Vector Charge and Magnetic Moment Form Factors of the Nucleon*

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The 2π contribution to the electromagnetic structure of the nucleon has been recalculated by use of a new method for evaluating the left-hand cut of the Frazer-Fulco amplitudes. Excellent agreement with experimental data has been obtained by using the experimental values for the position and the width of the ρ meson.

IN this paper, we present the results of a theoretical study of the vector part of the electromagnetic structure of the nucleon which differs in two important respects from previous calculations on this subject.[1–3] Firstly, we use a more reliable method for evaluating the contributions of the left-hand cut of the Frazer-Fulco amplitudes. This is achieved by representing the distant part of the left-hand cut by means of two poles, whose positions are determined *a priori* by the method of Balázs,[4] and whose residues are determined by following the normalization procedure of Ball and Wong.[3] Secondly, we have been able to get good agreement with the experimental data on the vector electromagnetic form factors, using currently acceptable experimental values of the position and width of the ρ meson and with no free parameters except for subtraction constants representing contributions of high-mass intermediate states. This is especially significant in view of an impression which seems to be prevalent that the mass of the ρ meson is too high, and its width is too small, to account for the nucleon electromagnetic structure.

The vector electromagnetic form factors of the nucleon are given by

$$G_i^V(t)=\frac{1}{\pi}\int_4^\infty \frac{g_i^V(t')dt'}{t'-t}. \tag{1}$$

Here, the 2π contribution is

$$g_i^{V(2\pi)}(t)=-(eq^3/2E)\frac{D_1(0)}{D_1^*(t)}\Gamma_i(t), \tag{2}$$

$$\Gamma_1(t)=(m/p_-^2)[(E^2/m\sqrt{2})f_-^{1(-)}(t)-f_+^{1(-)}(t)], \tag{3}$$

$$\Gamma_2(t)=(1/2p_-^2)[f_+^{1(-)}(t)-(m/\sqrt{2})f_-^{1(-)}(t)], \tag{4}$$

where we are using the usual notation.[5] D_1 is the denominator function of the $J=1$, $T=1$ $\pi\pi$ amplitude. It we now neglect the right-hand inelastic cut of Γ_iD_1, if satisfies the dispersion relation

$$\Gamma_iD_1(t)=\frac{1}{\pi}\int_{-\infty}^{a} dt'\frac{D_1(t')\,\mathrm{Im}\Gamma_i(t)}{t'-t}, \tag{5}$$

where $a=4(1-(1/4m^2))$. In the interval $0<t<a$, the only contribution to $\mathrm{Im}\Gamma_i$ comes from the nucleon pole in the crossed channel, and this can be calculated exactly. For $t<0$, there are contributions from elastic πN scattering and from inelastic processes. These, however, can be seen to be unimportant down to $t\approx-10$ from the fact that the first important contribution is from the 3-3 state of the πN channel; and if one makes the very good approximation of replacing the 3-3 state absorptive part by a δ function,[1] this contribution is nonzero only for $t\leqslant-11$.

We therefore rewrite Eq. (5) as

$$\Gamma_iD_1(t)=\frac{1}{\pi}\left(\int_{-8}^{a}dt'+\int_{-\infty}^{-8}dt'\right)\frac{D_1(t')\,\mathrm{Im}\Gamma_i(t')}{t'-t}, \tag{5'}$$

and in the first integral we use the approximation

$$\mathrm{Im}\Gamma_i(t)=[\mathrm{Im}\Gamma_i(t)]_N, \tag{6}$$

where $[\mathrm{Im}\Gamma_i(t)]_N$ are the nucleon-pole contributions given by Eqs. (2.6a) and (2.6b) of Frazer and Fulco.[1] We now make the change of variables $t=4(\nu+1)$, $t'=4(\nu'+1)$ in the first integral, and $t'=4(1-1/x)$ in the second integral, and rewrite (5′) as

$$\Gamma_iD_1(\nu)=\frac{1}{\pi}\int_{-3}^{-1/4m^2}d\nu'\frac{[\mathrm{Im}\Gamma_i(\nu')]_ND_1(\nu')}{\nu'-\nu}+\frac{1}{\pi}\int_0^{1/3}dx\frac{\mathrm{Im}\,\Gamma_i(-x^{-1})D_1(-x^{-1})}{x(1+\nu x)}. \tag{7}$$

We shall denote the first integral here by f_{Ni}. In the second integral, following Balázs,[4,6] we approximate the kernel $1/(1+\nu x)$ by

$$\frac{1}{1+\nu x}\approx\frac{1}{0.14}\left(\frac{6.25(x-0.02)}{6.25+\nu}-\frac{50(x-0.16)}{50+\nu}\right), \tag{8}$$

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[1] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

[2] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960) and Phys. Rev. Letters **5**, 386 (1960).

