# Energetic Stability of Persistent Currents in a Long Hollow Cylinder*

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This paper represents an extension of the work of Schafroth in which the carriers of electricity in a superconductor are assumed to constitute an ideal Bose gas. The long-range electromagnetic interactions are fully taken into account by the method of the self-consistent field. The corresponding general equations are obtained from a variational principle applied to the total energy of the system, and the change of the total energy due to individual particle transitions is considered. Axially symmetric solutions are found for the geometry of a long hollow cylinder and for the case in which all particles are in the same lowest single-particle level compatible with the angular momentum $l'\hbar$ around the cylinder axis. This case represents a state of the system in which $l'$ flux quanta are trapped in the cylinder. Assuming the lattice to be at $T=0$, such a state will be stable against single-particle transitions if the corresponding energy change of the system is positive. This state need not be the state of lowest energy, but it would have a practically infinite lifetime if the stability criterion against single-particle transitions is satisfied. Indeed, its energy could only be lowered by a simultaneous transition of a number of particles comparable to their total number, which would be an extremely rare event. It is shown that stability depends on the fields $H_1$ and $H_2$ in the cylinder hole and outside the cylinder, respectively. For $H_1=H_2$, stability requires that these fields be less than $H^*=4\pi n\mu_0$, where $n$ is the density of the bosons and $\mu_0$ is the Bohr magneton. For $H_2=0$, it requires that $H_1<(H_c)_T$ where $(H_c)_T$ depends upon the hole radius $r_1$ and the wall thickness $d$. For $d\gg r_1$, one finds $(H_c)_T=H^*$, while for $d\ll r_1$, one has $(H_c)_T=(d/r_1)H^*$. The difference between $(H_c)_T$ and $H^*$ for $d\ll r_1$ is further investigated, and it is shown that the case for which $H^*>H_1>(H_c)_T$ may nevertheless experimentally appear to be stable since the stability conditions are strongly modified considering that the transfer of very small but finite amounts of energy to the lattice may not occur during the time of a measurement.

## I. INTRODUCTION

THE recent discovery[1,2] of magnetic flux quanta $hc/2e$ has furnished an important clue toward the basic understanding of superconductivity. The idea of quantized trapped flux in a multiply connected superconductor was first suggested by London[3]; his arguments were based upon the behavior of the wave function in the field-free region inside a superconductor and indicated that the trapped flux should appear in integer units of $hc/e$. These arguments seemed misleadingly to apply to the flux surrounded by any ring-shaped quantum mechanical system, but Byers and Yang[4] have clarified the particular properties of the superconductor which are essential for flux quantization. Their work shows that the free energy of the conduction electrons, although generally a periodic function of the trapped flux with period $hc/e$, exhibits negligible variations in the case of a normal conductor. In contrast, the existence of pronounced periodic minima of this function and, hence, the appearance of flux quantization must be considered to be peculiar to the superconductive state. Furthermore, the observed magnitude of the flux quantum indicates the occurrence of such minima not only at integer but also at integer plus one-half multiples of $hc/e$ and suggests as a simple interpretation that the carriers of electricity in a superconductor consist of electron pairs with an effective elementary charge twice as large as that of a single electron.

The relation of this feature to the pair correlation in the theory of Bardeen, Cooper, and Schrieffer,[5] has been pointed out by Byers and Yang,[4] Brenig,[6] and by Onsager[7] who, in the same connection, has also emphasized the usefulness of Schafroth's[8] model of the superconductor as a condensed Einstein-Bose gas of electron pairs. The same model was used by Blatt to show that it leads automatically to flux quantization and to a reduced magnitude of the flux quantum in very small superconductive rings.[9]

The present paper deals in some further detail with the properties of a charged boson gas, chosen as a model for the superconductive state of a metal. This choice was primarily motivated by a desire for mathematical simplicity, and additional simplifying assumptions will be introduced below to the extent to which they can be made while still retaining the essential features of the phenomena under consideration. The model does not account for some of the important properties of real superconductors, notably the observed existence of a gap in the energy spectrum, but we believe that it bears on reality in the description of

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electromagnetic effects and that the methods and results presented here are of sufficiently basic character to find their qualitative counterpart in a totally adequate theory.

The treatment is largely an extension of the earlier work of Schafroth.[8] It is thus assumed that one deals with an ideal gas of bosons in the sense that their short-range interactions are negligible, that only their long-range electromagnetic interactions need to be considered, and that these can be treated by the method of the self-consistent field. The underlying idea that the bosons originate from a formation of spinless "di-electronic molecules" is used by considering their charge "$e$" to be twice that of the electron.

The principal goal of this paper is to investigate the stability of the currents which are responsible for the Meissner effect and for the persistent flux trapped in the interior of a superconductive ring. It was shown by Schafroth that the Bose model leads to a Meissner effect if the gas is below the temperature at which Einstein-Bose condensation occurs, and if the magnetic field is less than a critical value. For simplicity, it will be assumed here that the gas is in thermal contact with a lattice which is at the temperature $T=0$. Transitions of the gas which lead to an increase of its energy are, in this case, excluded and the problem of stability reduces to the question whether or not there occur transitions accompanied by a decrease of its energy. It will be shown that for a long hollow cylinder of given inside and outside radius, the answer to this question depends upon the magnetic field both in the interior and the exterior region. For sufficiently weak fields, only the highly improbable simultaneous transition of a number of bosons comparable or equal to their total number can lead to a decrease of energy so that one deals with a metastability of the trapped interior flux which amounts to a practically indefinite persistency. Beyond certain critical fields, however, single boson transitions will be seen to become energetically possible, and their occurrence will presumably lead to a rapid collapse of the superconductive currents.

Starting with the expression for the total energy of a system described by this model, the second section of this paper deals with the equations for the self-consistent field and develops the formulas necessary to calculate changes of the energy. The particularly simple geometry of a long hollow cylinder is introduced in the third section, and axially symmetrical solutions of the equations are discussed. In the fourth section, the results are applied to investigate stability and flux quantization for cylinders of different wall thickness.

## II. TOTAL ENERGY AND THE EQUATIONS FOR THE SELF-CONSISTENT FIELD

The equations for the self-consistent field can be conveniently derived from an extremum principle applied to an appropriate form of the total energy of the system. We consider in this model a system of $N$ spinless bosons with mass $m$ and charge $e$[10] under the influence of an electromagnetic field which is partly of external origin and partly due to the average field produced by all particles. It shall be described by a vector potential $\mathbf{A}$ and a scalar potential $U/e$, so that $U$ represents the potential energy of a particle. As pointed out by Schafroth,[8] it is essential to include in $U$ the neutralizing effect of the lattice ions, and it is permissible for this purpose to attribute to them a uniformly distributed charge equal and opposite to the total charge of the bosons. It is further important to consider the effect of an external magnetic field, and the corresponding contribution to the vector potential $\mathbf{A}$ must be distinguished from that due to the currents in the superconductor itself; this latter "internal" contribution to the vector potential will be denoted by $\mathbf{A}_i$.

A stationary state of the system is to be characterized by the occupation numbers $N_s$ of individual particle states $s$, described by a complete set of orthogonal and normalized wave functions $\psi_s$. The corresponding total energy of the system can then be written in the form

$$E=\sum_s N_s\int\psi_s^*\mathfrak{H}\psi_s d\tau-n\int U d\tau+\frac{1}{8\pi}\int[\mathrm{curl}\mathbf{A}_i]^2 d\tau$$
$$-\frac{1}{8\pi e^2}\int[\mathrm{grad}U]^2 d\tau. \quad (1)$$

The first term in this expression represents the mechanical energy of the system, and the Hamiltonian operator for the individual particle is given by

$$\mathfrak{H}=\frac{1}{2m}\left(\frac{\hbar}{i}\mathrm{grad}-\frac{e}{c}\mathbf{A}\right)^2+U. \quad (2)$$

The second term accounts for the potential energy of the neutralizing ions, described by a uniform charge density $-ne$ inside the metal of volume $V$ where $n=N/V$ is the mean value of the number of bosons per unit volume. The third term is the energy stored in the magnetic field $\mathbf{H}_i=\mathrm{curl}\mathbf{A}_i$ which is produced by the motion of the charged bosons, and it accounts for the work done against their forces of mutual induction while the system was brought into the state under consideration. The last term represents the negative of the energy stored in the electric field $\mathbf{E}=-(1/e)\,\mathrm{grad}U$. This term has to be added since twice the positive amount of this energy is already contained in the potential part of the mechanical energy of the bosons together with the potential energy of the neutralizing ions. Equation (1), thus, correctly represents the total energy as the sum of the kinetic energy of the particles

[10] In accordance with the underlying idea of electron pairs, mentioned in the introduction, the charge and mass of these particles should be understood to be twice those of the electron.

and the electromagnetic energy of their accompanying fields, and it can assume only non-negative values.

The self-consistent field equations are obtained by demanding that the expression of Eq. (1) for the energy $E$ has an extremum against independent variations of the functions $\psi_s$, $U$, and $\mathbf{A}_i$. Indeed, the equation $\delta E/\delta\psi_s^*=0$, obtained upon variation of the conjugate complex of one of the wave functions $\psi_s$, is equivalent to the time-independent Schrödinger equation,

$$\mathfrak{H}\psi_s=E_s\psi_s, \tag{3}$$

for the individual particle states.[11] One obtains further from $\delta E/\delta U=0$ the Poisson equation

$$\nabla^2 U=-4\pi e\rho, \tag{4}$$

and from $\delta E/\delta\mathbf{A}_i=0$ the Maxwell equation

$$\text{curl}\mathbf{H}_i=\text{curl curl}\mathbf{A}_i=4\pi\mathbf{i}/c, \tag{5}$$

where, in view of the Hamiltonian $\mathfrak{H}$ of Eq. (2), the charge and current density, respectively, are given by[12]

$$\rho=e\left(\sum_s N_s\psi_s^*\frac{\partial\mathfrak{H}}{\partial U}\psi_s-n\right)=e\left(\sum_s N_s\psi_s^*\psi_s-n\right). \tag{6}$$

and,

$$\mathbf{i}=-c\,\text{Re}\left\{\sum_s N_s\psi_s^*\frac{\partial\mathfrak{H}}{\partial\mathbf{A}_i}\psi_s\right\}$$

$$=\frac{e}{m}\,\text{Re}\left\{\sum_s N_s\psi_s^*\left(\frac{\hbar}{i}\,\text{grad}-\frac{e}{c}\mathbf{A}\right)\psi_s\right\}. \tag{7}$$

To each given set of occupation numbers $N_s$ there belongs a definite value of the total energy. To obtain this value, one has to solve the Eqs. (3), (4), and (5) with the appropriate boundary conditions and with the chosen numbers $N_s$ appearing in the Eqs. (6) and (7) for the charge and current densities; inserting the solution for the wave functions and potentials, the energy of this particular state of the system is then found by evaluating the expression for $E$ given in Eq. (1). It is evident that this procedure may become prohibitively involved for an arbitrary geometry and arbitrary occupation numbers; however, it will be carried out in Sec. III for the simple geometry of the long hollow cylinder and for the case of greatest interest in this paper where all particles are in the same state.

To investigate stability, it is further of importance to calculate the change of energy of the system upon a certain change of the occupation numbers. Under otherwise general conditions, we shall first investigate the case where one particle is transferred from its initial state $s'$ to a final state $s''$ while all other particles remain in their original states. Partly, one has to consider here the explicit change $\Delta N_{s'}=-1$ and $\Delta N_{s''}=1$ of the occupation numbers $N_{s'}$ and $N_{s''}$, respectively, in the first part of the energy of Eq. (1). The corresponding change of $E$ is given by

$$\Delta E=\int\psi_{s''}^*\mathfrak{H}\psi_{s''}d\tau-\int\psi_{s'}^*\mathfrak{H}\psi_{s'}d\tau$$

or, because of Eq. (3) and of the normalization of the wave functions, by

$$\Delta E=E_{s''}-E_{s'}, \tag{8}$$

where the individual particle energies $E_{s'}$ and $E_{s''}$ are eigenvalues of the Hamiltonian $\mathfrak{H}$ of Eq. (2) with $\mathbf{A}$ and $U$ corresponding to the original occupation numbers. It is necessary, in addition, to consider the changes $\Delta\mathbf{A}$ and $\Delta U$ of these functions and, consequently, also the change $\Delta\psi_s$ of the wave functions which result from the transfer of the particle from state $s'$ to $s''$ since they may likewise affect the total energy of the system. The causes of these changes, however, are the changes of the charge and current densities which can be seen, from Eqs. (6) and (7), to be only of relative order of magnitude $1/N$, since all but one of the $N$ particles remain in their original states. The relative change of $\psi_s$, $\mathbf{A}$, and $U$ is, therefore, likewise of this order of magnitude, but this fact alone is not sufficient to justify their neglect, even for a very large total number $N$ of particles, since the change $\Delta E$ of Eq. (8) is itself of order $1/N$ in comparison to the total energy $E$. The true justification arises from the circumstance that the original value of $E$ was obtained from an extremum principle so that the contribution to its change which is linear in that of the functions to be varied has to vanish. The additional quadratic and higher terms are indeed of higher order in $1/N$, and their effect upon the energy change for a very large number $N$ of particles is thus completely negligible in comparison to the value $\Delta E$ of Eq. (8).

