bmatrix} A^0 \\ B^0 \end{bmatrix} = 6^{1/2}\begin{bmatrix} A^+ \\ B^+ \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} A' \\ B' \end{bmatrix} = 2\begin{bmatrix} A^- \\ B^- \end{bmatrix}. \tag{1.18}$$

In the expansions (1.17) only alternate terms are nonzero, for $f_{\pm}^{0J}=0$ if J is odd, and $f_{\pm}^{1J}=0$ if J is even, a consequence of the Pauli principle.

One is interested in the helicity amplitudes $f_{\pm}^{TJ}$ in the region just above the two-pion threshold, where the nucleons are virtual. They can be related to the phase shifts of $\pi\pi-\pi\pi$ scattering, for, applying the unitarity of the S matrix to the decomposition pictured in Fig. 2, one has

$$\Im\langle\pi\pi|N\bar{N}\rangle \sim \int d\Omega\langle\pi\pi|\pi\pi\rangle^*\langle\pi\pi|N\bar{N}\rangle. \tag{1.19}$$

That this formula is applicable in this unphysical region has been deomonstrated by Mandelstam[5]; also, it is exact below the four-pion threshold. It gives a simple expression for the $\pi\pi-\pi\pi$ phase shift, $\delta_J{}^T$:

$$\delta_J{}^T = \arg f_{\pm}^{TJ}. \tag{1.20}$$

The program of the following sections will be to calculate the invariant amplitudes from the $\pi N-\pi N$ phase shifts, using (1.13) through (1.16), in the channel I physical region, then to extrapolate to the channel III physical region, and to calculate the $\pi\pi-\pi\pi$ phase shifts, using (1.17) though (1.20).

[4] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

[5] S. Mandelstam, Phys. Rev. Letters **4**, 84 (1960).

2. DISPERSION RELATIONS

The invariant amplitudes can be considered to be the boundary values of analytic functions of two independent variables. The functions have cuts on the real axes[6]:

$$\begin{aligned} (M+1)^2 &< s_1 < \infty, \quad \text{I}, \\ (M+1)^2 &< s_2 < \infty, \quad \text{II}, \\ 4 &< s_3 < \infty, \quad \text{III}. \end{aligned}$$

These regions correspond to the "physical" regions of channels I, II, III; and the boundary values on these cuts, approaching through positive values of the imaginary part of the argument, are the invariant amplitudes, in the corresponding channels. The B amplitudes also have poles, at

$$s_1 = M^2, \quad s_2 = M^2,$$

that arise from single-nucleon intermediate states.

These analytic properties can be expressed by means of a double Cauchy integral, the Mandelstam representation:

$$A = \sum_{i=1}^{3}\left[\frac{\gamma^i}{M^2-s_i}+\frac{1}{\pi}\int ds_i'\frac{\rho_i(s_i')}{s_i'-s_i} + \frac{1}{\pi^2}\int\int ds_j'ds_k'\frac{\rho_{jk}(s_j',s_k')}{(s_j'-s_j)(s_k'-s_k)}\right]. \tag{2.1}$$

Here $A=(A^+,A^-,B^+,B^-)$, (ijk) is a cyclic permutation of (123), and

$$\gamma^1 = G^2(0011), \quad \gamma^2 = G^2(00-11), \quad \gamma^3 = 0.$$

G is the renormalized pion-nucleon coupling constant. In the Mandelstam representation, the spectral functions ρ_i, ρ_{jk} are real, and vanish outside well-defined regions. They are not all independent, because of the crossing symmetry. For the interchange of the two pions does not alter the nature of the reaction; and the scattering matrix element is invariant under the interchange. This implies the symmetry:

$$A(s_1s_2s_3) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} A(s_2s_1s_3). \tag{2.2}$$

Equation (2.1) can be used to give a dispersion relation for the invariant amplitudes, in the backward direction, as a function of ν. The amplitudes are defined on a two-sheeted surface, according to Eq. (1.12). The crossing relation (2.2) becomes

$$A(\nu^I) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} A(\nu^{II}), \tag{2.3}$$