[3] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **6**, 29 (1961).

[4] L. Balázs, Phys. Rev. (to be published).

[5] See reference 1. We are using the pion mass as the unit.

[6] It happens that the range of x and of ν over which we have to approximate the kernel is almost the same as that required by L. Balázs in his discussion of the $\pi\pi$ problem. Therefore we have chosen the same positions for the effective poles as he did. The effect of a variation in the position of the nearby effective pole is discussed in the Appendix.

which is a good approximation in the region $0 \leqslant x \leqslant \frac{1}{3}$ and $-1 \lesssim \nu \lesssim 10$. We thus get for $\Gamma_i D_1$ the two-pole expression

$$\Gamma_i D_1(\nu) = f_{Ni} + [\alpha_i/(\nu+6.25)] + [\beta_i/(\nu+50)], \quad \text{for} \quad -1 \lesssim \nu \lesssim 10. \qquad (9)$$

Here, we have a two-parameter expression for each of the $\Gamma_i D$'s. We fix the values of these parameters by using the normalization procedure of Ball and Wong, which is based on the fact that at $\nu = -1$ each $\Gamma_i(\nu)$ and its derivative can be calculated in terms of an integral over physical πN scattering amplitudes. The values of these quantities were calculated by Ball and Wong[3] and are reproduced in Table I. Ball and Wong evaluated these quantities by taking the fixed momentum-transfer dispersion relations without subtractions (set *a* in Table I), and also with one subtraction at πN threshold (set b). One knows now from one's knowledge of the asymptotic behavior of the amplitudes that no subtractions are needed in these dispersion relations.[7] Therefore, the difference between the values in set *a* and set *b* represents the uncertainty in the evaluation of the Γ_i's and their derivatives at $\nu = -1$. We can now solve Eq. (9) and the equation obtained by taking its derivative at $\nu = -1$, and get for α_i and β_i the expressions

$$\alpha_i = -0.63\left[[\Gamma_i D_1(-1) - f_{Ni}(-1)] + 49\left(\frac{\partial}{\partial \nu}(\Gamma_i D_1)_{\nu=-1} - f_{Ni}'(-1)\right)\right],$$

$$\beta_i = 54.88\left[[\Gamma_i D_1(-1) - f_{Ni}(-1)] + 5.25\left(\frac{\partial}{\partial \nu}(\Gamma_i D_1)_{\nu=-1} - f_{Ni}'(-1)\right)\right]. \qquad (10)$$

It now remains to evaluate $D_1(\nu)$ from one's knowledge of the position and width of the ρ meson. For this purpose, we write the $T=1$, $J=1$ $\pi\pi$ amplitude A_1 as

$$A_1(\nu) = (\nu + 1/\nu)^{1/2} e^{i\delta_1} \sin\delta_1 = N_1(\nu)/D_1(\nu), \qquad (11)$$

with

$$N_1(\nu) = a_0 + (\nu - \nu_0)\left(\frac{a_1}{\nu+6.25} + \frac{a_2}{\nu+50}\right), \qquad (12)$$

and

$$D_1(\nu) = 1 - \frac{(\nu-\nu_0)}{\pi}\int_0^\infty d\nu' \left(\frac{\nu'}{\nu'+1}\right)^{1/2} \frac{N_1(\nu')}{(\nu'-\nu)(\nu'-\nu_0)} \qquad (13)$$

$$= 1 - (\nu-\nu_0)[a_0 K(-\nu, -\nu_0) + a_1 K(-\nu, 6.25) + a_2 K(-\nu, 50)], \qquad (14)$$

[7] Virendra Singh and B. M. Udgaonkar, Phys. Rev. **123**, 1487 (1961); also Virendra Singh, thesis, University of California Radiation Laboratory Report UCRL-10254, 1962 (unpublished).

