Relationship between Velocity-Dependent Potentials and Hard-Core Potentials*

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It is shown that, outside the range of the velocity dependence, the two-body wave function resulting from a repulsive velocity-dependent force is exactly the same as that produced by an angular-momentum dependent potential outside a hard core.

THE purpose of this paper is to clarify the relationship between two-body hard-core potentials and repulsive velocity-dependent potentials. We shall show that, outside the range of the velocity dependence, the wave function resulting from a repulsive velocity-dependent force is exactly the same as that produced by a hard core and an angular-momentum dependent potential. The apparent core size obtained by dropping the angular-momentum dependent terms is larger for nonzero angular momentum than for zero, and is energy-independent for zero angular momentum.

As any but the simplest theory of nuclear forces involves contributions of a velocity-dependent type (or nonlocal, which is in effect the same), it seems likely that there are, in fact, velocity-dependent forces between nucleons. Relativistic corrections introduce (small) velocity-dependent forces. For example, the Dirac equation for an electron moving in a static potential may be reduced rigorously to a Schrödinger equation with a velocity-dependent potential.[1]

We will prove the above results by transforming the two-body Schrödinger equation with a velocity-dependent force of the type considered by Green[2] and Razavy, Field, Levinger, Rojo, and Simmons[3] into one of standard, nonvelocity-dependent type. A somewhat related investigation was reported by Bell[4] from, however, a rather different viewpoint. We start with the Schrödinger equation for a pair of nucleons

$$\{(1/4M)[\mu(r)p^2+2\mathbf{p}\cdot\mu(r)\mathbf{p}+p^2\mu(r)]+V(r)\}\psi(\mathbf{r})=E\psi(\mathbf{r}), \quad (1)$$

where $\mu(\infty)=1$. The condition for a repulsive force is $\mu\geq 1$. We have chosen this Hermitization for definiteness. It is the one implied by the Weyl correspondence[5] for the classical quantity $\mu(r)p^2$. Other Hermitizations differ from this one only by a function of r. Equation (1) may be rewritten as

$$\mu(r)\nabla^2\psi(\mathbf{r})+\boldsymbol{\nabla}\mu(r)\cdot\boldsymbol{\nabla}\psi(\mathbf{r})+\tfrac{1}{4}\psi(\mathbf{r})\nabla^2\mu(r)+(M/\hbar^2)[E-V(r)]\psi(\mathbf{r})=0. \quad (2)$$

As we have chosen μ and V to be functions of r alone, we may separate (2) into radial and angular components $Y_{lm}(\theta,\varphi)\psi_l(r)$. If we now make the transformation

$$r=r(\rho),\qquad \psi_l=v(\rho)u_l(\rho), \quad (3)$$

Eq. (2) goes into

$$\frac{\mu(r)v(\rho)}{[r'(\rho)]^2}\frac{d^2u_l}{d\rho^2}+\left[\frac{2v'(\rho)\mu(r)}{[r'(\rho)]^2}+\left(\frac{2\mu(r)}{r(\rho)r'(\rho)}+\frac{\mu'(r)}{r'(\rho)}-\frac{r''(\rho)\mu(r)}{[r'(\rho)]^3}\right)v(\rho)\right]\frac{du_l(\rho)}{d\rho}+\left[\tfrac{1}{4}[\nabla^2\mu(r)]v(\rho)+\frac{\mu(r)v''(\rho)}{[r'(\rho)]^2}+\left(\frac{2\mu(r)}{r(\rho)r'(\rho)}+\frac{\mu'(r)}{r'(\rho)}-\frac{r''(\rho)\mu(r)}{[r'(\rho)]^3}\right)v'(\rho)\right]u_l(\rho)-\frac{l(l+1)\mu(r)}{r^2(\rho)}v(\rho)u_l(\rho)+\frac{M}{\hbar^2}[E-V(r(\rho))]v(\rho)u_l(\rho)=0. \quad (4)$$

If we equate the coefficient of $du_l(\rho)/d\rho$ to zero, we may readily solve for $v(\rho)$. It is

$$v(\rho)=[r(\rho)]^{-1}[r'(\rho)/\mu(r)]^{1/2}. \quad (5)$$

Likewise, equating the coefficient of $v(\rho)d^2u_l/d\rho^2$ to unity, we obtain

$$r'(\rho)=[\mu(r)]^{1/2}. \quad (6)$$

Thus,

$$v(\rho)=[r(\rho)]^{-1}[\mu(r)]^{-1/4}. \quad (7)$$

Substituting (6) and (7) into (4) and dividing by $v(\rho)$, we obtain, after a little manipulation,

$$\frac{d^2u_l(\rho)}{d\rho^2}-\frac{l(l+1)\mu(r)}{r^2}u_l(\rho)+\frac{M}{\hbar^2}\left[E-V(r)+\frac{\hbar^2}{2M}\mu'(r)\left(\frac{\mu'(r)}{8\mu(r)}-\frac{1}{r}\right)\right]u_l(\rho)=0. \quad (8)$$

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[1] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1953), p. 129.