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

where the amplitudes are written as a function of ν only. A dispersion relation will be written on sheet I only; and the superscript will be omitted. On this sheet there is a cut:

$$0<\nu<\infty\,;$$

and the boundary value on it, approaching through positive values of $\mathfrak{I}\nu$, is the $\pi N-\pi N$ backward-invariant amplitude. There is also a cut

$$-\infty<\nu<-1\,;$$

and the boundary value, approaching through negative values of $\mathfrak{I}\nu$, is the $\pi\pi-N\bar{N}$ backward amplitude, as a function of ν, which is related to p and q by

$$-\nu=1+q^2=M^2+p^2. \tag{2.4}$$

There is also a Born single-nucleon pole at

$$\nu=\nu_0\equiv-1+1/4M^2.$$

The singularities are shown in Fig. 3.

It will be seen from Eqs. (1.17) and (1.18) that

$$pq=[(\nu+1)(\nu+M^2)]^{1/2}$$

factors out from the expansions of A^- and B^+, and it can be shown, by considering (2.1), that A^-/pq and B^+/pq have the same singularities as A^- and B^+. It is, in fact, more convenient to disperse these quantities. Accordingly, one defines

$$N(\nu)=\begin{bmatrix}A^+\\ \rho A^-\\ \rho B^+\\ B^-\end{bmatrix},\quad \rho=[(\nu+1)(\nu+M^2)]^{-1/2}, \tag{2.5}$$

and writes a dispersion relation for N, using the contour of Fig. 3. Introducing a subtraction at the pion-nucleon threshold, $\nu=0$, the relation is

$$\frac{N(\nu+i\epsilon)-N(0)}{\nu}=\frac{1}{\pi}\left[\int_0^\infty+\int_{-\infty}^{-1}\right]\frac{d\nu'}{\nu'-\nu}\frac{\mathfrak{I}N(\nu'+i\epsilon)}{\nu'}-\frac{\Gamma}{\nu_0(\nu-\nu_0+i\epsilon)}, \tag{2.6}$$

where

$$\Gamma=G^2\left[\frac{d\nu}{ds_1}\begin{pmatrix}0\\0\\ \rho\\1\end{pmatrix}\right]_{\nu=\nu_0}.$$

A complex quantity, the discrepancy, is defined by

$$\Xi(\nu)=\frac{N(\nu)-N(0)}{\nu}-\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\frac{\mathfrak{I}N(\nu')}{\nu'}+\frac{\Gamma}{\nu_0(\nu-\nu_0)}. \tag{2.7}$$

The notation follows that of Hamilton,[1] in that the discrepancy, like his, consists in those parts of the dispersion relation not directly calculable in terms of the πN phase shifts and the Born terms. Here, this is simply the contribution from the $\pi\pi$ cut; but in his case it consists in short-range contributions from the crossed πN reaction, and from the crossed Born pole, beside the $\pi\pi$ contribution. The Cauchy pole of the dispersion relation (at ν) can be placed in the πN physical region:

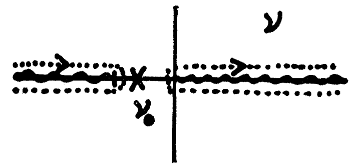

FIG. 3. The cuts and integration contours in the ν plane.

$$\Xi(\nu)=\frac{\mathfrak{R}N(\nu)-\mathfrak{R}N(0)}{\nu}-\frac{\mathfrak{P}}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\frac{\mathfrak{I}N(\nu')}{\nu'}+\frac{\Gamma}{\nu_0(\nu-\nu_0)},\quad \nu>0. \tag{2.8}$$

