

Mandelstam Representation and Regge Poles with Absorptive Energy-Dependent Potentials*

JOHN M. CORNWALL† AND MALVIN A. RUDERMAN
Department of Physics, University of California, Berkeley
 (Received May 28, 1962)

The Mandelstam representation in potential theory is proved for superpositions of absorptive, energy-dependent Yukawa potentials. The dependence of the potential $V(r,s)$ on the energy s is such that $V(r,s)$ is analytic in the s plane cut from s_0 to infinity, where s_0 is a threshold for inelastic processes. Such a potential is implied by the causality condition that the scattered wave cannot precede the incident wave. Further, it is shown that certain interactions nonlocal in both space and time can also be reduced to the above type of local, energy-dependent potential. The postulate of absorptivity [$\text{Im}V(r,s) \leq 0$] is crucial for the existence of the usual Mandelstam representation. The relationship of the absorptive potentials in the Schrödinger equation and in dispersion theory to unitarity is discussed. An analysis of partial waves with complex angular momenta is given from the point of view of the Green's function, and it is shown that absorptive energy-dependent causal potentials behave qualitatively like real energy-independent potentials, so far as Regge pole trajectories are concerned. Regge poles for emissive potentials can behave anomalously. Absorptive potentials may, however, give rise to new singularities in unphysical regions.

I. INTRODUCTION

THE Mandelstam representation, which remains unproven in field theory, has been proved for scattering by real energy-independent potentials which are superpositions of Yukawa potentials¹⁻⁵: the potential must be of the form

$$V(r) = -\frac{1}{\pi} \int_{m_0^2}^{\infty} dm^2 \rho_0(m^2) \frac{e^{-mr}}{r}. \quad (1)$$

The real static potential of Eq. (1) does not represent the most general kind of spatially local potential for the description of the elastic scattering of two particles within the framework of the Schrödinger equation; rather, one is led to consider energy-dependent potentials of the type

$$V(r,s) = -\frac{1}{\pi^2} \int_{m_0^2}^{\infty} dm^2 \rho_1(s,m^2) \frac{e^{-mr}}{r}, \quad (2)$$

where s is the total center-of-mass kinetic energy⁶ and ρ_1 is not necessarily real. Such energy dependent complex local potentials have long been utilized in the optical model; they have also been suggested by the full relativistic Mandelstam representation.⁷ In this work, the Mandelstam representation will be proved for potentials of the type (2), subject to certain condi-

tions of the weight function $\rho_1(s,m^2)$ to which one is led by requiring causality and over-all conservation of probability. The requirements to be placed on the potential lead to the following restrictions on $\rho_1(s,m^2)$: (a) $\rho_1(s,m^2)$ is, for every m , an analytic function of s in the s plane cut along the real axis from $s_0 > 0$ to infinity; (b) $\rho_1(s,m^2)$ is such that $|V(s,r)| < M/r^2$ for all s and r , where M is a fixed constant independent of s and r ; (c) $\text{Im}V(s,r) \leq 0$ for all r . The branch point at s_0 corresponds to an inelastic threshold; one hopes that a potential of the type (2) can represent quite generally the influence of inelastic processes on the elastic scattering amplitude. Justification for the representation (2), along with conditions (a) and (c), will be given in Sec. II; further, it will be shown that certain types of potentials which are nonlocal in both space and time variables can be given equivalent representations in the form (2).

The proof of the Mandelstam representation, just as in the ordinary potential case, proceeds in two steps. First, one proves the analyticity of the scattering amplitude $f(s,t)$ in s for t fixed inside the Lehmann ellipse; then, one proves that $f(s,t)$ for fixed s is analytic everywhere in the finite t plane. It is the fact that the potential can be written as a sum of Yukawa potentials that allows one to conclude the analyticity in t . These questions are taken up in Sec. III.

There are at least two methods of proving the energy dispersion relations: the Fredholm method, based on the work of Jost and Pais,⁸ and used by Khuri⁹ in his proof; and what will be called the Green's function method of Klein and Zemach.¹⁰ The essence of the Klein and Zemach argument is knowledge of the analytic properties in $k = s^{1/2}$ of the Green's function of the Schrödinger equation $G(\mathbf{r},\mathbf{r}';k)$. For a Hermitian Hamiltonian (real potential) these properties are well

* Research supported in part by the National Science Foundation.

† National Science Foundation Pre-doctoral Fellow, 1960-62.

¹ R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, *Ann. Phys. (New York)* **10**, 62 (1960).

² A. Klein, *J. Math. Phys.* **1**, 41 (1960).

³ T. Regge, *Nuovo cimento* **14**, 951 (1959).

⁴ T. Regge, *Nuovo cimento* **18**, 947 (1960).

⁵ A. Bottino, A. M. Longoni, and T. Regge, *Nuovo cimento* **23**, 954 (1962).

⁶ We use the following notation. Let \mathbf{k} , \mathbf{k}' be the initial and final momenta, respectively, of one of the scattered particles in the center-of-mass system. Then

$$s = k^2 = k'^2, \quad t = -|\mathbf{k} - \mathbf{k}'|^2 = 2s(1 - \cos\theta).$$

We use units such that $\hbar = 2M = 1$, where M is the reduced mass.

⁷ G. F. Chew and S. C. Frautschi, *Phys. Rev.* **124**, 264 (1961).

⁸ R. Jost and A. Pais, *Phys. Rev.* **82**, 840 (1951).

⁹ N. N. Khuri, *Phys. Rev.* **107**, 303 (1957).

¹⁰ A. Klein and C. Zemach, *Ann. Phys. (New York)* **7**, 440 (1959); C. Zemach and A. Klein, *Nuovo cimento* **10**, 1078 (1958).

known: $G(\mathbf{r}, \mathbf{r}'; k)$ is analytic in k except for poles in the lower half k plane, and poles along the positive imaginary axis corresponding to bound states. When G is expressed as a function of s , there appears a branch cut of the square root type along the positive real s axis, corresponding to the continuous spectrum. Provided conditions (a)–(c) given above are satisfied, the Green's function for a complex energy-dependent potential can be shown to have the same analyticity properties, except that there is an additional branch cut in the s plane, beginning at s_0 , the threshold for inelastic processes. Then, one may take over the Klein-Zemach arguments to prove the dispersion relations in energy.

The analyticity in momentum transfer has been proved by a number of authors¹⁻⁵ for Yukawa-type potentials, and these proofs may be taken over at once for the energy-dependent case. Once the Mandelstam representation is proved, it is possible to give an algorithm for computing the spectral function in terms of the complex potential so that a knowledge of the potential together with the Mandelstam representation is equivalent to the Schrödinger equation even without the usual unitarity condition.