While the preceding considerations refer to the transition of a single particle, they can be extended to an arbitrary change $\Delta N_s$ of the occupation numbers as long as this change does not materially affect the charge and current densities, i.e., as long as one deals with transitions in which the overwhelming majority of the particles is not involved. In this case, the change of energy is more generally given by

$$\Delta E=\sum_s\Delta N_s E_s, \tag{9}$$

where the eigenvalues $E_s$ are still those of the Hamil-

[11] The fact that the variation of the wave functions is constrained by the normalization $\int|\psi_s|^2d\tau=1$ is taken into account in the usual way by considering the eigenvalues $E_s$ as Lagrange multipliers, i.e., by subtracting from $E$ the sum $\sum_s N_s E_s\int|\psi_s|^2d\tau$. The variation of $\psi_s$ leads to the conjugate complex of Eq. (3). While the index $s$ does not originally refer to any specific ordering this may be achieved, for example, by labeling the states $s$ in the order of increasing eigenvalues $E_s$ of $\mathfrak{H}$.

[12] The symbol "Re" in Eq. (7) signifies the real part of the subsequent expression; in obtaining the variation of $\mathfrak{H}$ upon that of $\mathbf{A}_i$, it has to be noticed here that the total vector potential $\mathbf{A}$ appearing in Eq. (2) differs from the internal contribution $\mathbf{A}_i$ by the part $\mathbf{A}_e=\mathbf{A}-\mathbf{A}_i$, which is due to a given external magnetic field and, therefore, is not to be varied with $\mathbf{A}_i$.

tonian with $\mathbf{A}$ and $U$ determined by the fields corresponding to the original set of occupation numbers. A more rigorous derivation of Eq. (9) is presented in Appendix I. Including the next higher term of order $1/N$, it is shown there that

$$\Delta E=\sum_s \Delta N_s(E_s+\tfrac{1}{2}\Delta E_s), \tag{10}$$

where $\Delta E_s$ is the first-order change of the eigenvalue $E_s$ of the Hamiltonian $\mathfrak{H}$, caused by the modification of $\mathbf{A}$ and $U$, upon the change $\Delta N_s$ of the occupation numbers.

It is true that Eq. (9) as well as Eq. (10) lose their validity if the number of particles participating in a single transition is comparable to the total number $N$ since not only the second term in Eq. (10) but also all higher neglected terms are then appreciable. It will be seen in Sec. III that such transitions cannot, in the case of trapped flux, be excluded for energetic reasons, but it can be safely expected that their corresponding matrix elements are so exceedingly small as to amply account for the observed extreme stability of persistent currents.

## III. AXIALLY SYMMETRICAL SOLUTIONS FOR A LONG HOLLOW CYLINDER

The geometry of a long hollow cylinder has not only the advantage of mathematical simplicity but it also describes adequately the actual conditions of the recent experiments[1,2] on flux quantization. We shall investigate axially symmetrical solutions of the Eqs. (3), (4), and (5) for a circular cylinder of length $L$, very large compared to both its inner and its outer radius $r_1$ and $r_2$, respectively. The wall thickness of the cylinder will be denoted by

$$d=r_2-r_1,$$

and its cross sectional area by

$$a=\pi(r_2^2-r_1^2). \tag{11}$$

Cylindrical coordinates $r$, $\theta$, and $z$ shall be used where $r$ measures the distance from the cylinder axis, $\theta$ the angle around it, and $z$ the distance parallel to the axis.

Because of the assumed axial symmetry both the internal field $\mathbf{H}_i=\mathrm{curl}\mathbf{A}_i$, produced by the motion of the particles, and the total field $\mathbf{H}=\mathrm{curl}\mathbf{A}$ are parallel to the axis and depend only on $r$. Their magnitudes can therefore be written in the form

$$H_i(r)=\frac{1}{2\pi r}\frac{d\phi_i(r)}{dr}; \quad H(r)=\frac{1}{2\pi r}\frac{d\phi(r)}{dr}; \tag{12}$$

where $\phi_i$ and $\phi$ represent the corresponding amounts of flux contained within the radius $r$, and are related to the tangential components $A_i(r)$ and $A(r)$ of the vector potential through the equations

$$\phi_i(r)=2\pi r A_i(r); \quad \phi(r)=2\pi r A(r).$$

It is further convenient to measure flux in units of the "flux quantum"[13]

$$\phi^*=2\pi\hbar c/e \tag{13}$$

and to thus define the dimensionless quantities

$$\alpha_i(r)=\phi_i(r)/\phi^*; \quad \alpha(r)=\phi(r)/\phi^*. \tag{14}$$

For the same reasons of symmetry, the scalar potential and, hence, the potential energy $U$ of a particle depends only on $r$.

The state $s$ of an individual particle can then be characterized by two quantum numbers $k$ and $l$ which, multiplied by $\hbar$, represent the components parallel to the $z$ axis of the linear and angular momentum, respectively, and by an additional radial quantum number $q$. With the index $s$ replaced by the symbol $klq$, the corresponding wave function can thus be written in the form

$$\psi_{k,l,q}=\frac{1}{(La)^{1/2}}e^{i(kz+l\theta)}f_{ql}(r). \tag{15}$$

The function $f_{ql}$ has to satisfy the conditions

$$f_{ql}(r_1)=f_{ql}(r_2)=0 \tag{16}$$

at the radial boundaries of the metal[14] and the condition of normalization

$$2\pi\int_{r_1}^{r_2} f_{ql}^2(r)rdr=a. \tag{17}$$

The expression for the total energy of Eq. (1) has then the form

$$E=\frac{2\pi}{a}\sum_{klq}N_{klq}\int_{r_1}^{r_2}\left\{\frac{\hbar^2}{2m}\left[\left(\frac{df_{ql}}{dr}\right)^2+k^2f_{ql}^2+\frac{(l-\alpha)^2}{r^2}f_{ql}^2\right]+Uf_{ql}^2\right\}rdr-2\pi nL\int_{r_1}^{r_2}Urdr$$
$$+\frac{L}{4}\left(\frac{\hbar c}{e}\right)^2\int_0^\infty\frac{1}{r}\left(\frac{d\alpha_i}{dr}\right)^2dr-\frac{L}{4e^2}\int_0^\infty\left(\frac{dU}{dr}\right)^2rdr, \tag{18}$$

and the variation of the three functions $f_{ql}$, $U$, and $\alpha$ of $r$ results in the equations equivalent to Eqs. (3), (4),

[13] With the understanding that the symbol $e$ represents twice the charge of the electron, one has in the conventional notation $\phi^*=hc/2e$.

[14] Strictly, the boundary conditions at the ends of the cylinder demand that the wave function vanish likewise for the corresponding values of $z$, so that it should be described by standing rather than by running waves in the $z$ direction. It would be misleading, however, to achieve this by a $z$ dependence in the form of the factor $\sin(kz)$. As pointed out by Schafroth and further discussed below, the Coulomb forces demand that the square of the wave function be practically constant, so that the form chosen in Eq. (15) provides a far better and, in fact, perfectly adequate description of the actual conditions.

and (5):

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{df_{ql}}{dr}\right)+k^2 f_{ql}+\frac{1}{r^2}(l-\alpha)^2 f_{ql}\right]+Uf_{ql} = E_{klq}f_{ql}(r), \quad (19)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)=-4\pi e^2\left(\frac{1}{V}\sum_{klq}N_{klq}f_{ql}^2-n\right), \quad (20)$$

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\alpha_i}{dr}\right)=-\frac{4\pi e^2}{mc^2}\frac{1}{V}\sum_{klq}N_{klq}(l-\alpha)f_{ql}^2, \quad (21)$$

where $V=La$ is the volume of the metal. The right side of Eq. (20) and (21) has to vanish outside the metal. This condition is satisfied if $n$ is understood to have the constant value $N/V$ for $r_1<r<r_2$ and to be zero otherwise, since $f_{ql}$ automatically vanishes outside the metal.

To each solution of these equations, obtained for a given set of occupation numbers $N_{klq}$ and satisfying the appropriate boundary conditions, there belongs, according to Eq. (18), a definite value of the total energy $E$. In line with the preceding introduction, it is our goal to search among the manifold of solutions for those which describe the occurrence of energetically stable superconductive currents. With the lattice assumed to be at the absolute zero, one may be guided in this search by observing that the energy would reach its absolute minimum if one postulated the existence of an individual particle state $k=0$, $l=l'$, $q=0$ such that $f_{0l'}=1$ and where $l'$ is a given integer. Indeed, with all particles in this state and postulating further that $\alpha_i$ is constant, $U=0$, and that $\alpha$ has the constant value $\alpha=l'$ inside the metal, one would satisfy the Eqs. (19), (20), and (21), and $E$ would according to Eq. (18) attain its lowest value $E=0$. However, these postulates do not represent the proper solution of the problem, since the boundary conditions expressed by Eq. (16) do not permit the radial function $f$ to have a constant value throughout the metal. Besides, a constant value of $\alpha_i$ means according to Eqs. (12) and (14) that the magnetic field $\mathbf{H}_i$, produced by the currents in the metal, is everywhere zero and that the above postulates could, therefore, only refer to the trivial case of vanishing superconductive currents. Nevertheless, these postulates can be expected to agree with the actual properties of the looked-for solutions in several respects.

In the first place, it remains true that the choice $k=0$ is energetically most favorable, since any other choice would merely increase the individual particle energy $E_{klq}$ of Eq. (19) by the amount $\hbar^2k^2/2m$. Secondly, one may expect that there exists a definite value $l'$ of the quantum number $l$ which, for a given function $\alpha(r)$, will minimize the contribution of the term proportional to $(l-\alpha)^2$ on the right side of Eq. (18).[15] Finally, the solution $f_{ql}(r)$ of Eq. (19) which, for given $k$ and $l$, leads to the lowest value of $E_{klq}$ shares with the constant at least the feature that it has no nodes in the interval $r_1<r<r_2$. This individual particle state will again be denoted by $q=0$, and one may thus expect that energetic stability can be attained if all particles are in the state $k=0$, $l=l'$, $q=0$, i.e., if the occupation numbers are chosen to be

$$N_{klq}=N\delta_{k0}\delta_{ll'}\delta_{q0}. \quad (22)$$

The actual proof of these conjectures will be presented in the following section by investigating the conditions of stability and by showing that these conditions can indeed be fulfilled in the state of the system considered here, insofar as transitions leading to other states are accompanied with an increase of energy.

The particular choice of the occupation numbers given by Eq. (22) is uniquely characterized by the single quantum number $l'$. With the notation $f_{0l'}=f_0$, one obtains from Eq. (18) for the corresponding value of the total energy of the system,

$$E(l')=2\pi nL\int_{r_1}^{r_2}\left\{\frac{\hbar^2}{2m}\left[\left(\frac{df_0}{dr}\right)^2+\frac{1}{r^2}(l'-\alpha)^2f_0^2\right]+U(f_0^2-1)\right\}rdr+\frac{L}{4}\left(\frac{\hbar c}{e}\right)^2\int_0^\infty\frac{1}{r}\left(\frac{d\alpha_i}{dr}\right)^2dr-\frac{L}{4e^2}\int_0^\infty\left(\frac{dU}{dr}\right)^2rdr, \quad (23)$$

by using the relation $N=nV=nLa$ between the total number $N$ of particles and their mean number $n$ per unit volume. Using further in Eqs. (19), (20), and (21), the notations

$$E_{0l0}=\hbar^2\epsilon_l/2m, \quad (24)$$

$$U=\hbar^2u/2m,$$

$$4\pi ne^2/mc^2=1/\lambda^2, \quad (25)$$

$$16\pi me^2n/\hbar^2=1/\mu^4, \quad (26)$$

the corresponding equations for $f_0$, $u$, and $\alpha$ are

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0}{dr}\right)=\left[\frac{1}{r^2}(l'-\alpha)^2+u-\epsilon_{l'}\right]f_0, \quad (27)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=-\frac{1}{2\mu^4}(f_0^2-1), \quad (28)$$

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\alpha_i}{dr}\right)=\frac{1}{\lambda^2}(\alpha-l')f_0^2, \quad (29)$$

[15] This argument is essentially equivalent to the explanation of flux quantization given by Byers and Yang (reference 4), and the postulate $\alpha=l'$ in the interior of the metal means, in fact, that the cylinder contains the integer number $l'$ of flux quanta $\phi^*$. It may also be observed that the expression for the energy of Eq. (18) satisfies their general theorem of periodicity with the flux in the sense that it remains unchanged if $\alpha_i$ (and hence $\alpha$) is increased by an integer and the summation index $l$ relabeled by subtraction of this integer.

where, according to Eqs. (11), (16), and (17), $f_0$ has to satisfy the conditions

$$f_0(r_1)=f_0(r_2)=0$$

$$2\pi\int_{r_1}^{r_2}(f_0^2-1)rdr=0.$$

The somewhat inconvenient appearance of both $\alpha$ and $\alpha_i$ in Eq. (29) can be avoided by observing that the right side of this equation is proportional to the current density in the metal, responsible for the internal field $H_i$ of Eq. (12). The replacement of $\alpha_i$ by $\alpha$ on the left side merely adds a term corresponding to external currents which vanish in the region of space considered here. One is, therefore, allowed to rewrite Eq. (29) in the form of the differential equation

$$\frac{d}{r\,dr}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)=\frac{1}{\lambda^2}(\alpha-l')f_0^2 \tag{30}$$

for $\alpha$ alone.