Here the discrepancy is real, and can be calculated from the known $N(\nu)$. A method will be given, in the next section, whereby $\Xi(\nu)$ can be extrapolated from the region $(0,\infty)$ to the region $(-\infty,-1)$. Then the amplitudes $N(\nu)$ are given in $(-\infty,-1)$ by placing the Cauchy pole there:

$$\frac{\mathfrak{R}N(\nu)}{\nu}=\mathfrak{R}\Xi(\nu)+\frac{\mathfrak{R}N(0)}{\nu}+\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\frac{\mathfrak{I}N(\nu')}{\nu'}-\frac{\Gamma}{\nu_0(\nu-\nu_0)}, \tag{2.9}$$

$$\frac{\mathfrak{I}N(\nu)}{\nu}=\mathfrak{I}\Xi(\nu),\quad \nu<-1.$$

3. CONFORMAL MAPPING

The problem to be considered in this section is the extrapolation of $\Xi(\nu)$ from $(0,\infty)$ on to its cut $(-\infty,-1)$. The continuation is unique, and amounts to the solution of a Stieltjes integral equation. For, according to Eqs. (2.6) and (2.7),

$$\Xi(\nu)=\frac{1}{\pi}\int_{-\infty}^{-1}\frac{\mathfrak{I}N(\nu')}{\nu'}\frac{d\nu'}{\nu'-\nu}.$$

Putting

$$\alpha=-(\nu'+1),\quad \mathfrak{I}N(\nu')/\nu'=g(\alpha),$$

$$\beta=\nu+1,\qquad \Xi(\nu)=f(\beta),$$

one has

$$f(\beta)=\frac{1}{\pi}\int_0^\infty\frac{g(\alpha)}{\alpha+\beta}d\alpha, \tag{3.1}$$

which is the Stieltjes equation, the solution being[7]:

$$(2\pi/i)g(\beta)=\lim_{\epsilon\to0+}[f(-\beta+i\epsilon)-f(-\beta-i\epsilon)]. \tag{3.2}$$

[7] E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford University Press, New York, 1937), p. 318.

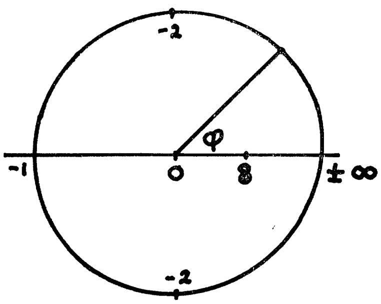

FIG. 4. The η plane.

Among several practical methods of soution, it was found that a conformal mapping technique was most satisfactory,[8–10] as it takes the errors of Ξ into account more satisfactorily, as will devolve.

The transformation is

$$\eta=\frac{K(\nu+1)^{n+1/2}-1}{K(\nu+1)^{n+1/2}+1},\quad \xi(\eta)=\Xi(\nu). \tag{3.3}$$

This maps the whole of the ν plane into the unit circle in the η plane. The cut $(-\infty<\nu<-1)$ is transformed into the unit circle $|\eta|=1$, while the region $(-1<\nu<\infty)$ becomes the interval $(-1<\eta<1)$. The elastic branch point at $\nu=-1$ is of a square-root type[11]; and is transformed away by the mapping. However, higher branch points survive in the η plane.

The discrepancy is expanded in a Maclaurin series:

$$\xi(\eta)=\sum_0^\infty a_n\eta^n, \tag{3.4}$$

where the a_n are determined by the knowledge of ξ on the real axis. The value on the circle is discovered by setting $\eta=e^{i\varphi}$:

$$\xi(e^{i\varphi})=\sum_0^\infty a_n e^{in\varphi}. \tag{3.5}$$

The discrepancy in the ν plane, on the cut, is then obtained by the substitution

$$K(-\nu-1)^{n+1/2}=(-)^n\cot\varphi/2. \tag{3.6}$$

The real constant K is arbitrary, and can be chosen to suit the discrepancy. The index, n, however, is prescribed by a condition at the $\pi\pi$ threshold. In accordance with Eq. (1.20),

$$\Re f_\pm{}^{TJ}\sim 1,\quad \Im f_\pm{}^{TJ}\sim q^{2J+1},\quad q\sim 0.$$

This implies, by virtue of Eqs. (1.17) and (2.5), that

$$\Re N_i\sim 1,\quad i=1,2,3,4,$$
$$\Im N_1\sim q,\quad \Im N_2\sim q^3,\quad \Im N_3\sim q^5,\quad \Im N_4\sim q\,,$$

where $N(\nu)=(N_1,N_2,N_3,N_4)$. This threshold behavior is automatically ensured by choosing n to be (0,1,2,1) in turn, for the four components of $N(\nu)$. The mapping is displayed, for $K=1$, $n=0$, in Fig. (4), where the figures are ν values.