In Sec. V, we also discuss complex angular moments from the point of view of the Green's function method. The Green's function for the complex potential case is the inverse of an operator $H = -\nabla^2 + V(s, \mathbf{r})$ which depends parametrically on the complex quantity s . Likewise, the Hamiltonian which determines the partial-wave amplitudes:

$$H_l = -\frac{d^2}{dr^2} + l(l+1)r^{-2} + V,$$

depends parametrically on l , where l is now allowed to take on arbitrary complex values. Exactly the same methods used to determine the analyticity in s of $(s-H)^{-1}$ may be used to study the analyticity in l of $(s-H_l)^{-1}$, whether or not the potential V is energy dependent. One has a unified approach to the analyticity of scattering amplitudes, based on the Green's function method, in both s and l . It is shown that, for absorptive potentials which satisfy conditions (a) and (c), the qualitative behavior of the Regge trajectories is the same as for the energy-independent case, whereas for emissive potentials [$\text{Im}V(s, \mathbf{r}) > 0$] one can expect strange, unphysical behavior of the trajectories. These general considerations are illustrated by the exactly soluble case of the Coulomb potential with complex strength.^{11,12}

II. PROPERTIES OF ENERGY-DEPENDENT INELASTIC POTENTIALS

An energy-dependent (inelastic) potential in the Schrödinger equation is equivalent to the introduction

¹¹ The real, energy-independent Coulomb potential has been studied by V. Singh (to be published).

¹² R. Oehme, Nuovo cimento **25**, 183 (1962).

of an interaction which is nonlocal in time, so that the scattered wave at a point is not necessarily emitted at the instant the incident wave arrives. The causality condition insists that the scattered wave is never emitted before the arrival of the incident wave.

The Schrödinger equation,

$$[-\nabla^2 + V(\mathbf{r}, s) - s]\phi(\mathbf{r}) = 0, \tag{3}$$

is equivalent to

$$i\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = -\nabla^2\psi(\mathbf{r}, t) + \int_{-\infty}^{\infty} dt' U(\mathbf{r}, t-t')\psi(\mathbf{r}, t'), \tag{4}$$

with

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-ist} \tag{5}$$

and

$$U(\mathbf{r}, t) = \frac{1}{2\pi} \int dr V(\mathbf{r}, s)e^{ist}. \tag{6}$$

The condition of microscopic time-causality requires that $\psi(\mathbf{r}, t)$ at time t depend only on its past history, and not on its future. This means that $U(\mathbf{r}, t-t')$ vanishes for $t' > t$ or

$$U(\mathbf{r}, t) = 0, \quad t < 0. \tag{7}$$

Since V is the Fourier transform of U we have the result that $V(\mathbf{r}, s)$ is regular in the lower half s -plane, $\text{Im}s < 0$. If we assume that $\text{Im}V(\mathbf{r}, s) = 0$ for s less than some positive threshold s_0 , it follows by the principle of reflection, $V^*(\mathbf{r}, s) = V(\mathbf{r}, s^*)$, that V can be continued analytically everywhere in the upper half s plane. Then $V(\mathbf{r}, s)$ is analytic everywhere in the s plane cut from s_0 to infinity, which is condition (a). We can use Cauchy's formula to represent V by an integral,

$$V(\mathbf{r}, s) = -\frac{1}{\pi} \int ds' \frac{\text{Im}V(\mathbf{r}, s')}{s' - s - i\epsilon} + V_0(\mathbf{r}), \tag{8}$$

if $s^{-1}V(\mathbf{r}, s) \rightarrow 0$ as $s \rightarrow \infty$.

Condition (c), the condition of absorptivity, follows readily from the consideration that the total particle flux cannot increase. Since the solution to (4) is assumed stationary, we have

$$(\partial/\partial t)|\psi(\mathbf{r}, t)|^2 = 0. \tag{9}$$

From the Schrödinger equation (4), one finds that (9) can be written

$$\nabla \cdot \mathbf{S} = 2 \text{Im}V(\mathbf{r}, s)|\phi(\mathbf{r})|^2, \tag{10}$$

where

$$\mathbf{S} = -i(\phi^* \nabla \phi - \phi \nabla \phi^*)$$

is the usual expression for the particle flux. The integrated flux through the sphere at infinity is then

$$2 \int d^3r \text{Im}V(\mathbf{r}, s)|\phi(\mathbf{r})|^2. \tag{11}$$

The integrated flux consists of two parts: the incoming wave and the outgoing wave. When inelastic processes complete with elastic processes, fewer particles leave the scattering region than enter it, because some are absorbed. It follows that the net integrated flux is negative, which implies that $\text{Im}V(\mathbf{r},s) \leq 0$. This is the absorptivity condition. It should be noted that this condition is to be interpreted as holding on the upper side of the branch cut in $V(\mathbf{r},s)$, since one supposes that a small positive imaginary part is added to s to insure outgoing scattered waves.

The Klein-Gordon equation can also always be cast into the form of Eq. (3) with $s = \omega^2 - \mu^2 = k^2$, so that as long as s is the square of the incident momentum the subsequent discussion is valid for the scattering of relativistic or nonrelativistic particles by spatially fixed potentials of the form of Eq. (1).

The interaction in (4) is nonlocal only in time. One might wish to consider also interactions which are nonlocal in space as well as in time. It is probably unnecessary to achieve such generality, according to the work of Chew and Frautschi,⁷ which is considered later; nevertheless, we can show that the dispersion relations hold for certain kinds of spatially nonlocal potentials. That it is possible to do this follows from the fact that a class of spatially nonlocal interactions can be replaced by local energy-dependent potentials which give the same wave functions.¹³ We consider, for example, a retarded interaction of the form

$$H_I\psi = \int d^3\mathbf{r}' \int dt' K(\mathbf{x}, t-t' - |\mathbf{x}-\mathbf{x}'|) \times \psi(\mathbf{x}', t') |\mathbf{x}-\mathbf{x}'|^{-1}. \quad (12)$$

We assume that $K(\mathbf{x},y)$ behaves like an ordinary potential in \mathbf{x} for every y ; it is well behaved at the origin and vanishes sufficiently fast at infinity.

Suppose $K(\mathbf{x},y) = 0$ for $y < 0$. As before, $\psi(\mathbf{x}', t')$ is taken to be a stationary solution, with energy s . Let

$$K(\mathbf{x},y) = \frac{1}{2\pi} \int ds k(x,s) e^{isy}. \quad (13)$$

The interaction becomes

$$H_I\psi = \int d^3\mathbf{x}' k(\mathbf{x},s) e^{-is|\mathbf{x}-\mathbf{x}'|} \psi(\mathbf{x}') |\mathbf{x}-\mathbf{x}'|^{-1}. \quad (14)$$

We can write symbolically $\psi(\mathbf{x}') = e^{i\mathbf{q} \cdot (\mathbf{x}-\mathbf{x}')} \psi(\mathbf{x})$, where $\mathbf{q} = -i\nabla_{\mathbf{x}}$, and find

$$H_I\psi = 4\pi k(\mathbf{x},s) (\nabla^2 + s^2)^{-1} \psi(\mathbf{x}). \quad (15)$$

The Schrödinger equation now reads

$$[-\nabla^2 - s - 4\pi k(\mathbf{x},s) (\nabla^2 + s^2)^{-1}] \psi(\mathbf{x}) = 0. \quad (16)$$

We determine a function $V(\mathbf{x},s)$, such that

$$[\nabla^2 + s] \psi(\mathbf{x}) = V(\mathbf{x},s) \psi(\mathbf{x}), \quad (17)$$

which leads at once to the equation

$$V(\mathbf{x},s) = -4\pi k(\mathbf{x},s) [V(\mathbf{x},s) - s + s^2]^{-1}. \quad (18)$$