Besides the radii $r_1$ and $r_2$, there enter into the problem the two characteristic lengths $\lambda$ and $\mu$ defined by Eqs. (25) and (26), respectively. The first is the well-known penetration depth of London; assuming the number and mass of the bosons to be comparable to those of the conduction electrons in the metal, i.e., with $m\cong10^{-27}$ g, $n\cong10^{22}/\text{cm}^3$, it is $\lambda\cong5\times10^{-6}$ cm. With the same assumptions, the second characteristic length is found to be of the order of magnitude of the Bohr radius, i.e., $\mu\cong10^{-8}$ cm. The fact that this length, which appears in Eq. (28), is small compared to $\lambda$ has the important consequence, pointed out by Schafroth,[8] that the radial function $f_0$ changes within a distance of order of magnitude $\mu$ from the value $f_0=0$ at the radii $r_1$ and $r_2$ to the practically constant value $f_0=1$ in the interior of the metal.[16] For $\mu\ll\lambda$, one finds, therefore, in the interior of the metal ($r-r_1\gg\mu$ and $r_2-r\gg\mu$)

$$f_0=1. \tag{31}$$

By a suitable choice of the arbitrary constant in the potential energy $U$ of a particle and, hence, in $u$ it follows then from Eq. (27) that

$$u=-(1/r^2)(l'-\alpha)^2, \tag{32}$$

$$\epsilon_{l'}=0 \tag{33}$$

and from Eq. (30)

$$\frac{d}{r\,dr}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)=\frac{1}{\lambda^2}(\alpha-l'). \tag{34}$$

[16] A similar behavior of the wave function $\psi$ occurs at the ends of the cylinder. It is for this reason that the exponential $z$ dependence has been chosen in Eq. (15); except for a distance of order $\mu$ from the ends, it represents the correct behavior of $|\psi|^2$. Further, if one assumes that only a fraction of the conduction electrons form Bose pairs, i.e., if one inserts for $n$ a smaller number than the one used above, one finds that $\mu$ is even smaller in comparison to $\lambda$ since one has from Eq. (25) and (26) $\mu/\lambda\sim n^{1/4}$.

Writing further

$$\alpha=l'+\beta, \tag{35}$$

the corresponding equation for $\beta$ can be readily solved if one further assumes both $r_1\gg\lambda$ and $r_2\gg\lambda$. One has then

$$\beta=c_1e^{-(r-r_1)/\lambda}+c_2e^{(r-r_2)/\lambda} \tag{36}$$

where the constants $c_1$ and $c_2$ are determined by the values of the field inside and outside of the cylinder. A more detailed discussion of the functions $f_0$, $u$, and $\beta$ near the boundary is presented in Appendix II.

The results obtained for $f_0$, $u$, and $\alpha=l'+\beta$ can be used to find the total energy $E(l')$ from Eq. (23). For later purposes, we are particularly interested in this value for the case of a vanishing external field, i.e., for $\alpha=\alpha_i$. The dominant part is the energy of the magnetic field $H_1$ inside the cylinder given by the second integral on the right side of Eq. (23).[17]

Since $\alpha_i$ depends quadratically upon $r$ between $0\leq r\leq r_1$ and reaches for $r=r_1$ very nearly its maximum value $\alpha_i=\alpha=l'$, one may write $\alpha_i=l'(r/r_1)^2$ and hence,

$$E(l')=l'^2(\hbar c/e)^2(L/2r_1^2). \tag{37}$$

## IV. STABILITY

The results of the preceding sections shall now be used to investigate the energetic stability of the particular configuration where all particles are assumed to be in the same individual state $k=0$, $l=l'$, $q=0$. Following the general discussion of this problem in Sec. II, we shall first ask for the change $\Delta E$ of the total energy of the system upon transition of a single particle to a different individual particle state $k''$, $l''$, $q''$. Since the energy of the initial state has been chosen according to Eq. (33) to be zero, $\Delta E$ is given by Eq. (8) to be simply the individual particle energy $E_{k''l''q''}$ of its new state, determined by finding the corresponding eigenvalue of Eq. (19) with the same functions $U$ and $\alpha$ used in the initial state. The necessary and sufficient condition for energetic stability against this transition demands that the lowest of these eigenvalues be still positive.

It is, therefore, evident from Eq. (19) that one has to choose $k''=0$ since any other choice would lead to a

[17] This can be seen by noting first that the third integral of Eq. (23) can, by means of the Poisson equation, be combined with the part in the first integral proportional to $U$, thereby reducing the contribution of the former by the factor $\frac{1}{2}$; this contribution is negligibly small, since $f_0^2-1$ differs from zero only in the immediate vicinity of order $\mu$ of the boundaries. The part in the first integral proportional to $(df_0/dr)^2$ is negligible for the same reason. One should further notice that in the absence of an external field one has $\alpha_i=\alpha$, and in view of equations (35) and (36) $\alpha-l'=-\lambda d\alpha/dr$ near the boundary of $r=r_1$; because of Eq. (25), the part proportional to $(l'-\alpha)^2$ can then be seen to be of the same order of magnitude as the contribution to the second integral arising from the interior of the metal, i.e., within a distance of order $\lambda$ from the boundary at $r=r_1$. Keeping only the part of the second integral within the limits of integration $r=0$ and $r=r_1$, one commits thus a negligible relative error of order of magnitude $\lambda/r_1$ in the determination of $E(l')$.

higher eigenvalue. For the same reason, the radial function $f_{q''l''}$ must be that which has no node in the interval $r_1<r<r_2$. The corresponding quantum number $q$ shall again be denoted by $q''=0$ so that it is only the azimuthal quantum number $l$ which undergoes a change, given by

$$p=l''-l'. \tag{38}$$

With the notations used in Eq. (27), and with the likewise corresponding notation $f_{0l''}=f_p$, one obtains thus from Eq. (19)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)-\left[\frac{1}{r^2}(l''-\alpha)^2+u\right]f_p=-\epsilon_{l''}f_p. \tag{39}$$

It is further convenient to use the notation introduced in Eq. (35) and (38), so that $l''-\alpha=p-\beta$, and to introduce the additional notation

$$u+(1/r^2)(l'-\alpha)^2=u+\beta^2/r^2=w \tag{40}$$

and

$$\epsilon_{l''}=-\eta_p. \tag{41}$$

One has then, from Eq. (39),

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)-\frac{1}{r^2}(p^2-2p\beta)f_p-wf_p=\eta_pf_p. \tag{42}$$

The transition considered so far is distinguished by the property that it leads to the least increase of energy among all those single-particle transitions in which the azimuthal quantum number changes by the amount $p$. The increase of energy of the system due to any one of these transitions satisfies, therefore, the inequality

$$\Delta E\geq(\hbar^2/2m)\epsilon_{l''}=-(\hbar^2/2m)\eta_p, \tag{43}$$

and none of them is energetically possible unless $\eta_p>0$. Conversely there exist single-particle transitions leading to a decrease of energy as long as one finds among all possible values of $p$ one or more such that $\eta_p>0$. The original state of the system with $k=0$, $l=l'$, and $q=0$ for all particles is, therefore, energetically stable against any single-particle transition, if the determining functions $w$ and $\beta$ of Eq. (42) have the property that none of the resulting values $\eta_p$ is positive. A special case of this equation is for $p=0$, i.e., where $=ll'''$, so that one has from Eqs. (33) and (41) $\eta_0=0$. Since all single-particle transitions with $p=0$ lead necessarily to an increase of energy, the condition for stability hinges solely upon those values of $\eta_p$ for which $p\neq0$. Denoting the highest of these values by $\eta_{\max}$, this condition can be formulated by demanding

$$\eta_{\max}<0, \tag{44}$$

and the "critical" functions $w$ and $\beta$ at which the transition between stability and instability occurs are those for which

$$\eta_{\max}=0. \tag{45}$$

While this criterion for energetic stability has been derived for single particle transitions, it retains its validity according to Eq. (9) for all transitions which do not involve a number of particles comparable to the total number $N$. Classifying these transitions according to the total number of particles,

$$\Delta N_p=\sum_{kq}\Delta N_{kl''q},$$

which change their azimuthal quantum number from the original value $l'$ to the final value $l''=l'+p$, the result of Eq. (43) for the increase of energy of the system can be generalized within the realm of validity of Eq. (9) by the inequality

$$\Delta E\geq-(\hbar^2/2m)\sum_p\Delta N_p\eta_p,$$

with arbitrary positive integers $\Delta N_p$. Since $\eta_0=0$, one still has $\Delta E>0$ if and only if the condition for $\eta_{\max}$ of Eq. (44) is fulfilled, so that it represents indeed the general criterion for stability, while Eq. (45) represents equally generally the critical condition.

The determination of $\eta_p$ demands, rigorously, the solution of Eq. (42) for $f_p(r)$ with the boundary condition $f_p(r_1)=f_p(r_2)=0$. This problem can be considerably simplified by considering the properties of the function $f_0(r)$ for the special case $p=0$ discussed in the preceding section. Except for very small distances of the order of magnitude $\mu$ from the boundaries, one has from Eq. (31) $f_0=1$ and from Eq. (32) and (40) $w=0$ so that Eq. (42) has in this region the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)-\frac{1}{r^2}(p^2-2p\beta)f_p=\eta_pf_p. \tag{46}$$

Starting at the boundary, the effect of the function $w(r)$ consists for the case $p=0$ in rapidly changing $f_0(r)$ from a function with finite slope and vanishing value to a function with vanishing slope. On the other hand, the appearance of the additional terms $(1/r^2)(p^2-2p\beta)$ and $\eta_p$ for $p\neq0$ has only a negligible effect upon the change of the slope of $f_p(r)$ over a distance of the order $\mu$ so that, in very good approximation, this function likewise attains the slope zero in the immediate vicinity of the boundary. It is therefore permissible for the determination of $\eta_p$ to replace the differential Eq. (42) with the boundary condition $f_p(r_1)=f_p(r_2)=0$ by the differential Eq. (46) assumed to be valid throughout the metal with the boundary condition

$$f_p'(r_1)=f_p'(r_2)=0. \tag{47}$$

A more rigorous argument for this replacement is presented in Appendix III. Based upon the simplified problem of Eq. (46) and (47), the following subsections deal with the more detailed investigation of stability for several special cases.

### A. Trapped Flux for very Thin Walls. ($d\ll\lambda$)

We shall here assume that the wall thickness $d=r_2-r_1$ of the cylinder is small compared to the penetration

depth $\lambda$. With $x=r-r_1$ and expanding the more general expression for $\beta$ of Eq. (36) in powers of $x/\lambda$ one obtains up to quadratic terms

$$\beta=A(1+x^2/2\lambda^2)+Bx/\lambda, \tag{48}$$

where the two constants $A$ and $B$ are related to the constants $c_1$ and $c_2$ of Eq. (36) and are likewise determined by the values of the magnetic field $H$ for $r<r_1$ and for $r>r_2$. These two values will from now on be denoted by $H_1$ and $H_2$, respectively. It is convenient to measure all magnetic fields in units of the characteristic field $H^*=hc/4\pi\lambda^2 e$ or, from Eq. (25),

$$H^*=4\pi n\mu_0,$$

where[18]

$$\mu_0=e\hbar/2mc.$$

It was shown by Schafroth[8] that the above value $H^*$ corresponds to that of the critical field in the sense that in a uniform field $H>H^*$, his model exhibits field penetration rather than a Meissner effect. It will be seen later that it is also connected, in a somewhat different sense, with the field determined from the criterion of energetic stability.