[8] S. Ciulli and J. Fischer, Nuclear Phys. **24**, 465 (1961).
[9] W. R. Frazer, Phys. Rev. **123**, 2180 (1961).
[10] C. Lovelace (to be published).
[11] W. Zimmerman, Nuovo cimento **21**, 249 (1961).

The question of convergence on the unit circle arises. The Maclaurin series in η becomes a Fourier series in φ; and analyticity is not required for its convergence. According to Fourier's Theorem,[12] if

$$\int_0^\pi \xi(e^{i\varphi})d\varphi \tag{3.7}$$

converges absolutely, and

$$b_n=\frac{\Re}{\pi}\int_0^\pi \xi(e^{i\varphi})e^{-in\varphi}d\varphi,\quad n=0,1,\cdots, \tag{3.8}$$

then if φ is interior to an interval in which $\xi(e^{i\varphi})$ has limited total fluctuation,

$$\sum_0^\infty b_n e^{in\varphi}=\tfrac{1}{2}[\xi(e^{i(\varphi+0)})+\xi(e^{i(\varphi-0)})]. \tag{3.9}$$

If, further, $\xi(e^{i\varphi})$ is continuous in the same interval, the sum reduces to $\xi(e^{i\varphi})$; and the convergence to it is uniform with respect to φ. In fact, physical considerations allow one to assert both limited total fluctuation and continuity of $\xi(e^{i\varphi})$ at all points, except possibly $\varphi=0$, corresponding to infinite energy. The integral (3.7) becomes, in the ν plane:

$$\int_{-\infty}^{-1}\frac{K(2n+1)(-\nu-1)^{n-1/2}}{1+K^2(-\nu-1)^{2n+1}}\Xi(\nu)d\nu, \tag{3.10}$$

and this certainly exists. Hence the Fourier series (3.9) converges uniformly to $\xi(e^{i\varphi})$. To establish that the Maclaurin series (3.4) converges to $\xi(e^{i\varphi})$ on the unit circle, one must demonstrate the identities:

$$a_n=b_n,\quad n=0,1,2,\cdots. \tag{3.11}$$

The analyticity ensures that

$$\xi(\rho e^{i\varphi})=\sum_0^\infty a_n\rho^n e^{in\varphi},\quad 0<\rho\leq R<1, \tag{3.12}$$

the convergence being uniform. Since the discrepancy is defined on the cut by a limiting process, it follows that, given any $\epsilon>0$, there exists a $\delta>0$, such that

$$|\xi(e^{i\varphi})-\xi(\rho e^{i\varphi})|<\epsilon\quad\text{for all}\quad 0<1-\rho<\delta. \tag{3.13}$$

Suppose for the moment that this limit is uniform, that is, that δ does not depend on φ, The sum

$$\xi(e^{i\varphi})-\xi(\rho e^{i\varphi})=\sum_0^\infty(b_n-a_n\rho^n)e^{in\varphi},\quad 0<\nu<1, \tag{3.14}$$

converges uniformly, and so can be integrated term by term, giving

$$b_n-a_n\rho^n=\frac{\Re}{\pi}\int_0^\pi[\xi(e^{i\varphi})-\xi(\rho e^{i\varphi})]e^{-in\varphi}d\varphi. \tag{3.15}$$

Using (3.13), with the postulated uniformity, one has

$$|b_n-a_n\rho^n|<\epsilon,\quad 0<1-\rho<\delta, \tag{3.16}$$

[12] E. T. Whittaker and G. N. Watson, *Modern Analysis* (The Macmillan Company, New York, 1940).

which establishes the required result. However, the limit

$$\xi(\rho e^{i\varphi}) \to \xi(e^{i\varphi}) \tag{3.17}$$

is not expected to be uniform over the whole range of φ; but only over an interval including no branch points. Let (a,b) be such an interval, and let $g_n(\varphi)$ be defined so that

$$\frac{\Re}{\pi}\int_a^b g_n(\varphi)e^{ip\varphi}d\varphi = \delta_{np}. \tag{3.18}$$