We can solve for the potential explicitly:

$$V(\mathbf{x},s) = \frac{1}{2}(s - s^2) - [\frac{1}{4}(s - s^2)^2 - 4\pi k(\mathbf{x},s)]^{1/2}. \quad (19)$$

The negative sign in front of the square root is chosen so that $V(\mathbf{x},s)$ goes to zero like $k(\mathbf{x},s)$ as $x \rightarrow \infty$. For each "causal" $V(\mathbf{x},s)$ in Eq. (19) we obtain a retarded $k(\mathbf{x},s)$. The converse is not so automatic. Now suppose that $k(\mathbf{x},s)$ is such that the square root in the effective potential never vanishes (thus avoiding branch points in s whose location depends on \mathbf{x}), and that $k(\mathbf{x},s)$ is analytic in the entire s plane cut from s_0 to infinity. This potential is automatically absorptive, if the principal branch of the square root is chosen. Then $V(\mathbf{x},s)$ likewise has a branch point at s_0 . If $k(\mathbf{x},s)$ is sufficiently well behaved in \mathbf{x} , $V(\mathbf{x},s)$ can be approximated as closely as is desired by a sum of Yukawa potentials, and the proof to be given of the Mandelstam representation holds for a wide class of retarded interactions. Actually, the generalization to nonlocal interactions does not lead to much physical insight, since one does not know what to choose for the interactions; further, if the Mandelstam representation is sufficient, only local energy-dependent Yukawa potentials need be considered for the description of scattering on the energy shell.

III. PROOF OF THE DISPERSION RELATIONS

A. Absorptivity Condition in the Entire Complex Plane

We have seen that the potential¹⁴ $V(s)$ is an analytic function in the cut s plane, and that in this cut plane it obeys $V^*(s) = V(s^*)$. We decompose $V(s)$ into its real and imaginary parts

$$V(s) = u(s_1, s_2) + iw(s_1, s_2), \quad (20)$$

where $s = s_1 + is_2$. The absorptivity condition states that

$$\lim_{s_2 \rightarrow +0} w(s_1, s_2) \leq 0 \quad \text{for all } s_1. \quad (21)$$

We presume that $V(s)$ vanishes at infinity

$$\lim_{|s| \rightarrow \infty} u(s_1, s_2), \quad w(s_1, s_2) = 0. \quad (22)$$

Now both u and w are harmonic functions of s_1, s_2 in the upper half plane. The boundary of this half plane is the real axis plus the contour at infinity, and everywhere on this boundary we have $w(s_1, s_2) \leq 0$. Since a harmonic function never takes on its maximum value

¹³ W. E. Frahn, *Nuovo cimento* 4, 313 (1956); 5, 393 (1957); W. E. Frahn and R. H. Lemmer, *ibid.* 5, 523, 1564 (1957).

¹⁴ We suppress the dependence of $V(s, r)$ on r when convenient in this section.

in the interior of its domain of harmonicity, it follows that $w(s_1, s_2) \leq 0$ everywhere in the upper half plane.

The condition $V(s^*) = V^*(s)$ implies

$$\begin{aligned} u(s_1, s_2) &= u(s_1, -s_2), \\ w(s_1, s_2) &= -w(s_1, -s_2). \end{aligned} \tag{23}$$

It follows from (21), (22) that on the lower side of the real axis that

$$\lim_{s_2 \rightarrow -0} w(s_1, s_2) \geq 0,$$

and thus on the boundary of the lower half plane, $w(s_1, s_2) \geq 0$. But w is harmonic in this plane, and since a harmonic function never takes on its minimum value in the interior of its domain of harmonicity, $w(s_1, s_2) \geq 0$ everywhere in the lower half plane. The two results found so far may be combined into the single statement,

$$\text{sgn} w(s) = \text{sgn} \text{Im} V(s) = -\text{sgn} \text{Im} s, \tag{24}$$

true everywhere in the cut s plane. This is the condition of absorptivity generalized in the entire s plane.

It is convenient to translate our results into the k plane, where $k = s^{1/2}$ (the principal brancy of the square root is to be taken). Let $k = x + iy$. Then the potential as a function of k obeys

$$\begin{aligned} V(s) = V(k^2) &= u(x^2 - y^2, 2xy) + iw(x^2 - y^2, 2xy) \\ &\equiv U(x, y) + iW(x, y) \end{aligned}$$

and (23) becomes

$$\begin{aligned} U(x, y) &= U(-x, y), \\ W(x, y) &= -W(-x, y), \end{aligned}$$

or more simply

$$V^*(k^2) = V[(-k^*)^2]. \tag{25}$$

B. Analyticity of the Green's Function

We define the Green's operator by

$$G(s) = [s - H_0 - V(s)]^{-1} \tag{26}$$

in the s plane cut from 0 to infinity; the physical Green's operator is defined as the limit of $G(s)$ as $s_2 \rightarrow +0$. The Green's function is defined as the matrix elements of $G(s)$ in a coordinate representation:

$$G(\mathbf{r}, \mathbf{r}'; s) = \langle \mathbf{r} | G(s) | \mathbf{r}' \rangle. \tag{27}$$

Those points where $G(s)$ fails to exist, or is an unbounded operator, are known as the spectrum of $H(s) = H_0 + V(s)$. The points where $G(s)$ does not exist comprise the point spectrum and correspond physically to bound states, if they occur on the first s sheet (upper half k plane). The points where $G(s)$ is unbounded, or the continuous spectrum (there is no residual spectrum), correspond to scattering states. Suppose for a moment that V , and hence H , were independent of s . It is then a well-known result¹⁵ that the matrix elements of $G(s)$

are analytic functions of s , except for s in the spectrum of H . We shall show that, when H and V depend on s , a very simple modification of this result is true; the matrix elements of $G(s)$ are analytic functions of s in the resolvent set, except for the singularities of the matrix elements of $V(s)$. (The complement of the spectrum is known as the resolvent set.)

Our first task is to locate the spectrum of $H(s)$. We can indicate the importance of the absorptivity condition (24) by simple considerations which serve to locate the bound states. Suppose there exists a square-integrable solution $f(\mathbf{x}; s)$ of the Schrödinger equation:

$$[-\nabla^2 + V(\mathbf{x}; s) - s]f(\mathbf{x}; s) = 0. \tag{28}$$

Then also

$$[-\nabla^2 + V^*(\mathbf{x}; s) - s^*]f^*(\mathbf{x}; s) = 0. \tag{29}$$

Multiply (28) by f^* , (29) by f , subtract the two resulting equations, and integrate over all space to get

$$\int d^3\mathbf{x} |f(\mathbf{x}; s)|^2 [w(\mathbf{x}; s) - \text{Im} s] = 0. \tag{30}$$

For any s with $\text{Im} s \neq 0$, (30) is an untrue equation, since the quantity in brackets can never vanish, by virtue of the generalized absorptivity condition (24). It follows that bound states can occur only for $\text{Im} s = 0$, $w(\mathbf{x}; s) = 0$. The bound states are, therefore, just where they are if V were real and energy independent—on the negative real s axis.