With this notation one finds from Eqs. (12), (13), (14), and (35) that

$$\kappa=H/H^*=\frac{2\lambda^2}{r}\frac{d\beta}{dr}. \tag{49}$$

With

$$\kappa_{1,2}=H_{1,2}/H^*, \tag{50}$$

one obtains then from Eq. (48)

$$\kappa_1=\frac{2\lambda^2}{r_1}\left(\frac{d\beta}{dx}\right)_0=\frac{2\lambda}{r_1}B \tag{51}$$

and

$$\kappa_2=\frac{2\lambda^2}{r_2}\left(\frac{d\beta}{dx}\right)_d=\frac{2\lambda}{r_2}\left(\frac{d}{\lambda}A+B\right).$$

We shall consider here only the case of trapped flux in the absence of an external field $H_2$, i.e., we shall let $\kappa_2=0$ so that $A=-B\lambda/d$. One finds then from Eq. (48) and (51)

$$\beta=-\frac{r_1\kappa_1}{2d}\left[1-\frac{d^2}{2\lambda^2}+\frac{(x-d)^2}{2\lambda^2}\right]. \tag{52}$$

This equation will first be used to determine the quantity $\alpha_1=\alpha(r_1)$ which represents the flux in units of $hc/e$ at $r=r_1$. Since for $r<r_1$, the flux and, hence, $\alpha$ must be quadratic in $r$, and since its derivative must be continuous at $r=r_1$ so that $(d\alpha/dr)_{r_1}=(d\beta/dx)_0$, it follows from Eq. (51) that for $r\leq r_1$,

$$\alpha=\kappa_1 r^2/4\lambda^2.$$

With $\kappa_1=4\lambda^2\alpha_1/r_1^2$, one therefore has from Eq. (52) for $x=0$: $\beta=-r_1\kappa_1/2d=-\alpha_1(2\lambda^2/r_1 d)$ and the continuity of $\alpha=l'+\beta$ at this point demands

$$\alpha_1=l'-\alpha_1(2\lambda^2/r_1 d)$$

or

$$\alpha_1=l'/(1+2\lambda^2/r_1 d). \tag{53}$$

Except for terms of order $d/r_1$ which have been neglected, the total flux contained in the cylinder, i.e., its value at $r=r_2$, is the same as that at $r=r_1$; measured in units $hc/e$, this value is thus likewise given by Eq. (53). As might have been expected, this expression vanishes for $d=0$; for very small values of $r_1 d$ it exhibits the fact, noted also by Blatt[9] and Bardeen,[19] that the effective magnitude of the flux quantum is less than its full value $hc/e$.

The expression for $\beta$ of Eq. (52) will next be used to find the values of $\eta_p$ according to Eq. (46). For the thin ring considered here, one has $r-r_1=x\ll r_1$ so that this equation can be approximated by

$$d^2f_p/dx^2-(1/r_1^2)(p^2-2p\beta)f_p=\eta_p f_p. \tag{54}$$

Except for very large values of $\kappa_1$, the $x$ dependence of $\beta$ in Eq. (52) may be treated as a small perturbation. The unperturbed equation has then the solution $f_p=1$, and one has in first approximation

$$\eta_p=-(1/r_1^2)(p^2-2p\bar{\beta}),$$

where $\bar{\beta}$ represents the average value of $\beta$ in the interval $0\leq x\leq d$. Omitting a negligible contribution of relative order of magnitude $d^2/\lambda^2$, one has $\bar{\beta}=-r_1\kappa_1/2d$, and hence

$$\eta_p=-(1/r_1^2)[p^2+p\kappa_1(r_1/d)]. \tag{55}$$

The higher solutions of Eq. (54) correspond to values of $\eta$ separated from this expression by amounts of the order of magnitude $1/d^2$. In second approximation one obtains therefore an additional contribution of the order of magnitude

$$d^2\left(\frac{p}{r_1^2}\frac{r_1\kappa_1 d}{\lambda^2}\right)^2=\frac{p^2}{r_1^2}\left(\frac{\kappa_1 d^2}{\lambda^2}\right)^2.$$

Compared to the first term in the bracket of Eq. (55) this contribution is thus negligible unless $\kappa_1\simeq\lambda^2/d^2\gg 1$. We shall here not be concerned with such large values of $\kappa_1$, and it is thus permissible to base the further discussion of this case upon Eq. (55).

It is evident that with $\kappa_1>0$, positive values of $\eta_p$ and, hence, energetic instability of the system, can only occur if $p<0$. With $-p=1, 2, 3, \cdots$, instability requires $\kappa_1>-p(d/r_1)$; since the smallest value is reached for $-p=1$, the criterion for stability can thus be stated as $\kappa_1\leqq d/r_1$, where the sign of equality refers to the 'critical' condition. With Eq. (50), the critical

[18] With the value $n\cong 10^{22}$ cm$^{-3}$ used to estimate $\lambda$, we obtain $H^*\cong 10^3$ G.

[19] J. Bardeen, Phys. Rev. Letters **7**, 162 (1961).

value of the field $H_1$ in the interior of the cylinder is thus given by

$$H_1=H^*(d/r_1). \tag{56}$$

Although this result has been derived only for a thickness $d\ll\lambda$, it will be seen later that it also holds for thicker rings, provided that $d\ll r_1$. It indicates that the critical field for flux trapping in thin rings can, according to this model, be very much smaller than the value $H^*$ obtained by Schafroth from different arguments, and the significance of this conclusion will be further discussed below.

### B. Stability for Wall Thickness $d\gg\lambda$

We consider here the opposite limiting case from the one discussed in A, and far more common in practice, where the wall thickness is large compared to the penetration depth. We will also assume throughout that $r_1\gg\lambda$. The discussion has to be based upon the form of $\beta$ given by Eq. (36), and we shall begin by expressing the constants $c_1$ and $c_2$ in terms of the $H_1$ and $H_2$ inside and outside the cylinder, respectively. Neglecting terms proportional to $e^{-d/\lambda}$, one has here from Eqs. (49) and (50) $\kappa_1=-(2\lambda/r_1)c_1$, $\kappa_2=(2\lambda/r_2)c_2$. Inserting the corresponding values of the constants into Eq. (36), it is, therefore,

$$\beta=-\frac{r_1\kappa_1}{2\lambda}e^{-(r-r_1)/\lambda}+\frac{r_2\kappa_2}{2\lambda}e^{(r-r_2)/\lambda}. \tag{57}$$

The quantization of the flux trapped inside the cylinder is directly evident from the fact that this expression is appreciable only within distances of the order $\lambda$ from the boundaries of the metal and approaches rapidly the value $\beta=0$ in its interior. In view of Eq. (35), the flux in units of $hc/e$ trapped within the interior of the metal is thus given by

$$\alpha=l',$$

i.e., the flux is indeed the integer $l'$ times the flux quantum $hc/e$.[20]

The principal use of Eq. (57) is, however, that of investigating the stability criterion by means of the values of $\eta_p$ obtained from Eq. (46).

We thus have to consider the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)-\frac{1}{r^2}\left[p^2+\frac{p}{\lambda}(r_1\kappa_1e^{-(r-r_1)/\lambda}-r_2\kappa_2e^{(r-r_2)/\lambda})\right]f_p=\eta_pf_p. \tag{58}$$

For each value of $p$, it can be solved with the boundary conditions of Eq. (47) to give $\eta_p(\kappa_1,\kappa_2)$. The critical condition for stability is given by $\eta_p(\kappa_1,\kappa_2)=0$, and for given $\kappa_2$ and $p<0$, any further increase of $\kappa_1$ will raise $\eta_p$ and thus lead to instability.

In order to discuss Eq. (58), it is convenient to consider first the form of $f_p$ in the interior of the metal ($r-r_1$ and $r_2-r\gg\lambda$) where $\beta=0$. It satisfies here the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)=\left(\frac{p^2}{r^2}+\eta_p\right)f_p,$$

with the general solution

$$f_{p0}=C_pJ_p(i\eta_p^{1/2}r)+C_{-p}N_p(i\eta_p^{1/2}r), \tag{59}$$

where $J_p$ and $N_p$ are Bessel functions of the first and second kind.

In the vicinity of the boundary at $r=r_1$, the term with $\kappa_2$ in Eq. (58) is proportional to $e^{-d/\lambda}$ and thus negligible, so that one has here

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)=\left\{\frac{1}{r^2}\left[p^2+\frac{pr_1\kappa_1}{\lambda}e^{-(r-r_1)/\lambda}\right]+\eta_p\right\}f_p,$$

with the problem to find a solution which has a vanishing derivative at $r=r_1$, and which approaches asymptotically the function $f_{p0}$ of Eq. (59) for $r-r_1\gg\lambda$. If we write

$$f_p=f_{p0}+f_{p1},$$

this means that the function $f_{p1}$ must appreciably differ from zero only for values of $x=r-r_1$ which are comparable to or small compared with $\lambda$, and that it must satisfy the boundary condition

$$f_{p1}'(r_1)=-f_{p0}'(r_1).$$

The differential equation for $f_{p1}(x)$ can then be written in the form

$$f_{p1}''=\left(\frac{p^2}{r_1^2}+\eta_p\right)f_{p1}+\frac{p\kappa_1}{r_1\lambda}e^{-x/\lambda}(f_{p0}+f_{p1}), \tag{60}$$

with the boundary condition for $x=0$

$$f_{p1}'(0)=-f_{p0}'(r_1). \tag{61}$$

If the conditions $|p|\ll(r_1/\lambda)$ and $|\eta_p|\ll(1/\lambda^2)$ are satisfied, then we may consistently assume that $|f_{p1}|\ll|f_{p0}|$ and $|(p^2/r_1^2+\eta_p)f_{p1}|\ll|f_{p1}''|$. It is also permissible to replace $f_{p0}$ in Eq. (60) by its value $f_{p0}(r_1)$ at $r=r_1$, since $f_{p0}$ is slowly varying within the distance $\lambda$ from the boundary. One has then the simplified equation

$$f_{p1}''=(p\kappa_1/r_1\lambda)f_{p0}(r_1)e^{-x/\lambda}, \tag{62}$$

with the solution

$$f_{p1}(x)=(p\kappa_1\lambda/r_1)f_{p0}(r_1)e^{-x/\lambda} \tag{63}$$

chosen in order to satisfy the requirement to vanish

[20] Although it appears in different form, this proof of flux quantization is in accordance with the more general energetic arguments of Byers and Yang[4] since, in last analysis, it has been obtained from the minimum principle for the total energy, given in Eq. (23).

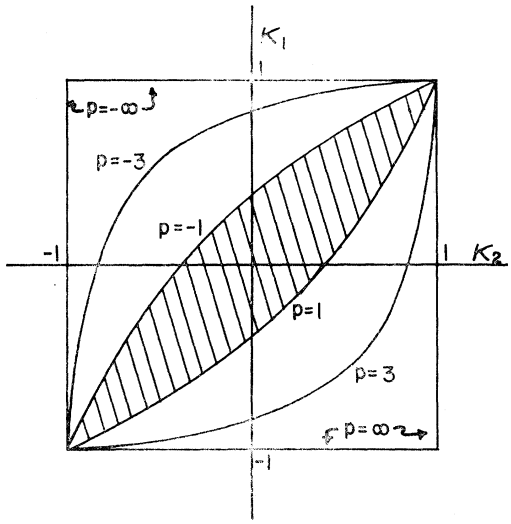


FIG. 1. Graphic representation of stability conditions. The hyperbolic arcs are labeled according to $p$, the change of the azimuthal quantum number. The shaded area represents the region of absolute stability obtained for $|p|=1$ and corresponds in the figure to a cylinder with a wall thickness approximately half as large as the inner radius. The values of $\kappa_1$ are considered to be continuously variable; because of flux quantization, they actually assume discrete values, separated by the small amount $4\lambda^2/r_1^2$. The hyperbolic arcs should, therefore, be replaced by the corresponding "staircase," but it would be too fine, for macroscopic cylinders, to be distinguishable from the smooth curves of this plot.

for $x\gg\lambda$.[21] This solution has further to satisfy the boundary condition (61), i.e., one has to demand

$$\kappa_1=(r_1/p)[f_{p0}'(r_1)/f_{p0}(r_1)]. \tag{64}$$

The same considerations which have been applied here to the boundary $r=r_1$ hold for $r=r_2$ with the result[22]

$$\kappa_2=(r_2/p)[f_{p0}'(r_2)/f_{p0}(r_2)]. \tag{65}$$

With the function $f_{p0}(r)$ given by Eq. (59), the last two equations represent two relations between the constants $C_p$ and $C_{-p}$. However, there appears actually only their ratio in these equations. By elimination of this ratio one obtains therefore a relation for $\eta_p$ as a function of $\kappa_1$, $\kappa_2$, and $p$. This relation can be written, in implicit form, as

$$\frac{(\kappa_1+1)N_{p+1}(z_1)+(\kappa_1-1)N_{p-1}(z_1)}{(\kappa_1-1)J_{p-1}(z_1)+(\kappa_1+1)J_{p+1}(z_1)}=\frac{(\kappa_2+1)N_{p+1}(z_2)+(\kappa_2-1)N_{p-1}(z_2)}{(\kappa_2-1)J_{p-1}(z_2)+(\kappa_2+1)J_{p+1}(z_2)}, \tag{66}$$

where $z_{1,2}=i\eta_p^{1/2}r_{1,2}$. The physical content of this equation can be grasped geometrically by noting that the function $\eta_p(\kappa_1,\kappa_2)$ can be represented as a surface constructed over the $\kappa_1$, $\kappa_2$ plane and intercepting the plane $\eta_p=$const on a curve determined by Eq. (66). This curve is a hyperbola that separates the $\kappa_1$, $\kappa_2$ plane into regions of stability and instability against transitions involving the transfer of the amount of energy $\hbar^2\eta_p/2m$ to the lattice.