The linear independence and square integrability of the functions $e^{in\varphi}$ in (a,b) assure the existence and boundedness of the $g_n(\varphi)$. Equation (3.14) is inverted by

$$b_n - a_n\rho^n = \frac{\Re}{\pi}\int_a^b [\xi(e^{i\varphi}) - \xi(\rho e^{i\varphi})]g_n(\varphi)d\varphi, \tag{3.19}$$

whence (3.16) follows, by using (3.18). Thus the Maclaurin series converges at all points of the circle, except the branch points. It should be noted that the $\pi\pi$ threshold, $\varphi=\pi$, is not a branch point in the η plane,[11] so that convergence occurs there.

4. ERRORS

The experimental information used consists in the pion nucleon phase shifts for S, P, and D waves, taken from the work of Woolcock.[13] The invariant amplitudes are calculated from these, using Eqs. (1.13) through (1.16), the expansions (1.14) being truncated at the D wave. The errors are estimated from those of Woolcock, and it is assumed that neglect of F waves causes negligible error.

It is necessary to estimate the error introduced by the conformal extrapolation. First, two objections to the mapping will be considered. The first is that while the high energy parts of the cuts in the ν plane contribute little to the low energy solution, this may not be the case in the η plane, where all parts of the $\pi\pi$ cut are at the same distance from the low energy region. The criticism is relevant to a method in which a dispersion relation is written down in the η plane, but it is not true that the useful properties of the ν plane are destroyed. For, in the ν plane, the high energy part of a once subtracted integral, from $|\nu|=U$ to ∞, behaves like U^{-1}. The corresponding behavior in the η plane, for the equivalent integral, is $U^{-(n+1/2)}$. Thus, if $n\geq 1$, the suppression of the high energy parts is more efficacious than it is in the ν plane; if $n=0$, it is not so, but the suppression still exists. The notion of distance as a factor reducing the importance of a cut is valid only when considered together with a lateral effect; and the relative importance of the two factors depends on the complex variable used.

The second point is that, since $\xi(\eta)$ is known only in the region

$$\eta \leq Y < 1,$$

Y representing the highest energy at which phase shifts are available, it may be that the true Maclaurin series should contain some very high powers, with appreciable coefficients. These would not be detected in the experimental curve $\xi(\eta)$; but they would clearly be important on the unit circle.

Before this point is considered, it would be well to point out that the difficulty lies in the dispersion relation, and not in the mapping as a technique of solution. The problem is that, in the equation

$$\Xi(\nu) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\Im\Xi(\nu')}{\nu'-\nu}d\nu', \quad \nu>0, \tag{4.1}$$

a violently oscillatory error of $\Im\Xi(\nu')$, $\nu'<-1$, only produces a small error of $\Xi(\nu)$, $\nu>0$, due to cancellation in the integral. All one can do, from a mathematical point of view, is to fit the experimental points by a curve $\Xi(\nu)$, lying within the band of errors of the points, and then devise some means of estimating $\Im\Xi(\nu')$, $\nu'<-1$, from this curve. Now if a polynomial in η is such a fit, then the corresponding Fourier series obtained by the extrapolation is identically the solution which, when integrated, gives the above fit. The mapping, as a trick for finding a solution which gives an acceptable discrepancy, has achieved what was required. No statement about the remainder term in the Fourier series, or the behavior of a_n as $n\to\infty$ is needed. Actually, an assumption merely of boundedness of the pion pion amplitude at infinite energy implies $|na_n| <$ constant.

The problem remains that, having found a solution which gives an acceptable discrepancy, one requires to estimate the error, in particular the possibliity of large oscillations. If $\Xi_0(\nu,)$ $\nu>0$, is the exact, unknown discrepancy, and $\Xi(\nu)$ is the approximation which is known, the error will be written

$$\Delta(\nu) = \Xi(\nu) - \Xi_0(\nu). \tag{4.2}$$

Then, if $\Im\Xi(\nu')$, $\nu'<-1$, is the known approximate solution, the true solution $\Im\Xi_0(\nu')$ is given by

$$\Im\Xi_0(\nu') = \Im\Xi(\nu') - \Im\delta(\nu'), \tag{4.3}$$

where $\Im\delta(\nu')$ satisfies

$$\Delta(\nu) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\Im\delta(\nu')}{\nu'-\nu}d\nu'. \tag{4.4}$$