These considerations unfortunately cannot be used to locate the continuous spectrum, since the wave functions $f(\mathbf{x}; s)$ are then non-normalizable, and one cannot perform the steps leading to (30). We therefore turn to a Hilbert space analysis of the spectrum, following Stone.¹⁵ The equations referring to Hilbert space which locate the bound states are a direct transcription of Eqs. (28)–(30). $H(s) - s$ fails to have an inverse if and only if there exist vectors f, f', g, h such that, with $f - g = h$,

$$\begin{aligned} [H(s) - s]f &= f', \\ [H(s) - s]g &= f', \\ [H(s) - s]h &= 0. \end{aligned} \tag{31}$$

The last of Eqs. (31) corresponds to (28), and one carries out simple manipulations in Hilbert space to arrive at an equation corresponding to (30). The Hilbert-space operator responding to $w(\mathbf{x}; s)$ is

$$w(s) \equiv (2i)^{-1} [V(s) - V^\dagger(s)]. \tag{32}$$

The continuous spectrum is located by a simple modification of the procedure which shows that the continuous spectrum of a Hermitian operator lies on the real axis.¹⁵ We assume that $G(s)$ exists—i.e., s is not in the point spectrum—and find those s for which $G(s)$ is a bounded operator. We may also assume without loss of generality that $w(s)$ is bounded—that

¹⁵ M. H. Stone, *Linear Transformations in Hilbert Space and Their Application to Analysis* (American Mathematical Society, New York, 1932), pp. 138/142.

is, for any vector g , $|(g, w(s)g)| \leq w_s(g, g) \equiv w_s \|g\|^2$. For those vectors g needed to construct the scattering amplitude, $w(s)$ is indeed bounded. In any case, if $w(s)$ is unbounded, so that the number w_s can become arbitrarily large, this only improves the boundedness of $G(s)$, as will be seen. The condition of absorptivity means that $w(s)$ is non-negative-definite in the lower half plane, non-positive-definite in the upper half plane.

We use the identity

$$A^{-1}(A-B)B^{-1} \equiv B^{-1} - A^{-1} \tag{33}$$

to find

$$G(s) - G^+(s) = -2i\{G^+(s)[w(s) - \text{Im}s]G(s)\}. \tag{34}$$

Then for any vector g ,

$$\begin{aligned} (g, [G(s) - G^+(s)]g) &= (g, G(s)g) - (G(s)g, g) \\ &= -2i(G(s)g, w(s)G(s)g) + 2i \text{Im}s \|G(s)g\|^2. \end{aligned} \tag{35}$$

We take the absolute value of both sides of (33) and use Schwarz's inequality,

$$|(G(s)g, g)|, \quad |(g, G(s)g)| \leq \|G(s)g\| \|g\|,$$

to come to

$$|\text{Im}s \|G(s)g\|^2 - (G(s)g, w(s)G(s)g)| \leq \|G(s)g\| \|g\|. \tag{36}$$

Because of the assumed boundedness of $w(s)$, there exists a number $w_s(g)$ which, in general, depends on the vector $G(s)g$, such that

$$(G(s)g, w(s)G(s)g) = w_s(g) \|G(s)g\|^2 \tag{37}$$

with $|w_s(g)| \leq w_s$ and

$$\begin{aligned} w_s(g) &\leq 0 \quad \text{for } \text{Im}s > 0 \\ w_s(g) &\geq 0 \quad \text{for } \text{Im}s < 0. \end{aligned} \tag{38}$$

Then (36) becomes

$$\|G(s)g\| \geq \|g\| |\text{Im}s - w_s(g)|^{-1}. \tag{39}$$

$G(s)$ is therefore a bounded operator except for those s 's for which the denominator of (39) can vanish. Here the condition of absorptivity becomes important, for by (38) it is assumed that the denominator can never vanish for $\text{Im}s \neq 0$. We note the analogy between (3) and (39): in both equations, $w(s) - \text{Im}s$ appears, and in both equations the nonvanishing of that quantity insures that s is in the resolvent set. Because of the absorptivity condition, we have for any g, s

$$|\text{Im}s - w_s(g)|^{-1} \leq |\text{Im}s|^{-1}. \tag{40}$$

It follows that

$$\|G(s)g\| \leq \|g\| |\text{Im}s|^{-1}, \tag{41}$$

and hence that $G(s)$ is bounded everywhere except on the real s axis. We can now see that the assumption of the boundedness of $w(s)$ is actually unnecessary, for, if $w_s(g)$ is allowed to increase without limit for a sequence of vectors g , the inequalities (39), (4) are fulfilled more and more strongly. The condition (41)

is exactly that which one finds for Hermitian operators,¹⁵ and it is now guaranteed that the continuous spectrum of $H(s)$ lies on the real s axis.

We define the domain D as the resolvent set of $G(s)$ minus the singular points of $V(s)$. We can now easily prove that the matrix elements of $G(s)$ are analytic in D . First, $G(s)$ has matrix elements continuous and single valued in D , since $G(s)$ is the bounded inverse of an operator $H(s)$ which has continuous and single-valued matrix elements in D . If $G(s)$ were not bounded, we could not draw this conclusion. Secondly, we may calculate the first derivative of $G(s)$ by a simple application of the identity (33), because $G(s)$ is bounded and continuous in D . We have

$$\begin{aligned} \frac{\partial G(s)}{\partial s} &= \lim_{\Delta s \rightarrow 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} \\ &= -G^2(s) + G(s) \frac{\partial V(s)}{\partial s} G(s), \end{aligned} \tag{42}$$

which is evidently single valued in D , since $V(s)$ is analytic in D . Then the matrix elements of $G(s)$ are bounded and single valued, and have single valued derivatives everywhere in D , and are therefore analytic functions of s in D . In the present case, $G(\mathbf{r}, \mathbf{r}'; s)$ is analytic in the s plane cut from 0 to infinity, with the exception of the bound-state poles on the negative real axis. We may write

$$G(\mathbf{r}, \mathbf{r}'; s) = \int ds' \frac{[\rho(\mathbf{r}, \mathbf{r}'; s)]}{s - s'} + \sum_B \frac{\phi_B(\mathbf{r})\phi_B(\mathbf{r}')}{s_B - s}, \tag{43}$$

where the s_B are the (negative) bound-state energies, and $\phi_B(\mathbf{r})$ are the bound-state wave functions, chosen to be real. The integral over the continuous spectrum exists for almost all \mathbf{r}, \mathbf{r}' .