The critical relation between $\kappa_1$ and $\kappa_2$ is obtained by putting $\eta_p=0$. By use of the expansion of the Bessel functions for small argument, we obtain from Eq. (66)

$$t_p(\kappa_1\kappa_2-1)=\kappa_1-\kappa_2. \tag{67}$$

with

$$t_p=(r_2^{2p}-r_1^{2p})/(r_2^{2p}+r_1^{2p})=\tanh(p\ln(r_2/r_1)). \tag{68}$$

The hyperbola determined by Eq. (67) has its center at the point $\kappa_1=-1/t_p$, $\kappa_2=1/t_p$, and its foci are situated on the line $\kappa_2=-\kappa_1$. For the three separate regions in the $\kappa_1$, $\kappa_2$ plane, separated from each other by the two branches of this hyperbola, $\eta_p$ can be seen to be positive in the region which does not contain the foci and thus represents the conditions under which the system is energetically unstable against the change $p$ of the azimuthal quantum number, while the other two regions represent the corresponding conditions of stability.[23]

While the critical relation of Eq. (67) involves for finite values of $|p|$ both the values of $\kappa_1$ and $\kappa_2$, there occurs a singular case in the limit $|p|\to\infty$.[24] In this limit, one has for $p\to\infty$ from Eq. (68) $t_\infty=1$ with the consequence that Eq. (67) is satisfied either for $\kappa_1=-1$ or for $\kappa_2=1$, irrespective of the value of the other variable; correspondingly, it follows for $p\to-\infty$, and

[21] One can directly verify that the solution (63) justifies the assumptions made in obtaining Eq. (62) from Eq. (60). Indeed, the ratio $|(p^2/r_1^2+\eta_p)f_{p1}|/|f_{p1}''|=[(p\lambda/r_1)^2+\eta_p\lambda^2]$ is small compared to unity if $|p|\ll r_1/\lambda$ and $|\eta_p|\ll 1/\lambda^2$. The same statement holds for the ratio $|f_{p1}|/|f_{p0}|=|p||\kappa_1|\lambda/r_1$, since it will be seen later that $\kappa_1\leq 1$ with the sign of equality for $|p|\gg 1$. For $|p|\lambda/r_1$ comparable to unity, the solution of Eq. (58) for the case $\eta_p=0$ and $\kappa_2=0$ (trapped flux) can be obtained by introducing $z=e^{-x/\lambda}$ as a new variable. This solution involves Bessel functions of order $\pm 2|p|\lambda/r_1$, and the requirement that $f_p$ vanish in the interior ($x\gg\lambda$) permits only the positive order. The boundary condition (47) further requires that $f_p'(r_1)=0$, and this condition gives a value of $|\kappa_1|$ for each value of $|p|$. The resulting values of $|\kappa_1|$ are found to increase monotonically with increasing $2|p|\lambda/r_1$, with the limiting value $\kappa_1=1$ for $2|p|\lambda/r_1\to 0$. Thus, here these values join the results obtained from Eq. (63) for $|p|\gg 1$ but still $2|p|\lambda/r_1\ll 1$. Since we are interested in the lowest values of $|H_1|$ and, hence, of $|\kappa_1|$ which lead to instability, the solution (63) is valid for the case $\kappa_2=0$, since larger values of $|p|$ always lead to larger values of $|H_1|$.

[22] Writing here $f_p=f_{p0}+f_{p2}$ and $x=r_2-r$, one has from $f_{p2}'(r_2)=-f_{p0}'(r_2)$ the boundary condition $-f_{p2}'(0)=-f_{p0}'(r_2)$ for $x=0$. Applied to the function $f_{p2}(x)$, which differs from $f_{p1}(x)$ by replacing in Eq. (63) $r_1$ by $r_2$ and $\kappa_1$ by $-\kappa_2$, this second boundary condition results in Eq. (65).

[23] This can be seen by first verifying that the point $\kappa_1=\kappa_2=0$ lies in one of the regions containing a focus. It follows on the other hand from Eq. (58) that for $\kappa_1=\kappa_2=0$

$$\eta_p\int f_p^2 r dr=-\int[(df_p/dr)^2+p^2/r^2]r dr,$$

i.e., $\eta_p<0$, which ascertains stability in this region. The assignment of instability and stability to the region without focus and that containing the other focus, respectively, follows from the fact that starting from the above stable region, the sign of $\eta_p$ changes once in passing over to the former and once more, in continuing to the latter.

[24] Physically, this means of course merely that $|p|$ becomes very large compared to unity, and this condition is compatible with the other condition $|p|\ll r_1/\lambda$ necessary for the validity of Eq. (67).

hence $t_{-\infty}=-1$, that either $\kappa_1=1$ or $\kappa_2=-1$. In order to have stability in both these limits, one is, therefore, confined within the square with corners at $\kappa_1=\pm 1$ and $\kappa_2=\pm 1$.[25] Within this square, the region of stability for each pair of values $p$ and $-p$ consists of the slice limited by two symmetrical hyperbolic arcs with end corners at the points $\kappa_1=\kappa_2=1$ and $\kappa_1=\kappa_2=-1$. This slice is the narrower, the smaller $|p|$. Demanding stability for all values of $|p|$ limits, therefore, the region of absolute stability to the slice corresponding to the smallest possible value $|p|=1$.

These results are graphically represented in Fig. 1 with the shaded area representing the region of absolute stability. To discuss them, we consider first the case where the fields inside and outside the cylinder are equal, i.e., where $\kappa_1=\kappa_2=\kappa$. It is seen that the straight line representing this equality lies inside the stability region as long as $|\kappa|\leq 1$. This case corresponds to the Meissner effect in the sense that it refers to the expulsion of the homogeneous field from the interior of the metal. Our findings thus indicate that this effect is properly described by the state of the system discussed in the preceding section, but that this state becomes unstable when the absolute value of the field $H=\kappa H^*$ reaches the critical value for the Meissner effect

$$(H_c)_M=H^*. \tag{69}$$

For higher values of the field, the system will tend towards a new stable state by giving energy to the lattice. Our discussion does not describe the detailed manner in which this transition takes place nor the final state which will be ultimately reached. However, it is important to note that coming from the other direction, Schafroth has demonstrated the possibility of another stable situation. Assuming a homogeneous field inside the metal, he showed that it can have a nonzero value as long as $H>H^*$, i.e., that there exists in this case another stable state, corresponding to field penetration into the metal. It seems reasonable to consider the stable state found here for $H<H^*$ and Schafroth's state for $H>H^*$ as alternatives, the former representing Meissner effect, the latter field penetration. In fact, one is forced to this conclusion if one assumes that for any common value $H$ of the field inside and outside the cylinder there exists one and only one energetically stable state of the system.

A totally different behavior of the metal is found if the fields $H_1$ and $H_2$ inside and outside the cylinder are no longer considered to be equal, thus demanding a persistent current to flow in the metal. In this case, one has to consider the total shaded area of Fig. 1, determined by the choice $|p|=1$ and, hence, limited according to Eq. (67) and (68) by the two hyperbolic arcs representing the equation

$$t(\kappa_1\kappa_2-1)=\pm(\kappa_1-\kappa_2), \tag{70}$$

where

$$t=(r_2^2-r_1^2)/(r_2^2+r_1^2)=\tanh[\ln(r_2/r_1)], \tag{71}$$

and where the + or − sign in Eq. (70) corresponds to $p=1$ or $p=-1$, respectively. It is true that for very thick cylinders, i.e., for $r_2\gg r_1$ and, hence, $t=1$, the stability region fills the whole square, so that one deals with a stable persistent current as long as neither of the two values $\kappa_1$ or $\kappa_2$ exceeds unity, i.e., as long as

$$|H_{1,2}|<H^*.$$

The limits of stability are, however, narrower for cylinders of finite thickness and impose an upper limit, less than its absolute maximum $2H^*$, upon $|H_1-H_2|$. This difference from the Meissner effect is best seen by considering the case $\kappa_2=0$, i.e., the case of trapped flux in the absence of an external field. It follows here from Fig. 1 and Eq. (70) that stability is limited by the critical condition

$$\kappa_1=\pm t$$

and thus ceases to exist when the absolute value of the field $H_1=\kappa_1 H^*$ reaches the critical value for flux trapping,

$$(H_c)_T=tH^*. \tag{72}$$

Since for cylinders with finite wall thickness $t<1$, one has $(H_c)_T<(H_c)_M$, i.e., the critical magnitude of the field inside the cylinder is more restrictive for flux trapping than for the Meissner effect. This restriction is particularly pronounced in the case $d\ll r_1$, i.e., for thin cylinders with a wall thickness small compared to the inner radius.[26] With $r_2=r_1+d$, one has here in good approximation $t=d/r_1$ from Eq. (71) and thus from Eq. (72)

$$(H_c)_T=H^*(d/r_1). \tag{73}$$

Although it was derived under different assumptions, this result agrees with that of Eq. (56), obtained for "very" thin cylinders where only the case of trapped flux with $H_2=0$ was considered. Thus, the ciritical field for flux trapping for a given inner radius $r_1$ is the smaller the thinner the cylinder wall. Nevertheless, this does not prevent the possibility of trapping a very large number $l'$ of flux quanta if one deals with macroscopic dimensions. Indeed, the trapping of a single quantum corresponds to a field in the interior given by $H_1=4\lambda^2H^*/r_1^2$; with $\lambda\cong 5\times 10^{-6}$ cm and even for $r_1\cong 10^{-3}$ cm and $d/r_1\cong 10^{-2}$, this value is still about

[25] This degeneracy corresponds physically to the fact that there exist for large values of $|p|$ separate solutions of $f_p$ which decrease rapidly towards the interior, coming either from one or the other boundary. This fact is reflected in the function $f_{p0}$ of Eq. (59) by the alternate choice $C_p=0$ or $C_{-p}=0$. Dependent upon the sign of $p$, Eqs. (64) and (65) lead then to the same conclusion as those presented above. The fact that the interior of the square corresponds to stability follows from the fact that it contains the point $\kappa_1=\kappa_2=0$ which was shown before to represent a stable situation.

[26] While Sec. A dealt with "very thin" cylinders, i.e., with the case $d\ll\lambda$, the case of "thin" cylinders discussed here refers to the condition $\lambda\ll d\ll r_1$, which is evidently easy to fulfill for cylinders where both wall thickness and inner radius are of macroscopic dimensions.

one hundred times smaller than $(H_c)_T$, i.e., about one hundred flux quanta can be stably trapped even under these very unfavorable assumptions.