TABLE I. Values of Γ_i and derivatives at $\nu = -1$.[a]

	$\Gamma_1(-1)$	$\Gamma_1'(-1)$	$\Gamma_2(-1)$	$\Gamma_2'(-1)$
Set *a*	-0.140	0.1544	0.0051	-0.0182
Set *b*	-0.141	0.1720	-0.0042	-0.03488 +an uncertain *D*-wave contribution.

[a] Prime on Γ_i denotes derivative with respect to ν.

where K's are known functions, as defined by Chew and Mandelstam.[8]

In writing (12), we have used a subtracted dispersion relation for N_1, with ν_0 (taken as -2) as the subtraction point, and replaced the distant left-hand cut by two poles in the same way as discussed above for $\Gamma_i D_1$. The nearby part of the left-hand cut is known to be weak[4] and has been neglected. The parameters a_0, a_1, and a_2 are now determined by using

$$N_1(\nu) = 0 \quad \text{at} \quad \nu = 0, \qquad (15)$$

$$\text{Re}\, D_1(\nu) = 0 \quad \text{at} \quad \nu = \nu_R, \qquad (16)$$

$$\frac{\text{Re}\, D_1(\nu)}{N_1(\nu)} = \frac{\nu_R - \nu}{\gamma\nu} \quad \text{at} \quad \nu = \nu'$$

with

$$|\nu' - \nu_R| \ll \nu_R. \qquad (17)$$

Here ν_R and γ are defined by the shape of A_1 in the resonance region, viz.,

$$A_1 = \frac{\gamma\nu}{\nu_R - \nu - i\gamma[\nu^3/(\nu+1)]^{1/2}}.$$

We take $\nu_R = 6.5$, corresponding to a resonance energy of 767 MeV,[9] and $\gamma = 0.2$, which corresponds to a width $\Gamma = 120$ MeV, the width Γ being defined as in Button *et al.*[9] We actually use (17) at $\nu' = 5.5$. Thus, the values of the a_i that we get are

$$a_0 = 0.559, \quad a_1 = -3.30, \quad a_2 = 12.40.$$

Having thus determined the $\pi\pi$ amplitude, we evaluate α_i and β_i, using (10), and then $\Gamma_i D_1$ from (9). The form factors $G_i^V(t)$ are then obtained by using (2) and (1).

At this point, we make the approximation, also used by previous workers in this field, of replacing $1/|D_1(\nu)|^2$ by a δ function.[10] We thus write

$$\frac{1}{|D_1(\nu)|^2} = \frac{\pi\gamma[\nu_R(\nu_R+1)]^{1/2}}{N_1^2(\nu_R)} \delta(\nu - \nu_R). \qquad (18)$$

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[9] J. Button, G. R. Kalbfleish, G. R. Lynch, B. C. Maglić, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. **126**, 1858 (1962).
[10] We have made sure that an exact evaluation of the integral, without making a δ-function approximation, does not materially alter the result.

Substitution of (18) into (1) gives

$$G_i^{V(2\pi)}(t)=\frac{\lambda_i}{1-(t/t_R)}, \tag{19}$$

where

$$\lambda_i=-2e\gamma D_1(0)\frac{\nu_R^2}{t_R}\left(\frac{D_1\Gamma_i}{N_1^2}\right)_{\nu=\nu_R}. \tag{20}$$

If we now use the set *a* for values of $\Gamma_i(-1)$ and $\Gamma_i'(-1)$ from Table I, we get from Eq. (20)

$$\lambda_1=1.07(e/2);\quad \lambda_2=0.88(e/2m).$$

From (19), these are the values of the 2π contribution to the vector charge and magnetic moment, respectively, to be compared with the total experimental values $(e/2)$ and $1.85(e/2m)$, respectively. We notice, however, that substantial cancellations between the pole terms of Eq. (9) make the values of λ_i rather sensitive to the values assumed for $\Gamma_i(-1)$, and more particularly of $\Gamma_i'(-1)$; e.g., an increase of $\Gamma_2'(-1)$ by 10%, keeping $\Gamma_2(-1)$ unaltered, increases λ_2 to $1.36(e/2m)$, or more than 50%. As discussed earlier, the values of $\Gamma_i(-1)$ and $\Gamma_i'(-1)$ are certainly not determined with a very great accuracy from the available experimental information in the πN channel. We have, therefore, thought it more appropriate to turn the problem around and to ask if the experimental values of λ_i are, or are not, consistent with the values of Γ_i and Γ_i' at $\nu=-1$, within their present uncertainty. For this purpose, we take the expressions