Mathematically, even if $\Delta(\nu)$ is a small quantity, $\delta(\nu')$ could be a large, rapidly oscillating function. It is necessary, therefore, to supplement the mathematics by a physical assumption. It is known that $\Xi(\nu')$, $\nu'<-1$, describes the $\pi\pi$ amplitude; and one can reject a highly oscillating solution. Thus, if one's initial solution, $\Xi(\nu')$, $\nu'<-1$, is slowly varying, one expects the error,

[13] W. S. Woolcock, thesis, 1961 (unpublished).

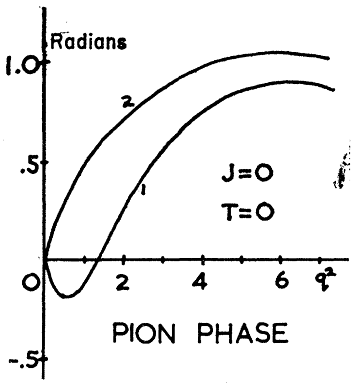

FIG. 5. The S wave.

$\delta(\nu')$, to be slowly varying too. One is at liberty to choose a simple functional behavior of $\Delta(\nu)$, and calculate $\delta(\nu')$. A reasonable, and eminently tractable assumption is that the error of the discrepancy is a constant percentage of itself. Then

$$\Delta(\nu)=c\Xi(\nu),\quad \nu>0, \tag{4.5}$$

and it follows that

$$\delta(\nu')=c\Xi(\nu'),\quad \nu'<-1. \tag{4.6}$$

Thus one expects about the same percentage error in the solution as in the discrepancy.

It is worth while pointing out that a constant error in the rescattering integral,

$$\frac{\mathfrak{P}}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\frac{\Im N(\nu')}{\nu'},$$

which could arise because of faulty assumptions about the very high energy part of the integral, causes no error in $N(\nu')$, $\nu'<-1$.

The partial-wave expansions of the invariant amplitudes (1.17) are truncated at the D wave. Together With Eq. (2.5) this allows one to write the components of $N(\nu)$, $\nu<-1$:

$$\begin{aligned}(p^2/2\pi)(\tfrac{3}{2})^{1/2}N_1&=-f^{00}+5(pq)^2\{M(\tfrac{3}{2})^{1/2}f_-^{02}-f_+^{02}\},\\(p^2/6\pi)N_2&=\tfrac{1}{2}M(\tfrac{3}{2})^{1/2}f_-^{11}-f_+^{11},\\(1/10\pi)N_3&=f_-^{02},\\(1/3\sqrt{2}\pi)N_4&=f_-^{11}.\end{aligned} \tag{4.7}$$

There are here four equations, and five unknown helicity amplitudes, and f_+^{02} cannot be determined. This difficulty could be removed by writing a dispersion relation for the derivative of A^+ with respect to momentum transfer, although the procedure might be subject to large error. The difficulty is circumvented below by neglecting f_+^{02}, although still retaining f_-^{02}.

If the D-wave contribution to N, in Eqs. (4.7) is neglected entirely, Eq. (1.20) gives

$$\begin{aligned}\delta_0^0&=\arg N_1,\\\delta_1^1&=\arg N_2=\arg N_4,\\\delta_2^0&=\arg N_3.\end{aligned} \tag{4.8}$$

To obtain a more accurate expression for the S wave, f_-^{02} can be eliminated between the first and third of Eqs. (4.7). f_+^{02} is still neglected. This gives

$$\delta_0^0=\arg[N_1-Mq^2N_3]. \tag{4.9}$$

The positive helicity D wave occurs in the expression for N_1 with a coefficient of about $\frac{1}{8}$, compared with unity for the negative helicity amplitude. If the two amplitudes are of about the same size, it can be shown that neglect of f_+^{02} has no significant effect. This, however, may not be the case, for the corresponding ratio between the positive and negative helicity P waves, derived from N_2 and N_4, is about 4, agreeing with the known ratio of the nucleon form factors, to which these amplitudes are connected. If f_+^{02} were four times f_-^{02}, the S-wave pion-phase shift derived from the analysis would be very slightly affected at the highest energies considered. If it were any greater than this, the results could be significantly altered at lower energies, too.