Until now, the discussion of the consequences of the stability criterion as formulated in Eq. (45) has been based upon purely energetic arguments. For the case of flux trapping, we have shown that the state in which all particles have the same set of quantum numbers is stable against single-particle transitions if $(H_c)_T > H_1$. Actually, even in this case we deal with a metastable state of very long life. The metastability of the persistent currents in the general case has been emphasized by London[27] and can be understood by a consideration of Eq. (37) for the total energy of the system. It follows from this expression that the state of the system corresponding to the trapping of a finite and, possibly, even large number $l'$ of flux quanta is not the state of lowest energy of the system, since this would correspond to $l'=0$, i.e., to the absence of any persistent current. A transition of the system in which all particles would simultaneously decrease their azimuthal quantum number by the same amount and lead to a lower common value of $l'$ would thus be accompanied by a decrease of $E(l')$, i.e., such a transition would be energetically possible. However, this process is extremely improbable and does not endanger the practically unlimited stability of the persistent current, even for a value of $l'$ which is so large that the corresponding field $H_1$ inside the cylinder surpasses the critical value $(H_c)_T$ for flux trapping. Considering such a state to be realized at an initial time,[28] it would, however, be unstable with a far reduced lifetime due to the fact that transitions which involve only a single particle would be energetically possible. In the absence of any strict selection rules prohibiting such transitions, it must be assumed that they will occur and lead eventually to a new stable state of the system. One cannot *a priori* exclude the possibility that such a stable state can be reached when only a small fraction of all particles have made transitions from their originally common individual particle state. However, it is shown in Appendix IV, that at least up to quadratic terms in the fractional change of the occupation numbers, one cannot reach a new state of the system which would be stable against further small changes. It is therefore reasonable to assume that single-particle transitions will continue to occur until all particles are finally again in a common individual state with the azimuthal quantum number $l'$ sufficiently lowered from its initial value to reach stability, i.e., until the field $H_1$ inside the cylinder has reached the critical value $(H_c)_T$ of Eq. (72).

[27] F. London, *Superfluids* (Dover Publishing Company, Inc., New York, 1960), Vol. 1, p. 51.

[28] Provided that $H_1<(H_c)_M=H^*$, this situation could be achieved by a preceding Meissner effect, i.e., by first providing a field $H_2=H_1$ outside the cylinder and then suddenly turning off this field.

It is to be stressed that this behavior contradicts the usual expectation that the flux inside the hole of a metal cannot change as long as the metal remains in the superconductive state. The fact that the Schafroth model does not exclude such changes is directly connected with the fact that, according to Eq. (69) and (72), it leads to a difference between the critical fields for the Meissner effect and for flux trapping. Starting, e.g., with equal fields inside and outside the cylinder, both slightly below $(H_c)_M$, then letting the outside field go to zero would result in a reduction of the inside flux by a factor $1/t$.

The existence of this difference between the critical fields $(H_c)_M$ and $(H_c)_T$ raises questions in regard to the lifetime of the state with $(H_c)_T<H_1<(H_c)_M$ and $H_2=0$. One deals here with transient effects which will be discussed in the following subsection. Since the difference between the two critical fields is most pronounced for small values of $d/r_1$, the discussion will be restricted to this case.

### C. Transient Effects in Thin-Walled Cylinders: ($\lambda \ll d \ll r_1$)

In the previous subsection we were concerned with the stable state which the system will tend to approach if initially $H_1>(H_c)_T$. We concluded that the transition of a single particle with a change $p$ of its azimuthal quantum number will eventually occur if it thereby supplies any amount of positive energy $(\hbar^2/2m)\eta_p$ to the lattice, assumed to be at the absolute zero,[29] and that this process will continue until $H_1=(H_c)_T$. Nevertheless, the fulfillment of the condition $H_1>(H_c)_T$ need not necessarily lead to a lifetime so short as to be easily observed. The dynamics may make it impossible for the lattice to absorb the energy given up by the particle, or the absorption process may have a very small probability. Although its occurrence cannot be ruled out on grounds of general selection rules, it appears plausible to think that the absorption of very small amounts of energy is highly improbable for a lattice at low temperatures. On the other hand, the result of Eq. (73) for thin rings was based upon the occurrence of processes with vanishing energy transfer, and it will be shown below to be essentially modified, even if the accompanying energy transfer $(\hbar^2/2m)\eta_p$ is only very small, instead of vanishing.

For this purpose, we rewrite Eq. (58) replacing the variable $r$ by $x=r_1-r$, in the form, valid for $d\ll r_1$, of the differential equation

$$f_p''-\left[\frac{p^2}{r_1^2}+\eta_p+\frac{p}{\lambda r_1}(\kappa_1 e^{-x/\lambda}-\kappa_2 e^{(x-d)/\lambda})\right]f_p=0.$$

[29] Because of Eqs. (24) and (41), $(\hbar^2/2m)\eta_p$ represents the *loss* of energy of the particle which is, therefore, supplied to the lattice.

Corresponding to Eq. (59), the general solution in the interior of the metal is given here by

$$f_{p0}(x)=C_p\exp\left[\left(\frac{p^2}{r_1^2}+\eta_p\right)^{1/2}x\right]+C_{-p}\exp\left[-\left(\frac{p^2}{r_1^2}+\eta_p\right)^{1/2}x\right].$$

The considerations which led to Eqs. (64) and (65) remain valid and result here in the equations

$$\kappa_1=(r_1/p)[f_{p0}'(0)/f_{p0}(0)]$$

and

$$\kappa_2=(r_1/p)[f_{p0}'(d)/f_{p0}(d)].$$

By elimination of $C_p/C_{-p}$ from these two equations, one obtains finally in analogy to Eqs. (67) and (68)

$$t_p'(\kappa_1'\kappa_2'-1)=\kappa_1'-\kappa_2' \tag{74}$$

and

$$t_p'=\pm\tanh(p'd/r_1) \tag{75}$$

with

$$p'=(p^2+\eta_p r_1^2)^{1/2} \tag{76}$$

and

$$\kappa_{1,2}'=\kappa_{1,2}p/p'. \tag{77}$$

With the approximation $\ln(r_2/r_1)=d/r_1$, valid for thin rings, Eqs. (74) and (75) are identical with Eqs. (67) and (68), respectively, if $\eta_p=0$ with the corresponding result of Eq. (73) that the critical field for flux trapping is the smaller, the smaller the wall thickness $d$. Any dependence upon $d$ disappears, however, as soon as one demands energetic stability only against those transitions in which the energy absorbed by the lattice is merely large compared to the amount

$$(\Delta E)_{\min}=\hbar^2/2md^2, \tag{78}$$

i.e., if

$$\eta_p\gg 1/d^2. \tag{79}$$

Even for $d\cong 10^{-3}$ cm, this energy is exceedingly small and corresponds to transition frequencies of the lattice of no more than about one megacycle per second. While absorption processes involving such low frequencies in the lattice spectrum are not impossible, they may well be sufficiently improbable to require macroscopic times in order to become effective.

With the condition of Eq. (79) satisfied, one obtains from Eq. (75) for any value of $p$, $t_p'=\pm 1$, and thus from Eq. (74) independently from each other either the values $\kappa_1'=-1$, $\kappa_2'=1$ or $\kappa_1'=1$, $\kappa_2'=-1$. Replacing $\kappa_{1,2}$ by $\kappa_{1,2}'$, this corresponds to the square of Fig. 1, i.e., to the case $p=\pm\infty$. On the other hand, one has here $\kappa_{1,2}=\kappa_{1,2}'(p^2+\eta_p r_1^2)^{1/2}/p$, i.e., the smallest magnitudes of $\kappa_{1,2}$ are obtained for $|p|\to\infty$ so that in this limit $\kappa_{1,2}=\kappa_{1,2}'$. The corresponding shaded area in Fig. 1 would thus fill the whole square, just as it would for $\eta_p=0$ in the case where $d/r_1\gg 1$.[30] Even a thin cylinder would thus show the "expected" behavior of a single critical field value,

$$H_c=H^*,$$

both for Meissner effect and for flux trapping, provided that the observations are not extended over a time interval which is so long that the lattice can effectively absorb even energies comparable to or small compared with the amount $(\Delta E)_{\min}$ of Eq. (78). It is impossible to predict the magnitude of these time intervals without detailed reference to the specific mechanism of the absorption processes.[31] Unless they are either prohibitively short or prohibitively long, a marked time-dependence of critical field values for flux trapping would be predicted by Schafroth's model.

## V. CONCLUSIONS

While the model discussed here might be considered quite unrealistic, it is nevertheless remarkable that it exhibits features which strikingly resemble those of real superconductors. Some of the results, and particularly their more quantitative aspects, may of course be merely consequences of a too far-going idealization and may be neither confirmed by experiment nor by a truly realistic theory; however, we believe that qualitatively they will turn out to agree with reality. We further believe that the manner in which the problem of stability has been approached and the decisive importance attributed to electromagnetic effects are not restricted to this special model but that they are essential for the general understanding of superconductive phenomena.

## APPENDIX I

### Change of the Total Energy

We shall compare the total energy $E^a$ and $E^b$ of two states $a$ and $b$ of the system, characterized by the sets of occupation numbers $N_s{}^a$ and $N_s{}^b$, respectively. The corresponding solutions of the Eqs. (3), (4), and (5) shall have been found for both sets with the result that the functions $\psi_s$, $U$, $\mathbf{A}_i$, and $\mathbf{A}$ are given by $\psi_s{}^{a,b}$, $U^{a,b}$,

[30] It may be noted that even the case $d/r_1=\infty$ can be realized for a solid cylinder, since one has here $r_1=0$. Of course, one deals here only with the possibility of a Meissner effect and thus only with a single critical field $H_c=H^*$.

[31] As an example of such a mechanism, one may consider the single-phonon processes which explain the conductivity of normal metals. Since a change $p$ of the azimuthal quantum number is associated with a wavelength $2\pi r_1/|p|$ of a sound wave around the cylinder, the corresponding energy of the phonon is given by $\hbar v|p|/r_1$, where $v$ is the sound velocity; being the energy absorbed by the lattice, this amount must be equated to $\hbar^2\eta_p/(2m)$, thus giving $\eta_p=2mv|p|/(\hbar r_1)$. To fulfill the condition of Eq. (79) even for the smallest possible value $|p|=1$, it is thus sufficient to demand $d/r_1\gg[\hbar/(2mvr_1)]^{1/2}$. Since $\hbar/(2mv)\cong 10^{-5}$ cm, and if one assumes, e.g., $r_1\cong 10^{-1}$ cm, this means $d/r_1\gg 10^{-2}$, i.e., merely $d\gg 10^{-3}$ cm. As long as only single-phonon processes are considered, even quite thin rings can therefore behave as if they were infinitely thick. Only multiple-phonon processes can here be responsible for a deviation from this behavior, and one can well understand that at low temperatures such processes may require a considerable time to become effective.

$\mathbf{A}_i^{a,b}$, and $\mathbf{A}^{a,b}$, respectively. Let further

$$N_s^b-N_s^a=\Delta N_s;\quad \psi_s^b-\psi_s^a=\Delta\psi_s;$$
$$U^b-U^a=\Delta U;\quad \mathbf{A}_i^b-\mathbf{A}_i^a=\mathbf{A}^b-\mathbf{A}^a=\Delta\mathbf{A}, \tag{A1}$$

$$\Delta\mathfrak{H}^{a,b}=-\frac{e}{2mc}\left\{\Delta\mathbf{A}\cdot\left[\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}^{a,b}\right]+\left[\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}^{a,b}\right]\cdot\Delta\mathbf{A}\right\}+\Delta U. \tag{A2}$$

In view of Eq. (2) the individual particle Hamiltonians $\mathfrak{H}^{a,b}$ of the two states with eigenvalues $E_s^{a,b}$ are related by

$$\mathfrak{H}^b=\mathfrak{H}^a+\Delta\mathfrak{H}^a+(e^2/2mc)(\Delta\mathbf{A})^2, \tag{A3}$$

$$\mathfrak{H}^a=\mathfrak{H}^b-\Delta\mathfrak{H}^b+(e^2/2mc)(\Delta\mathbf{A})^2. \tag{A4}$$

With these notations, one can express the total energy $E^b$ of state $b$ through Eq. (1) in terms of quantities referring to the state $a$ and vice versa. Through partial integration and keeping only terms linear and quadratic in $\Delta N_s$, $\Delta\psi_s$, $\Delta U$, and $\Delta\mathbf{A}$, one finds thus

$$E_b=E_a+\sum_s\Delta N_s\int\psi_s^{a*}(\mathfrak{H}^a+\Delta\mathfrak{H}^a)\psi_s^a d\tau+\int\Delta U\left\{\frac{1}{4\pi e^2}\nabla^2U^a+\sum_s N_s^a\psi_s^{a*}\psi_s^a-n\right\}d\tau$$
$$+\int\Delta\mathbf{A}\cdot\left\{\frac{1}{4\pi}\operatorname{curl}\operatorname{curl}\mathbf{A}_i^a-\frac{e}{2mc}\sum_s N_s^a\left[\psi_s^{a*}\left(\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}^a\right)\psi_s^a+\psi_s^a\left(-\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}^a\right)\psi_s^{a*}\right]\right\}d\tau$$
$$+\sum_s(N_s^a+\Delta N_s)\int(\Delta\psi_s^*\mathfrak{H}^a\psi_s^a+\psi_s^{a*}\mathfrak{H}^a\Delta\psi_s)d\tau+\sum_s N_s^a\int\left(\Delta\psi_s^*\mathfrak{H}^a\Delta\psi_s+\Delta\psi_s^*\Delta\mathfrak{H}^a\psi_s^a+\psi_s^{a*}\Delta\mathfrak{H}^a\Delta\psi_s\right.$$
$$\left.+\frac{e^2}{2mc}(\Delta\mathbf{A})^2\psi_s^{a*}\psi_s^a\right)d\tau+\frac{1}{8\pi}\int[\operatorname{curl}\Delta\mathbf{A}]^2d\tau-\frac{1}{8\pi e^2}\int[\operatorname{grad}\Delta U]^2d\tau. \tag{A5}$$