It should be noted that $G(\mathbf{r}, \mathbf{r}'; s)$ is really a function of $k = s^{\frac{1}{2}}$, just as the free-particle Green's function is. It is for this reason that there is a branch cut along the real s axis. We therefore write $G(\mathbf{r}, \mathbf{r}'; k)$ for the Green's function. Because of (25), one has

$$G^*(\mathbf{r}, \mathbf{r}'; k) = G(\mathbf{r}', \mathbf{r}; -k^*). \tag{44}$$

C. The Klein-Zemach Method

To finish the proof of the energy dispersion relations, one simply uses the known results of Klein and Zemach,¹⁰ which for the sake of completeness, we sketch here. Let the potential be

$$\begin{aligned} V(\mathbf{r}, s) &= -\frac{1}{\pi} \int_{m_0^2}^{\infty} dm^2 \rho_0(m^2) e^{-m\mathbf{r}} r^{-1} \\ &\quad - \frac{1}{\pi^2} \int_{m_0^2}^{\infty} ds' dm^2 \frac{\rho_1(s', m^2)}{s' - s} e^{-m\mathbf{r}} r^{-1} \\ &\equiv V_0(\mathbf{r}) + V_1(\mathbf{r}, s). \end{aligned} \tag{45}$$

The matrix elements of V are:

$$\begin{aligned} \langle \mathbf{k} | V(s) | \mathbf{k}' \rangle &\equiv \frac{1}{4\pi} \int d^3r e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(\mathbf{r},s) \\ &= -\frac{1}{\pi} \int dt' \frac{\rho_0(t')}{t'-t} - \frac{1}{\pi^2} \int dt' ds' \frac{\rho_1(s',t')}{(t'-t)(s'-s)} \\ &\equiv -V(s,t) \equiv -V_0(t) + V_1(s,t), \end{aligned} \tag{46}$$

where, as usual, $t = -|\mathbf{k}-\mathbf{k}'|^2 = -2s(1-\cos\theta)$ and $s = k^2$. One may then write for the scattering amplitude (with $-t < 4m_0^2$)

$$\begin{aligned} f(s,t) \equiv F(k,t) &= V(s,t) - \frac{1}{4\pi} \int d^3r d^3r' e^{-i\mathbf{k}'\cdot\mathbf{r}'} \\ &\quad \times V(\mathbf{r}',s) G(\mathbf{r}',\mathbf{r};k) V(\mathbf{r},s) e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{47}$$

Because of (25), (44), and the invariance of the Green's function to $\mathbf{r}, \mathbf{r}' \rightarrow -\mathbf{r}, -\mathbf{r}'$, we have

$$F^*(k,t) = F(-k^*,t) \quad (t \text{ real}). \tag{48}$$

We may isolate the bound-state pole terms explicitly, by using (43) for the Green's function:

$$\begin{aligned} F(k,t) &= V(s,t) - \frac{1}{4\pi} \int d^3r d^3r' ds' e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')} \\ &\quad \times \frac{V(\mathbf{r}',s)\rho(\mathbf{r},\mathbf{r}';s')V(\mathbf{r},s)}{s'-s} \\ &\quad + \sum_B (s-s_B)^{-1} [R(s_B,t,s) - R(s_B,t,s_B)] \\ &\quad + \sum_B (s-s_B)^{-1} R(s_B,t,s_B), \end{aligned} \tag{49}$$

where

$$\begin{aligned} R(s_B,t,s) &= -\frac{1}{4\pi} \int d^3r d^3r' e^{i(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}')} \\ &\quad \times V(\mathbf{r},s)\phi_B(\mathbf{r})\phi_B(\mathbf{r}')V(\mathbf{r}',s). \end{aligned} \tag{50}$$

Note that only the last term in (49) contains any poles, and that, since $V(\mathbf{r},s)$ is real for $s < 0$, $R(s_B,t,s_B)$ is real.

The exponentials in (47) may be written

$$\mathbf{k}\cdot\mathbf{r} - \mathbf{k}'\cdot\mathbf{r}' = -\frac{1}{2}\Delta\cdot(\mathbf{r}+\mathbf{r}') + \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}') (s-\frac{1}{4}t)^{1/2},$$

where $\Delta = \mathbf{k}' - \mathbf{k}$, $\mathbf{n} = (\mathbf{k}' + \mathbf{k})/|\mathbf{k}' + \mathbf{k}|^{-1}$. For t fixed and real, Δ is fixed and has real magnitude. Because the potential vanishes like e^{-m_0r} , the integral in (47) converges uniformly in k for real k under the stated condition on t : $0 > t > -4m_0^2$. Further, the integrand in (47) is analytic in the upper half k plane, except for the bound-state poles on the imaginary axis, which have been separated off in (49). Define the integrands

of (49) through

$$F(k,t) = \int d^3r d^3r' \langle \mathbf{r}' | T(k,t) | \mathbf{r} \rangle \tag{51}$$

$$R(s_B,t,s_B) = \int d^3r d^3r' \langle \mathbf{r}' | R(s_B,t,s_B) | \mathbf{r} \rangle.$$

We can write the following dispersion relation:

$$\begin{aligned} \frac{1}{\pi i} \int_{\Gamma} k' dk' \frac{\langle \mathbf{r}' | T(k',t) | \mathbf{r} \rangle}{k'^2 - k^2} &= \langle \mathbf{r}' | T(k,t) | \mathbf{r} \rangle \\ &\quad + \sum_B \frac{\langle \mathbf{r}' | R(s_B,t,s_B) | \mathbf{r} \rangle}{s_B - s}, \end{aligned} \tag{52}$$

where the contour Γ includes the real k axis and the semicircle at infinity in the upper half plane. If now we integrate (52) over \mathbf{r}, \mathbf{r}' we would have a dispersion relation for the scattering amplitude, provided that we can exchange the order of integration. Let us suppose that this is legitimate for the moment. Then

$$F(k,t) = \sum_B \frac{R(s_B,t,s_B)}{s_B - s} + \frac{1}{\pi i} \int_{\Gamma} k' dk' \frac{F(k',t)}{k'^2 - k^2}. \tag{53}$$

If the Green's function behaves well enough, the only contribution from the semicircle at infinity is $V_0(t)$. The energy-dependent part of the potential has dropped out of sight, but this is no handicap to writing the dispersion relations.

Finally, by using the relation (48), we come to the usual dispersion relation

$$\begin{aligned} F(k,t) \equiv f(s,t) &= \sum_B \frac{R(s_B,t,s_B)}{s_B - s} + V_0(t) \\ &\quad + \frac{1}{\pi} \int ds' \frac{\text{Im}f(s',t)}{s' - s} \end{aligned} \tag{54}$$

of which the imaginary part is an identity, since V_0 and R are real. The mathematical details which we have passed over here, such as interchanging orders of integration and treating the semicircle at infinity, proceed exactly as in the Klein-Zemach paper,¹⁰ because the energy-dependent part of the potential vanishes as $|s| \rightarrow \infty$, and is not repeated here.

D. The Momentum Transfer Dispersion Relations

Because the potential (45) is a combination of Yukawa potentials, one can prove that $f(s,t)$ is analytic in the finite t plane for fixed s , with the exception of poles and branch cuts on the real t axis, whose location is determined by the range of the potential.¹⁻⁵ Blankenbecler, Goldberger, Khuri, and Treiman¹ have proved analyticity in t with the Fredholm method,⁸ while Klein² has given a proof based on a study of the Born

series for $f(s,t)$. These methods do not allow one to discuss the behavior of $f(s,t)$ as $t \rightarrow \infty$. Regge and co-workers³⁻⁵ have been able to show that there are no essential singularities at infinity in the t plane for a wide class of potentials, by using the very interesting method of complex angular momenta. These proofs are not repeated here (the Regge work will be discussed in Sec. V); we only point out that it is evident that the dependence of the potential on s cannot effect the analyticity of $f(s,t)$ in t . The final result is that one comes to a double-dispersion formula for $f(s,t)$;

$$f(s,t) = V_0(t) + \sum_B \frac{g_B(t)}{s_B - s} + \frac{1}{\pi^2} \int ds' dt' \frac{\rho(s',t')}{(s'-s)(t'-t)}. \quad (55)$$

We do not consider the question of subtractions. The residues $g_B(t)$ are polynomials in t of deg l_B , where l_B is the angular momentum of the bound state. Again, notice that the potential $V_1(s,t)$ is not explicitly exhibited, but, by (46), $V_1(s,t)$ can be expressed as a double integral of the type occurring in (55) and hence may be explicitly exhibited if desired.