5. RESULTS

The results are displayed as graphs of phase shifts, in Figs. 5, 6, and 7. The abscissa in each case is q^2, in units of the pion mass squared, and the ordinate is in radians. The error bands, based on the input experimental errors, are 10% for the plus, 25% for the minus combinations. Some of the parameters extracted have larger errors, however, for example the resonance half-width, as indicated below.

The $T=0$, $J=0$, $\pi\pi$ interaction shows a low-energy repulsion, but is predominantly attractive over the range $0<q^2<7$. The scattering length is -0.4 ± 0.1; and the phase shift passes through zero at $q^2=1.35\pm0.20$. The minimum in the negative going part is $-10°$; and a

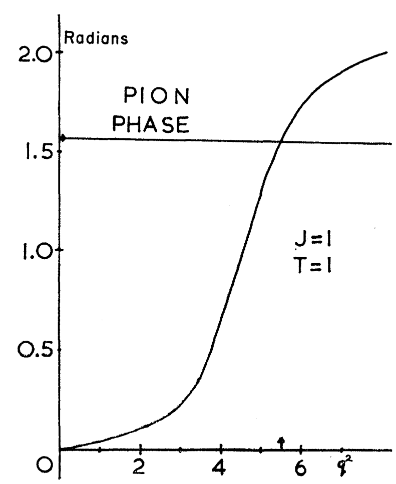

FIG. 6. The P wave.

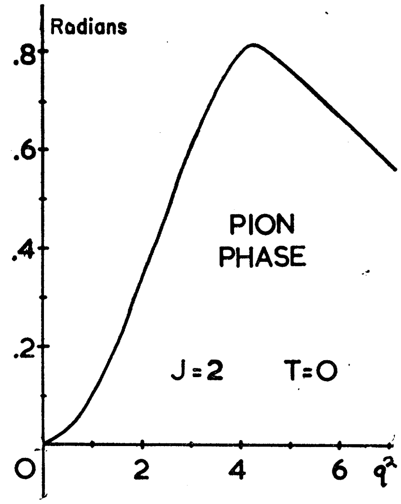

FIG. 7. The D wave.

positive maximum of 50° is reached at $q^2=6$. Since most approaches to the $\pi\pi$ problem, based on a solution of the $\pi\pi$ equations in a self-consistent way, give a positive definite phase shift, it is of interest to see what are the circumstances under which such a solution would be acceptable in the present method.

It is found that if the plus combination of the small p waves, that is, $f_{1-}^{1/2}+2f_{1-}^{3/2}$, were altered from Woolcock's values by two of his standard deviations at threshold, then the negative going part of the S-wave pion-pion phase shift would be eliminated, giving a scattering length of $+0.6$. Therefore, it would seem that an accurate determination of the small $\pi N-\pi N$ phase shifts is necessary before a definite statement about the very low energy $\pi\pi-\pi\pi$ S wave can be made. This conclusion agrees with that of Hamilton *et al.*[14] The two versions of the S wave are shown in Fig. 5 (curves 1 and 2, respectively).

The P-wave estimate, coming from the minus combinations, has a larger error. There are two values, from N_2 and N_4, and these agree, within the errors. The scattering length is 0.030 ± 0.015; and a resonance is found at $q^2=5.5\pm1.0$. The half-width is subject to large error, since it depends on the derivative of the real part of the amplitude, as it passes through zero. The estimate is 200 ± 160 MeV. The discrepancy is not very sensitive to the resonance, since the effect is smoothed out by the dispersion integral. The phase shift is shown in Fig. 6.

The $T=2$, $J=0$ interaction is attractive, with a scattering length of 0.10 ± 0.03. The phase shift is shown in Fig. 7, where it will be seen that there is a peak of about 45° at $q^2=4.2$.

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[14] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick (to be published).