The curly brackets in the second and third integral vanish because of the Poisson and Maxwell Eqs. (4) and (5), respectively, together with the Eqs. (6) and (7) for $\rho$ and $\mathbf{i}$. In view of the Schrödinger equation $H^a\psi_s^a=E_s^a\psi_s^a$ and its conjugate complex, the fourth integral can be written as

$$E_s^a\int(\Delta\psi_s^*\psi_s^a+\Delta\psi_s\psi_s^{a*})d\tau$$
$$=E_s^a\left[\int(\psi_s^{a*}+\Delta\psi_s^*)(\psi_s^a+\Delta\psi_s)d\tau-\int\psi_s^{a*}\psi_s^a d\tau-\int\Delta\psi_s^*\Delta\psi_s d\tau\right]. \tag{A6}$$

Because of Eq. (A1), the first integral in the square bracket of Eq. (A6) is equal to

$$\int\psi_s^{b*}\psi_s^b d\tau$$

and, hence, equal to

$$\int\psi_s^{a*}\psi_s^a d\tau$$

since normalization requires both these integrals to be equal to unity. Combining the remaining term in (A6) with the fifth integral in Eq. (A5) and neglecting the higher-order term, one has therefore

$$E^b=E^a+\sum_s\Delta N_s\left[\int\psi_s^{a*}(\mathfrak{H}^a+\Delta\mathfrak{H}^a)\psi_s^a d\tau\right]$$
$$+\sum_s N_s^a\int\left\{\Delta\psi_s^*(\mathfrak{H}^a-E_s^a)\Delta\psi_s+\Delta\psi_s^*\Delta\mathfrak{H}^a\psi_s^a+\psi_s^{a*}\Delta\mathfrak{H}^a\Delta\psi_s+\frac{e^2}{2mc^2}(\Delta\mathbf{A})^2\psi_s^{a*}\psi_s^a\right\}d\tau$$
$$+\frac{1}{8\pi}\int[\operatorname{curl}\Delta\mathbf{A}]^2d\tau-\frac{1}{8\pi e^2}\int[\operatorname{grad}\Delta U]^2d\tau. \tag{A7}$$

The corresponding expression for $E^a$ is obtained by replacing on the right side throughout the upper index $a$ by $b$ and, because of Eqs. (A1) to (A4), the symbol $\Delta$ by $-\Delta$. Subtracting the equation thus obtained from Eq. (A7) for $E^b$, it follows then that

$$E^b-E^a=E^a-E^b+\sum_s\Delta N_s\int\{\psi_s^{a*}\mathfrak{H}^a\psi_s^a+\psi_s^{b*}\mathfrak{H}^b\psi_s^b\}d\tau$$
$$+\sum_s\int\left\{\Delta\psi_s^*[N_s^a(\mathfrak{H}^a-E_s^a)-N_s^b(\mathfrak{H}^b-E_s^b)]\Delta\psi_s+\Delta\psi_s^*[N_s^a\Delta\mathfrak{H}^a\psi_s^a-N_s^b\Delta\mathfrak{H}^b\psi_s^b]\right.$$
$$+[N_s^a\psi_s^{a*}\Delta\mathfrak{H}^a-N_s^b\psi_s^{b*}\Delta\mathfrak{H}^b]\Delta\psi_s+\Delta N_s[\psi_s^{a*}\Delta\mathfrak{H}^a\psi_s^a-\psi_s^{b*}\Delta\mathfrak{H}^b\psi_s^b]$$
$$\left.+\frac{e^2}{2mc}(\Delta\mathbf{A})^2[\psi_s^{a*}\psi_s^a-\psi_s^{b*}\psi_s^b]\right\}d\tau. \tag{A8}$$

While the individual terms in the second integral of (A8) are themselves of order $\Delta^2$ (i.e., quadratic in the numbers $\Delta N_s$), it has to be noticed that all the square brackets represent differences between corresponding $a$ and $b$ terms so that they are one order higher in $\Delta$ than the individual terms. This whole integral is thus of order $\Delta^3$ and therefore negligible to the order $\Delta^2$. To this order, and with the Schrödinger Eq. (3) for $a$ and $b$, one may thus write Eq. (A8) in the form

$$\Delta E=E^b-E^a=\tfrac{1}{2}\sum\Delta N_s(E_s{}^a+E_s{}^b).$$

Writing

$$E_s{}^b=E_s{}^a+\Delta E_s,$$

it is sufficient to calculate $\Delta E_s$ in first-order perturbation theory and to keep only terms of order $\Delta$ in the perturbation energy. Omitting from now on the index $a$, referring to the "original" state of the system, one has therefore from Eqs. (A2) and (A3)

$$\Delta E_s=\int\psi_s{}^*\Big\{-\frac{e}{2mc}\Big[\Delta\mathbf{A}\cdot\Big(\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}\Big)+\Big(\frac{\hbar}{i}\operatorname{grad}-\frac{e}{c}\mathbf{A}\Big)\cdot\Delta\mathbf{A}\Big]+\Delta U\Big\}\psi_s d\tau \quad \text{(A9)}$$

and

$$\Delta E=\sum_s\Delta N_s(E_s+\tfrac{1}{2}\Delta E_s), \quad \text{(A10)}$$

in agreement with Eq. (10) of the text.

## APPENDIX II

### The Functions $f_0$, $u$, and $\beta$ near the Boundary

With $\epsilon_{l'}=0$, $\beta=\alpha-l'$ and

$$w=u+\beta^2/r^2,$$

Eqs. (27), (28), and (30) can be written in the form

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{df_0}{dr}\Big)=wf_0, \quad \text{(A11)}$$

$$\frac{1}{r}\frac{d}{dr}\Big[r\frac{d}{dr}(w-\beta^2/r^2)\Big]=-\frac{1}{2\mu^4}(f_0{}^2-1), \quad \text{(A12)}$$

$$r\frac{d}{dr}\Big(\frac{1}{r}\frac{d\beta}{dr}\Big)=\frac{\beta}{\lambda^2}f_0{}^2. \quad \text{(A13)}$$

We shall restrict ourselves to the vicinity of the boundary at $r=r_1$. With $x=r-r_1$ and $x\ll r_1$, the differential operators can then be replaced by $d^2/dx^2$ and $r$ can be replaced by $r_1$. For sufficiently large values of $x$ and small values of $\mu$ we expect, according to Eqs. (31) and (32), the solutions $f_0=1$, $w=0$, and $\beta=\beta_0$, where

$$\beta_0=c_1e^{-x/\lambda}+c_2e^{(x-d)/\lambda} \quad \text{(A14)}$$

agrees with the form of $\beta$ of Eq. (36) with $d=r_2-r_1$. As one approaches the boundary from the interior of the metal, there will appear increasingly large deviations from these values.

Writing

$$f_0=1+\varphi$$

and

$$\beta=\beta_0+\beta_1,$$

we shall carry out a perturbation calculation to the first order in $\varphi$, $\beta_1$, and $w$, valid for $x\gg\mu$ and correct to the lowest power in $\mu/\lambda\ll1$. To this order one has then

$$\varphi''=w, \quad \text{(A15)}$$

$$w''-\frac{1}{r_1{}^2}(\beta_0{}^2)''=-\varphi/\mu^4, \quad \text{(A16)}$$

$$\beta_1''-\beta_1/\lambda^2=2\beta_0\varphi/\lambda^2, \quad \text{(A17)}$$

where $\beta_0$ is given by Eq. (A14) and is a solution of the "zero-order" equation $\beta_0''=\beta_0/\lambda^2$. From Eqs. (A15) and (A16) follows the differential equation

$$\varphi^{\mathrm{IV}}+\varphi/\mu^4=\frac{4}{r_1{}^2\lambda^2}[c_1{}^2e^{-2x/\lambda}+c_2{}^2e^{2(x-d)/\lambda}] \quad \text{(A18)}$$

where Eq. (A14) has been used to express $(\beta_0{}^2)''$ as a function of $x$. Of the four solutions of the homogeneous equation proportional to $\exp[(1\pm i)x/\sqrt{2}\mu]$ and $\exp[(-1\pm i)x/\sqrt{2}\mu]$, only the last two fulfill the requirement to vanish for $x\gg\mu$ while the first two have to be excluded. The appropriate solution of Eq. (A18) is then

$$\varphi=Be^{-x/\sqrt{2}\mu}\cos\Big(\frac{x}{\sqrt{2}\mu}+\psi\Big)+\frac{4\mu^4}{r_1{}^2\lambda^2}(c_1{}^2e^{-2x/\lambda}+c_2{}^2e^{2(x-d)/\lambda}). \quad \text{(A19)}$$

Instead of the factor $\mu^4$ in the last term, there appears actually a factor $\mu^4/[1+(2\mu/\lambda)^4]$ which, however, would affect the result only to higher orders in $\mu/\lambda$.

By inserting this expression into Eq. (A15), one finds further

$$w=\frac{B}{\mu^2}e^{-x/\sqrt{2}\mu}\sin\Big(\frac{x}{\sqrt{2}\mu}+\psi\Big)+\frac{16\mu^4}{r_1{}^2\lambda^4}(c_1{}^2e^{-2x/\lambda}+c_2{}^2e^{2(x-d)/\lambda}),$$

and from Eq. (A17)

$$\beta_1=2c_1e^{-x/\lambda}\Big[-\frac{\mu^2}{\lambda^2}Be^{-x/\sqrt{2}\mu}\sin\Big(\frac{x}{\sqrt{2}\mu}+\psi\Big)+\frac{4\mu^4}{\lambda^2r_1{}^2}\Big(\frac{c_1{}^2}{8}e^{-2x/\lambda}+c_2{}^2\frac{x}{2\lambda}e^{2(x-d)/\lambda}\Big)\Big]+2c_2e^{(x-d)/\lambda}\Big[-\frac{\mu^2}{\lambda^2}Be^{-x/\sqrt{2}\mu}\sin\Big(\frac{x}{\sqrt{2}\mu}+\psi\Big)+\frac{4\mu^4}{\lambda^2r_1{}^2}\Big(-c_1{}^2\frac{x}{2\lambda}e^{-2x/\lambda}+\frac{c_2{}^2}{8}e^{2(x-d)/\lambda}\Big)\Big], \quad \text{(A20)}$$

where, in the first term of the square brackets, higher-order terms in $\mu/\lambda$ have been again neglected. The corresponding results are obtained in the vicinity of the boundary at $r_2$, if one replaces on the right side of these equations $x$ by $d-x$, the index 1 by 2 and vice versa.

The first term on the right side of Eq. (A19) shows the expected behavior to rapidly disappear for $x \gg \mu$. For $x$ comparable to $\mu$ it loses its validity and would have to be replaced by an expression to be obtained by "backward" integration of the nonlinear Eqs. (A11), (A12), and (A13) down to $x=0$; the result depends upon the constants $B$ and $\psi$ which would then have to be determined from the normalization of $f_0$ and from the boundary condition that for $x=0$, one has $f_0=0$ and hence $\varphi=-1$. Nevertheless, our approximation remains qualitatively correct for $x \cong \mu$; it therefore permits the conclusion that $|B|$ must be of order unity and that the correction to $f_0=1$, arising from this term, is indeed vanishingly small at any appreciable distance from the boundary.

The second term in Eq. (A19) does not show this rapid decrease with increasing distance from the boundary. While it is proportional to the fourth power of the small quantity $\mu$, its relative importance is also determined by the magnitude of the constants $c_1$ and $c_2$ and hence by that of the magnetic field inside and outside the cylinder. As long as these fields do not greatly exceed their critical values, $c_1$ and $c_2$ are shown in Sec. IV not to exceed the order of magnitude $r_1/\lambda$ so that the second term contributes no more than approximately $(\mu/\lambda)^4$ to the value $f_0=1$ in the interior of the metal. For $n \cong 10^{22}/\text{cm}^3$, this amounts only to about $10^{-8}$ and to even less for smaller values of $n$.