IV. COMPLEX POTENTIALS AND UNITARITY

In the absence of inelastic scattering the potential to be used in the Schrödinger equation and that derived from field theory or from the Mandelstam representation are identical. Inelasticity and the consequent use of complex potentials gives rise to differences in the definition of the potential which depend upon its use. The relationship of the imaginary part of the potential to the unitarity condition is significantly different for the Schrödinger equation potential and that suggested by Chew and Frautschi⁷ from the point of view of the Mandelstam representation. For the Schrödinger equation we insert the exact eigenfunction $\varphi(\mathbf{r})$ which satisfies

$$\varphi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + r^{-1} e^{i\mathbf{k}r} f(s,t) \quad (56)$$

into Eq. (10) and integrate over all space to find

$$\text{Im}f(s,0) = k(4\pi)^{-1} \int d\Omega' |f(s,t')|^2 - \frac{1}{4\pi} \int d\mathbf{r} \text{Im}V(s,\mathbf{r}) |\varphi(\mathbf{r})|^2. \quad (57)$$

The second term on the right involves the exact wave function $\varphi(\mathbf{r})$ over all space. The field theoretic-dispersion relation potential gives rise to a different and simpler relation. There are two conventional ways of constructing such an effective potential: via the Low equation in field theory; and via the full Mandelstam representation.⁷ The two prescriptions give the same result.

We consider first the problem of defining the potential from the viewpoint of conventional field theory. Let

there be two types of point interactions (i.e., potentials) in the theory. One is, say, a four-point elastic potential V^{el} . Also there are interactions which are inelastic, i.e., vertices which involve particles other than those in the elastic channel. Such vertices we denote by V^{in} . The elastic scattering process consists of all processes which have the same two final particles as initial particles. This includes, of course, other processes than those in which V^{el} alone occurs. We wish to decide on a modification of V^{el} which allows us to discuss the elastic problem as if there were no inelastic channels. To do this, we separate the full T matrix, which describes all channels, into four submatrices:

$$T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix},$$

where 1 refers to the elastic channel, 2 to the inelastic channels. Because of time-reversal invariance, T is symmetric in the indices 1, 2. We now wish to formulate our problem in terms of a new potential V' which is in general a complex, energy-dependent operator, and the elements T^{11} . To do so, we use the equation,

$$T = V + TG_0T^\dagger, \quad (58)$$

where $G_0 = (s - H_0 + i\epsilon)^{-1}$ and take matrix elements of this equation between elastic states:

$$T_{ab}{}^{11} = V_{ab}{}^{\text{el}} + \sum_c \frac{T_{ac}{}^{11}T_{cb}{}^{11\dagger}}{s - s_c + i\epsilon} + \sum_d \frac{T_{ad}{}^{12}T_{db}{}^{21\dagger}}{s - s_d + i\epsilon}. \quad (59)$$

It is evident that this equation is equivalent to a purely elastic Low equation with a modified potential:

$$T_{ac} = V'_{ab} + \sum_c \frac{T_{ac}{}^{11}T_{cb}{}^{11\dagger}}{s - s_c + i\epsilon}, \quad (60)$$

where

$$V'_{ab} = V_{ab}{}^{\text{el}} + \sum_d \frac{T_{ad}{}^{12}T_{db}{}^{21\dagger}}{s - s_d + i\epsilon}. \quad (61)$$

Since V^{el} is Hermitian,

$$V'_{ab} - V'_{ab}{}^\dagger = -2\pi i \sum_d \delta(s - s_d) T_{ad}{}^{12} T_{db}{}^{21\dagger} \equiv 2i \text{Im}V'_{ab}. \quad (62)$$

It is evident from (62) that $\text{Im}V'$ is a negative-definite Hermitian operator, and is therefore automatically absorptive.

It is the spirit of the S -matrix theory that only those relations which involve matrix elements on the physical shell are to be introduced. In that sense, Eqs. (58)–(61) are suspect. But (62), which gives the imaginary part of the potential, is a legitimate relation. If one now assumes that the potential is an analytic function of the energy s on the basis of the causality arguments given above, one can express the entire potential V' in terms of on-the-physical-shell quantities only, by using the

Cauchy formula :

$$V'_{ab}(s) = V_{ab}{}^{el} + \frac{1}{\pi} \int ds' \frac{\text{Im} V'_{ab}(s')}{s' - s - i\epsilon}. \quad (63)$$

Now suppose that both V^{el} and $\text{Im}V(s)$ can be written as sums of Yukawas with weight functions $-\pi^{-1}\rho_0(m^2)$, $-\pi^{-2}\rho_1(s, m^2)$, respectively. Let state s have momentum \mathbf{k} , state b momentum \mathbf{k}' , with $k = k' = s^{\frac{1}{2}}$. Then we may rewrite (63) as

$$V'_{ab}(s) \equiv -V'(s, t) = -\frac{1}{\pi} \int dt' \frac{\rho_0(t')}{t' - t} - \frac{1}{\pi^2} \int ds' dt' \frac{\rho_1(s', t')}{(s' - s)(t' - t)}. \quad (64)$$

For physical s ,

$$\text{Im}V'(s, t) = +\frac{1}{\pi} \int dt' \frac{\rho_1(s, t')}{t' - t}. \quad (65)$$

Chew and Frautschi⁷ have given a definition of a generalized potential based on the Mandelstam representation. Let the elastic scattering channel in which we are interested by that in which s is the energy variable, and t the momentum transfer. The t channel contributes the direct forces, and the third channel gives the exchange forces. Since we ignore exchange forces, we assume all spectral functions referring to this third channel are zero. The Chew-Frautschi definition of the weight function for the direct potential is

$$V''(s, t) = D(s, t) - \frac{1}{\pi} \int ds' \frac{\rho^{el}(s', t)}{s' - s}, \quad (66)$$

where $D(s, t)$ is the absorptive part in the t channel, and $\rho^{el}(s, t)$ is the elastic double spectral function. We have

$$D(s, t) = \rho_0(t) + \frac{1}{\pi} \int ds' \frac{\rho(s', t)}{s' - s}, \quad (67)$$

where $\rho(s, t)$ is the full double spectral function. We may identify the weight function (66) with that in (64) by writing

$$\rho(s, t) = \rho^{el}(s, t) + \rho_1(s, t). \quad (68)$$

Evidently $\rho_1(s, t)$ is the inelastic contribution to the spectral function, and according to the Cutkosky rules¹⁶ $\rho_1(s, t)$ is given by a sum over graphs with inelastic intermediate states. But by using (62) and (65) one sees that the same identification is to be made in the field-theoretic case, and the Chew-Frautschi potential V'' is the same as the field-theoretic potential V' . Instead of the unitarity relation of Eq. (57) we have for

the potential $V' = V''$

$$\text{Im}f(s, 0) = k(4\pi)^{-1} \int d\Omega' |f(s, t')|^2 - \frac{1}{4\pi} \int d\mathbf{r} \text{Im}V'(s, r) \quad (69)$$

Here the right side does not involve a knowledge of the exact wave function everywhere and Eq. (69) together with the Mandelstam representation determines f by the same procedures as are applicable for the real potential. When only V but not V' is given, Mandelstam¹⁷ has shown that f may still be found by an analogous but altered algorithm.