[2] A. M. Green, Nuclear Phys. (to be published).

[3] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962) and O. Rojo and L. M. Simmons, *ibid.* **125**, 273 (1962), and references therein.

[4] J. S. Bell, *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961* (Heywood & Company, London, 1961), p. 373.

[5] G. A. Baker, Jr., Phys. Rev. **109**, 2198 (1958).

This equation is now of the usual static-potential type. Matrix elements between states $\psi_a(r)$, $\psi_b(r)$ of (2) are related to those between the corresponding states $u_a(\rho)$, $\mu_b(\rho)$ of (8) by

$$\int_0^\infty \psi_a^*(r)S\psi_b(r)r^2dr=\int_0^\infty u_a^*(\rho)Su_b(\rho)d\rho, \tag{9}$$

where S is any operator, and ρ is related to r by the integration of (6),

$$\rho=\int_0^r [\mu(r)]^{-1/2}dr. \tag{10}$$

Outside the range of the velocity-dependent potential, $\mu(r)=1$, and hence in that region

$$\rho=r-a,\quad u_l=\psi_l, \tag{11}$$

where

$$a=\int_0^\infty \{1-[\mu(r)]^{-1/2}\}dr. \tag{12}$$

To study the apparent effect of the velocity-dependent force let us introduce

$$R=\rho+a. \tag{13}$$

In the velocity-dependent potential-free region, $R=r$. Equation (8) becomes [let $w_l(R)=u_l(R-a)$]

$$\frac{d^2w_l(R)}{dR^2}-\frac{l(l+1)}{R^2}w_l(R)+\frac{M}{\hbar^2}\biggl\{E-\biggl[V(r(R-a)) +\frac{\hbar^2}{M}l(l+1)\left(\frac{\mu(r(R-a))}{[r(R-a)]^2}-\frac{1}{R^2}\right)-\frac{\hbar^2}{2M}\mu'(r(R-a)) \times\left(\frac{\mu'(r(R-a))}{8\mu(r(R-a))}-\frac{1}{r(R-a)}\right)\biggr]\biggr\}w_l(R)=0, \tag{14}$$

subject to the boundary condition $w_l(a)=0$. We see at once from the form that this is exactly the problem of a potential (enclosed in large square brackets) outside a hard core of radius a. However, the effective potential is angular-momentum dependent. As the coefficient of $l(l+1)$ in the potential can easily be shown [from (10)–(13)] to be non-negative, to drop it to obtain an angular-momentum independent potential is equivalent to adding an attractive potential. The effect of an attractive potential is to bend the wave function more rapidly toward the axis and hence $w_l(R)=0$ for a value of $R>a$, its original intercept. Thus, we have shown that the velocity-dependent force given by (1) is exactly equivalent to an angular-momentum dependent potential outside a hard core.

Dropping the angular-momentum dependent terms we have a hard core of state-dependent radius; however, the radius is always at least as large as given by (12).

The $l(l+1)$ factor must, of course, be replaced by the Legendre differential operator if the wave function is not an angular momentum eigenfunction. We remark that a hard core could be exactly simulated, outside the range of the velocity dependence, by subtracting this term from the original Hamiltonian. This remark is equivalent to Bell's[4] result that a slightly more complicated velocity dependence than contained in (1) simulated a hard core exactly. If a potential $W(R)$ outside a hard core of radius a is used to fit two-body scattering data, then by reversing the steps which led to (14) we see that the family of velocity-dependent potentials characterized by the arbitrary function $\mu(r)$, which are implied by

$$\biggl\{\mu(r)\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{l(l+1)}{[\rho(r)+a]^2} +\boldsymbol{\Delta}\mu(r)\cdot\boldsymbol{\nabla}+\tfrac{1}{4}\nabla^2\mu(r)+\frac{M}{\hbar^2}\biggl[E-W(\rho(r)+a) -\frac{\hbar^2}{2M}\mu'(r)\left(\frac{\mu'(r)}{8\mu(r)}-\frac{1}{r}\right)\biggr]\biggr\}\psi_l(r)=0, \tag{15}$$

where $\rho(r)$ and a are given by (10) and (12), give identical results for the two-body problem. Identical results for the two-body problem do not, of course, necessarily imply identical results for the many-body problem.

We have investigated the effective size of the core implied by the velocity-dependent potentials of Rojo and Simmons.[3] They take

$$\mu(r)=1+se^{-r/\beta}. \tag{16}$$

Thus, from (10) and (12),

$$\rho(r)+a=r+2\beta\ln\left[\frac{1+(1+se^{-r/\beta})^{1/2}}{2}\right]$$

$$a=2\beta\ln\left[\frac{1+(1+s)^{1/2}}{2}\right]. \tag{17}$$

Using their values of $s=10.0$ and $\beta=1/3.6$ f, we compute $a=0.43$ f. This size is quite comparable with, for instance, that found by Gammel and Thaler.[6]

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[6] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957).