While a behavior similar to that of $\varphi$ appears also in $w$ and $\beta_1$, it is seen from Eq. (A20) that $\beta=\beta_0$ represents a good approximation not only for $x \gg \mu$, but that even for $x \cong \mu$ the relative contribution of $\beta_1$ does not exceed the order of magnitude $(\mu/\lambda)^2$, i.e., that it never causes a correction of more than about one part in $10^4$. We are thus led to the conclusion that equations (31) and (32) represent excellent approximations for $x \gg \mu$ and that Eq. (36) well approximates the true behavior of $\beta=\alpha-l'$ throughout the metal, including the boundaries.

## APPENDIX III

### Replaced Differential Equation and Boundary Condition for $f_p(r)$

To discuss the differential equation (42) for $f_p(r)$, the temporary abbreviation

$$K_p(r)=\eta_p+(1/r^2)(p^2-2p\beta) \tag{A21}$$

will be used so that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_p}{dr}\right)-wf_p=K_pf_p. \tag{A22}$$

For the special case $p=0$, one has $\eta_p=0$. Therefore, also $K_0=0$, and hence

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_0}{dr}\right)-wf_0=0. \tag{A23}$$

Instead of $f_p(r)$, a new function $g_p(r)$ will now be introduced such that

$$f_p=f_0g_p.$$

Since $f_0(r)$ satisfies the proper boundary condition to vanish for $r=r_1$ and $r=r_2$, $f_p(r)$ will satisfy the same boundary condition provided that $g_p$ remains finite at the boundaries. Substitution of this expression for $f_p$ into Eq. (A22) leads by means of Eq. (A23) to the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(rf_0^2\frac{dg_p}{dr}\right)=K_pf_0^2g_p \tag{A24}$$

for $g_p$, and by integration

$$rf_0^2\frac{dg_p}{dr}=\int_{r_1}^{r}K_pf_0^2g_pr'dr'. \tag{A25}$$

The absence of an additional constant of integration on the right side of Eq. (A25) is necessitated by the fact that with finite values of $g_p$ and $dg_p/dr$ and with $f_0=0$ at the boundary, both sides of this equation vanish for $r=r_1$. Therefore, we obtain

$$g_p'(r)=\frac{dg_p}{dr}=\frac{1}{rf_0^2}\int_{r_1}^{r}K_pf_0^2g_pr'dr'.$$

As $r$ approaches the value $r_1$, the integral vanishes more rapidly than $f_0^2$, i.e., the derivative of $g_p$ vanishes for $r=r_1$. Similar considerations hold for the boundary at $r_2$, and the boundary conditions for $g_p$ are therefore

$$g_p'(r_1)=g_p'(r_2)=0. \tag{A26}$$

On the other hand, it was shown in Appendix II that to a very good approximation one has $f_0=1$ in the interior region of the metal exclusive of distances comparable to $\mu$ from the boundaries. In this region one has, from Eq. (A24) and with $K_p$ from Eq. (A21),

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg_p}{dr}\right)-\frac{1}{r^2}(p^2-2p\beta)g_p=\eta_pg_p. \tag{A27}$$

Except for the fact that they refer to the radial function $g_p$ instead of $f_p$, Eqs. (A27) and (A26) are seen to be identical with Eqs. (46) and (47) in the text, respectively. It is true that at distances comparable to $\mu$, Eq. (A27) is to be replaced by Eq. (A24) with $f_0 \neq 1$, so that, rigorously, there appears on the left side of Eq. (A27) the additional term

$$P=-\frac{1}{r}\frac{d}{dr}\left[r(f_0^2-1)\frac{dg_p}{dr}\right]-(f_0^2-1)K_pg_p.$$

Having found a solution of Eq. (A27), the effect of this perturbation upon $\eta_p$ is in first order

$$\Delta\eta_p=\left(\int_{r_1}^{r_2} Pg_p r dr\right)\Big/\left(\int_{r_1}^{r_2} g_p^2 r dr\right).$$

This correction to the eigenvalues $\eta_p$ as well as that to the eigenfunctions $g_p$ is negligibly small since $P$ assumes appreciable values only for distances from the boundary comparable to or small compared with $\mu$.

## APPENDIX IV

### Quadratic Corrections to the Total Energy Due to Changes in the Occupation Numbers

We shall consider the more accurate expression for the energy change $\Delta E$ of the system given in Eq. (10) of the text and in Eq. (A10) of Appendix I. Originally all particles shall be in the individual state characterized by the quantum numbers $k=q=0$, $l=l'$, and it will be assumed that after a transition there will be

$$\Delta N_p=\nu_p N$$

particles in the individual state which differs from the original one merely by the change $p$ of the azimuthal quantum number and is thus characterized by the radial function $f_p(r)$. The numbers $\nu_p$, representing the fractional changes of occupation numbers, obey the relation

$$\sum_p \nu_p=0 \tag{A28}$$

and shall further satisfy the condition $\nu_p\ll 1$. With the notations of Eqs. (24) and (41) and denoting by

$$\Delta E_p=-(\hbar^2/2m)\Delta\eta_p$$

the change of the individual particle energy $E_p=-(\hbar^2/2m)\eta_p$, one has then from Eq. (A10)

$$\Delta E=-\frac{N\hbar^2}{2m}\sum_p \nu_p(\eta_p+\tfrac{1}{2}\Delta\eta_p). \tag{A29}$$

From Eq. (A9), we obtain

$$\Delta\eta_p=-\frac{2\pi}{a}\int_{r_1}^{r_2}\left\{\frac{2(\beta-p)\Delta\beta}{r^2}+\Delta u\right\}f_p^2 r dr, \tag{A30}$$

with the notations used in Secs. III and IV, and with $\Delta\beta$ and $\Delta u$ as the changes of $\beta$ and $u$, corresponding to $\Delta\mathbf{A}$ and $\Delta U$ in Eq. (A9). The differential equation for $\Delta\mathbf{A}$ is obtained from that for $\mathbf{A}$ by replacing on the right side of Eq. (5) the current density $\mathbf{i}$ by its change due to the increase of $N_s$ by $\Delta N_s$, and by replacing $\mathbf{A}$ by $\Delta\mathbf{A}$ on the right side of Eq. (7). Keeping only linear terms in these changes, the corresponding differential equation for $\Delta\beta$ is then given by

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\Delta\beta}{dr}\right)=\frac{1}{\lambda^2}(\Delta\beta-\textstyle\sum_p \nu_p p f_p^2), \tag{A31}$$

and it is permissible, in this order, to neglect the change $\Delta f_p$ of $f_p$ in Eqs. (A30) and (A31). On the other hand, it is possible and advisable to express $\Delta u$ in terms of the change $\Delta f_0$ of $f_0$. Keeping again only linear terms, one has from Eq. (42) for $p=0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Delta f_0}{dr}\right)-w\Delta f_0=(\Delta w+\Delta\eta_0)f_0,$$

since $\eta_0=0$. As discussed in the text and in Appendix II, it is possible because of the strong Coulomb effects to use Eqs. (31) and (32), i.e., to let in this equation $f_0=1$ and $w=u+\beta^2/r^2=u+(1/r^2)(l'-\alpha)^2=0$. With $\Delta w=\Delta u+(2\beta\Delta\beta/r^2)$, one obtains thus

$$\Delta u=-\frac{2\beta\Delta\beta}{r^2}+\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Delta f_0}{dr}\right)-\Delta\eta_0. \tag{A32}$$

Just as in the original state, the Coulomb forces have also in the new state the principle effect that the charge density of the particles must neutralize that of the ions. In terms of the numbers $\nu_p$, this means that

$$f_0^2+\textstyle\sum_p \nu_p f_p^2=1$$

with the above result $f_0=1$ for the special case where $\nu_p=0$. It is again permissible in this order to neglect the change $\Delta f_p$ of the function $f_p$ in the sum, since it contains already the small numbers $\nu_p$, but it is necessary to write for the first term in this equation $f_0^2\cong 1+2\Delta f_0$ with the result that

$$\Delta f_0=-\tfrac{1}{2}\textstyle\sum_p \nu_p f_p^2.$$

Therefore, from Eqs. (A30) and (A32)

$$\Delta\eta_p=\frac{2\pi}{a}\int_{r_1}^{r_2}\left\{\frac{2p\Delta\beta}{r^2}+\frac{1}{2}\sum_{p'}\nu_{p'}-\frac{1}{r}\frac{d}{dr}\left(r\frac{df_{p'}^2}{dr}\right)\right\}f_p^2 r dr+\Delta\eta_0, \tag{A33}$$

where in the last term the normalization

$$(2\pi/a)\int f_p^2 r dr=1$$

has been used.

Except for terms of the order $\lambda/r_1$, the functions $f_p$ can be considered to vary only relatively little over distances of order of magnitude $\lambda$, so that for $\lambda/r_1\ll 1$, Eq. (A31) is solved by

$$\Delta\beta=\textstyle\sum_p \nu_p p f_p^2.$$

Inserting this expression into the expression for $\Delta\eta_p$ of Eq. (A33) and the latter into Eq. (A29), one finally

obtains

$$\Delta E=-\frac{\hbar^2}{2m}\Big[\sum_p \nu_p\eta_p+\frac{2\pi}{a}\int_{r_1}^{r_2}\Big\{\frac{1}{r^2}(\sum_p \nu_p p f_p^2)^2 -\Big(\sum_p \nu_p f_p\frac{df_p}{dr}\Big)^2\Big\}rdr\Big], \quad \text{(A34)}$$

since the additive constant $\Delta\eta_0$ in Eq. (A33), upon summation over $p$, gives a vanishing contribution in virtue of Eq. (A28). Among the numbers $\nu_p$, it is only $\nu_0$ which is negative since particles have only been removed from the state $p=0$; on the other hand, the value $p=0$ gives a vanishing contribution to the sums in Eq. (A34) since $\eta_0=0$ and $(df_0/dr)=0$ so that all the numbers $\nu_p$ in this equation are either positive or zero.

It will now be assumed that $H_2=0$ and $(H_c)_M>H_1>(H_c)_T$, i.e., $\kappa_2=0$ and $1>\kappa_1>t$. The system has been found, in this case, to be energetically unstable against some of the single-particle transitions, which means that for some (negative) values of $p$, one has $\eta_p>0$; the occurrence of such transitions is indicated by a finite (positive) value of the corresponding numbers $\nu_p$ while all the other numbers $\nu_p$ vanish. The first term in the square bracket of Eq. (A34) is then positive, corresponding to a decrease of the total energy of the system, and the question to be answered is whether the additional quadratic terms in the numbers $\nu_p$ can cause $\Delta E$ to have a minimum. Any further change of these numbers would then lead to an increase of the total energy, i.e., one would deal with a new stable state of the system, reached after a small fraction of all particles had left their original individual state.

Clearly such a situation cannot arise if the total contribution of the quadratic terms in Eq. (A34) is positive, i.e., if it has the same sign as the linear term. While the first term in the curly bracket under the integral is indeed positive, the second is negative and might prevail.

In order to see that this is not the case, one has to consider the variation of the derivative $(df_p/dr)$ in the interval $r_1<r<r_2$. Starting from the value zero for $r=r_1$, it reaches its maximum at a distance of order of magnitude $\lambda$ from this boundary; in view of the considerations of Sec. IV which lead to Eq. (64) of the text, it satisfies at that point the condition

$$df_p/dr=(\kappa_1 p/r)f_p.$$

At this point of the interval of integration the negative term in Eq. (A34) thus differs from the positive term by the factor $\kappa_1^2<1$, so that the former prevails. For larger values of $r$, this is even more the case, since $(df_p/dr)$ will decrease until it reaches again the value zero at $r=r_2$. We are thus led to the conclusion that the total contribution of the integral and, hence, of the quadratic terms in $\nu_p$ of Eq. (A34) is necessarily positive, so that these terms lead to an even larger decrease of the energy than that caused by the linear term. The possibility of a new stable state, reached by a small fractional change of the total number $N$ of particles which originally occupy the same individual state, is thereby excluded.

It is not *a priori* certain that the inclusion of higher order terms which would become important for values of $\nu_p$ comparable to unity could not invalidate this conclusion. It seems reasonable, however, to interpret our result for $\nu_p\ll 1$ to generally indicate that the system will continue to make transitions until the field $H_1$ in the interior has decreased to the value $H_1=(H_c)_T$, and that at this stage stability is regained by all particles being again in a common individual state with a smaller value of the azimuthal quantum number than the one characteristic of the original state.