The magnitude of the potential $\text{Im}V'$ cannot be specified arbitrarily without violating unitarity for the scattering amplitude $f(s, t)$. It depends upon an exact detailed knowledge of the inelastic amplitudes. Except for the conditions (a)-(c) there is no such restriction on the magnitude of $\text{Im}V$.

V. REGGE POLES WITH ABSORPTIVE POTENTIALS

We begin with a brief review of known results. Partial-wave amplitudes are defined by

$$f_l(s) = \frac{1}{2} \int_{-1}^1 dx P_l(x) f_l[s, -2s(1-x)], \quad (70)$$

where l is integral. Regge and co-workers³⁻⁵ have studied the question of defining a function $f(l, s)$ which is analytic in l except for poles, and which coincides with $f_l(s)$ for integral l . The Regge results allow one to replace the partial-wave expansion

$$f(s, t) = \sum_l (2l+1) f_l(s) P_l[1 + (t/2s)] \quad (71)$$

by an integral,

$$f(s, t) = -\frac{1}{2}i \int_C \frac{f(l, s)}{\sin\pi l} P_l\left(-1 - \frac{t}{2s}\right) (2l+1) dl, \quad (72)$$

where the contour C encloses all the poles of $\sin\pi l$ on the positive real l axis. If now $f(l, s)$ has only poles in the plane $\text{Re}l > -1/2$, the contour may be unfolded to lie along the line $\text{Re}l = -1/2$, plus a semicircle at infinity (which we assume gives no contribution). In the process of unfolding, the poles of $f(l, s)$ can be treated explicitly:

$$f(s, t) = -\frac{1}{2}i \int_{i\infty-\frac{1}{2}}^{i\infty-\frac{1}{2}} \frac{f(l, s)}{\sin\pi l} P_l - \left(1 - \frac{t}{2s}\right) (2l+1) dl + \sum_n R_n P_{l_n} \left(-1 - \frac{t}{2s}\right) \frac{(2l_n+1)}{\sin\pi l_n}, \quad (73)$$

where R_n are the residues of the poles l_n of $f(l, s)$ in $\text{Re}l > -1/2$. The integral vanishes as $|t| \rightarrow \infty$, and

¹⁷ S. Mandelstam (private communication to G. F. Chew).

¹⁶ R. E. Cutkosky, Phys. Rev. Letters 4, 624 (1960); J. Math. Phys. 1, 429 (1960).

one can therefore deal with the behavior of $f(s,t)$ as $|t| \rightarrow \infty$ by knowing something about the locations of the poles, since $P_{l_n}[-1-(t/2s)] \sim t^{\text{Re}l_n}$ for large t , fixed s .

Let us suppose for the moment that $f(l,s)$ has the following properties: $f(l,s)$ is analytic in $\text{Im}s \neq 0$ for all real l with $\text{Re}l > -1/2$; for all s , $f(l,s)$ is analytic in $\text{Re}l > -1/2$, except for a finite number of simple poles. [The affixes of the poles in l move with s ; there is some relation $l=l_n(s)$, where $l_n(s)$ is analytic in s , which gives the affixes of the poles.] Also assume that $f(l,s) \rightarrow 0$ along any ray in $\text{Re}l > -1/2$. Then the Mandelstam representation can be proved from (73) for potentials of the type $V_0(r)$ in (45).⁵ The essence of the proof (which we shall not give in detail here) is that the integral in (73) converges and defines an analytic function of $\cos\theta = 1+t/2s$ with the exception at most of the cut $\text{Re}\theta = 0$, or $\cos\theta$ real and > 1 . This assertion follows from the asymptotic behavior

$$P_l(-\cos\theta)(\sin\pi l)^{-1} \sim e^{-\text{Re}\theta|\text{Im}l|}$$

as $|\text{Im}l| \rightarrow \infty$ with $\text{Re}l = -1/2$. On the other hand, $f(s,t)$ is known to be analytic in the Lehmann ellipse,¹ which intersects the real $\cos\theta$ axis at $\cos\theta = 1+m_0^2/2s$, and hence $f(s,t)$ has a cut in

$$1+m_0^2/2s < \cos\theta < \infty$$

or

$$m_0^2 < t < \infty.$$

The bound-state poles occur when $l_n(s_B) = m_B$, m_B a non-negative integer, for some $s_B < 0$. The bound-state poles may be explicitly exhibited, and yield

$$\sum \frac{R_n(s_B) P_{m_B}(1+t/2s_B)(2m_B+1)}{l_n'(s_B)(s-s_B)},$$

where

$$l_n'(s_B) = \left. \frac{d}{ds} l_n(s) \right|_{s=s_B}.$$

We study the domain of analyticity of $f(l,s)$ by the Green's function method. This domain has already been deduced by Regge,³⁻⁵ and we have nothing new to say about real energy-independent potentials (although the Green's function method may give an intuitively clearer picture than the original ground-breaking Regge work). The motivation of the present work is twofold: first to put the analyticity in l of $f(l,s)$ on the same basis as the analyticity in s , and secondly, to provide a transparent means of studying the qualitative influence of complex energy-dependent potentials on Regge pole trajectories.

We wish to study the Green's function of the operator

$$-\frac{d^2}{dr^2} + \frac{\alpha-1/4}{r^2} + V(r,s) - s \equiv H(s) - s + \alpha L \equiv A(\alpha,s), \quad (74)$$

with

$$H(s) = \frac{-d^2 - 1/4}{dr^2} + \frac{1}{r^2} + V(r,s), \quad L = \frac{1}{r^2}, \quad \alpha = (+1/2)^2.$$

The parameter α may now take on all complex values. There is an obvious analogy between this problem and the energy-dependent potential problem; in both cases, we are studying operators which depend on a parameter.

We denote the (outgoing) Green's function for (74) by

$$\langle r' | A^{-1}(\alpha,s) | r \rangle = \langle r' | G(\alpha,s) | r \rangle = G(r',r; \alpha,s). \quad (75)$$

The Green's function is a function of $l = \alpha^{1/2} - 1/2$, just as it is a function of $k = s^{1/2}$, so we also write $G(r,r'; l,k)$ for the Green's function. Those values of l, k for which the Green's operator is bounded from the resolvent set S . First, we write down the scattering amplitude $f(l,s) \equiv F(l,k)$ in terms of $G(r,r'; l,k)$, and show that the only singularities of $F(l,k)$ come from the singularities of the Green's function, if l, k are constrained to the region $\text{Re}l > -1/2, \text{Re}k \neq 0$. Then we study the singularities of the Green's function.