$$\begin{aligned}F_{1V}&=-0.28+1.28/[1-(t/28)],\\ F_{2V}&=-0.32+1.32/[1-(t/28)],\end{aligned} \tag{21}$$

with $[F_{iV}(t)=G_i^V(t)/G_i^V(0)]$ as given by Hofstadter[11] in an attempt to fit the Stanford data with $t_R=28$. With our choice of $\nu_R=6.5$, i.e., $t_R=30$, these have to be changed slightly into[12]

$$\begin{aligned}F_{1V}&=-0.37+1.37/[1-(t/30)],\\ F_{2V}&=-0.41+1.41/[1-(t/30)].\end{aligned} \tag{22}$$

We thus must have

$$\begin{aligned}\lambda_1&=1.37G_1^V(0),\\ \lambda_2&=1.41G_2^V(0).\end{aligned}$$

With $G_1^V(0)=e/2$ and $G_2^V(0)=1.85(e/2m)$, one gets

$$\begin{aligned}\lambda_1&=1.37(e/2),\\ \lambda_2&=2.61(e/2m).\end{aligned} \tag{23}$$

Using set *a* for values of $\Gamma_1(-1)$ and $\Gamma_2(-1)$ from Table I, we then see that if we choose

$$\begin{aligned}\Gamma_1'(-1)&=0.1484,\\ \Gamma_2'(-1)&=-0.02475,\end{aligned} \tag{24}$$

we can reproduce the experimental values (23) of λ_i corresponding to a pole at $t_R=30$.[13] These values (24) for $\Gamma_i'(-1)$ lie very close to set *a* in Table I. We, therefore, conclude that the 2π contribution to the vector form factors can be explained in terms of a single vector meson with the current values of the mass and width of the ρ meson.[14]

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APPENDIX

In the above discussion we fixed the positions of the effective poles on the left-hand cut *a priori* at $\nu=-6.25$ and -50, respectively, so as to make a good approximation to the kernel in the region of interest. Once the positions of the poles are considered from this point of view, there is some, but not much, latitude allowed in the choice of these positions, and the results are not expected to depend much upon this choice, as already emphasized by Balázs.[4] We have verified that this is indeed the case. For this purpose we changed the position of the nearby pole only, since one expects the results to be more sensitive to the position of a nearby pole. It was found that a change in this position by as much as 100% does not change the values of $\Gamma_i'(-1)$ required to get the correct experimental values (23) of λ_i by more than 10%, which is certainly within the range of uncertainty of these values in Table I. For example, a choice of $\nu=-4$ for the position of the nearby pole in the case of the $\pi\pi$ amplitude leads to a value $\Gamma_1'(-1)=0.152$, to be compared to the value 0.1484 quoted in the text. Similarly a choice of $\nu=-4$ or $\nu=-10$ for the nearby pole of the Frazer-Fulco amplitudes leads to $\Gamma_1'(-1)=0.165$ and 0.139, respectively.

The reader may convince himself that $\nu=-4$ and -10 are rather extreme values from the point of view of approximating the kernel $1/(1+\nu x)$ in the region of interest.

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[11] R. Hofstadter, in *Proceedings of the Aix-en-Provence International Conference on Elementary Particles, 1961* (Centre d'Etudes Nucleaires de Saclay, Seine et Oise, 1961), R. R. Wilson, *ibid.* Vol. **2**, p. 21 (1961).

[12] The coefficients in Eq. (22) have been adjusted so that F_{iV} and F_{iV}' have the same values at $t=0$, as given by (21).

[13] We have kept $\Gamma_i(-1)$ fixed in view of the fact that λ_i are only weakly dependent on $\Gamma_i(-1)$. Further, as seen from Table I, $\Gamma_1(-1)$ is much better known than $\Gamma_1'(-1)$, and $\Gamma_2(-1)$ is very small.

[14] It is gratifying to note that the $\pi^+\pi^0$ mass difference also requires a $T=1$, $J=1$ meson of mass approximately 750 MeV. See S. K. Bose and R. E. Marshak, Nuovo cimento **25**, 529 (1962).