The partial-wave amplitude is

$$\begin{aligned} f(l,s) &\equiv F(l,k) \\ &= - \int dr r^2 j_l^2(kr) V(r,s) - \int dr dr' r r' j_l(kr) V(r,s) \\ &\quad \times G(r,r'; l,k) V(r',s) j_l(kr'). \end{aligned} \quad (76)$$

For potentials of the type (45), the first integral in (76) becomes

$$\begin{aligned} -V(l,s) &= -(2s)^{-1} \left\{ \int_{m_0^2} dm^2 \rho_0(m^2) Q_l(1+m^2/2s) \right. \\ &\quad \left. + \int dm^2 ds' \frac{\rho_1(s',m^2)}{s'-s} Q_l(1+m^2/2s) \right\} \end{aligned} \quad (77)$$

where the Q_l 's are Legendre functions of the second kind.

As a function of l , $V(l,s)$ is regular in $\text{Re}l > -1/2$, and has poles at $l = -1, -2, \dots$. These poles are not of interest to us. As a function of s , there is the usual left-hand branch cut beginning at $s = -m_0^2/4$ for integral l ; otherwise, there is a branch point in s at the origin. To study the analytic properties of the second integral in (76), the following facts about the $j_l(kr)$ are useful: The j_l 's are entire functions of l , with branch points in the argument kr at $kr = 0$ for nonintegral l . For $\text{Re}l > -1/2$ ¹⁸

$$|j_l(z)| \leq C |z|^{\text{Re}l|\text{Im}z|}. \quad (78)$$

(Better bounds than this exist, but we shall not need them.) If l, k are in S , the Green's function is bounded, which means that we can replace the Green's function

¹⁸ G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1952), 2nd ed., p. 49.

in (76) by a constant, for purposes of determining whether or not the integral converges. Since the potential behaves like $e^{-m_0 r}$, the inequality (78) implies that the integrals in (76) formally converge only for $|\text{Im}k| < m_0$, or $\text{Res} > -m_0^2$. We can, however, use a modification of a trick of Bottino, Longoni, and Regge⁵ to define (76) outside this range. Suppose that, in (74), we let r be complex. The potential $V(r,s)$ is analytic in r and vanishes exponentially at infinity if $\text{Re}r > 0$. By Fuch's theorem,¹⁹ the solutions to (74) are regular in $\text{Re}r > 0$, and hence the Green's function $G(r,r';l,k)$ is analytic in $\text{Re}r > 0$, $\text{Re}r' > 0$. The Bessel functions and the potential are likewise regular in this region; hence it follows that the paths of integration in (76) may be taken along any rays $r = |r|e^{iz}$, $r' = |r'|e^{iz'}$, with $|z|$, $|z'| < \frac{1}{2}\pi$, for those values of k for which the integrand vanishes sufficiently fast at infinity. From (78), one easily finds that permissible values of k satisfy

$$|\text{Im}ke^{iz}| \leq m_0 \cos z, \quad |z| < \frac{1}{2}\pi. \quad (79)$$

In particular, those k are allowed for which $\text{Im}ke^{iz} = 0$ which, as z ranges from $-\frac{1}{2}\pi + \epsilon$ to $\frac{1}{2}\pi - \epsilon$ includes the entire k plane except for the imaginary axis. One also finds that (79) is satisfied with $k = \pm i|k|$, if $|k| < m_0$. Let the set D contain the points $\text{Re}l > -1/2$, and the entire k plane cut from im_0 to ∞ , and from $-im_0$ to $-\infty$. We conclude that the integrals in (76) converge uniformly in k, l and absolutely in the intersection of D with the resolvent set S . If l is integral, and l, k are in S , it follows that the second integral in (76) defines an analytic function of s in the s plane cut from $-m_0^2$ to $-\infty$. This shows, incidentally, that the entire discontinuity on the left-hand cut in the region $-m_0^2 < s < -\frac{1}{4}m_0^2$ is given by the first Born approximation. Of course, if l is nonintegral, there is a branch point at the origin of s , coming from the Bessel functions.

We have proved that the only singularities in $f(l,s)$ for l, s in D are those arising from the singularities of the Green's function, aside from the branch points in k for nonintegral l . Therefore, what remains is to study the singularities of the Green's function. As was the case for the complex potential, we may assume without loss of generality that both L and $w(s)$ are bounded, that is, there exist numbers Λ, w_s such that

$$\begin{aligned} |(g, Lg)| &\leq \Lambda \|g\|^2, \\ |(g, w(s)g)| &\leq w_s \|g\|^2, \end{aligned} \quad (80)$$

for any vector g . It follows from (76) and (78) that $L, w(s)$ are in fact bounded for those vectors g needed to construct the scattering amplitude.

We may then write

$$\begin{aligned} (g, Lg) &= \Lambda(g) \|g\|^2, \\ (g, w(s)g) &= w_s(g) \|g\|^2, \end{aligned} \quad (81)$$

¹⁹ E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford University Press, New York, 1935), p. 234.

with

$$\begin{aligned} 0 < \Lambda(g) &\leq \Lambda, \\ 0 \leq w_s(g) &\leq w_s, \quad \text{Im} s < 0; \\ 0 \geq w_s(g) &\geq -w_s, \quad \text{Im} s > 0. \end{aligned} \quad (82)$$

The last line is an expression of the absorptivity condition. We may now carry out the steps which led to (39) and find

$$\|G(\alpha, s)g\| \leq \|g\| |\text{Im} s - \text{Im} \alpha \Lambda(g) - w_s(g)|^{-1}. \quad (83)$$

Now the term $\text{Im} s - w_s(g)$ always has the sign of $\text{Im} s$, by (82). Because $\Lambda(g) > 0$, we come to the conclusion that $G(\alpha, s)$ is bounded in

$$\text{Im} s > 0, \quad \text{Im} \alpha > 0; \quad \text{and} \quad \text{Im} s < 0, \quad \text{Im} \alpha < 0. \quad (84)$$

We may then conclude, as in the case of the energy-dependent potential, that $G(r, r'; \alpha, s)$ is an analytic function of both α and s in the region given by (84). This is not the largest region of analyticity in (α, s) ; the domain of holomorphy can be found by a process of analytic completion under certain circumstances,⁵ e.g., the highest angular momentum for which there is a bound state is known. The domain of holomorphy is, in general, the largest region in which $f(l, s)$ is regular. This domain is exactly that which is found for real energy-independent potentials,⁵ which fact depends in an essential manner on the absorptivity condition. If $w_s(g)$ were to have signs opposite to those stated in (82), no general conclusion could be drawn at all.

It is interesting to specialize the parameters in (83). Let us take s real, $s < s_0$, so that $\text{Im} s, w_s(g)$ vanish identically. Then, from (83), $G(\alpha, s)$ is bounded and analytic whenever $\text{Im} \alpha \neq 0$, that is, everywhere in the α plane except for the real α axis. This confirms the results of Regge^{3,4} that $f(l, s)$ can have poles in the l plane for real s since the entire l plane corresponds to two sheets of the α plane, and $G(\alpha, s)$ may well be analytic on one sheet and have poles on the other. Analyticity on a full sheet of the α plane implies that there are no singularities in one or the other half planes in the l plane. As mentioned before, $G(\alpha, s)$ is really a function of $\alpha^{1/2}$, and should therefore have a branch cut along the positive α axis. The unboundedness of $G(\alpha, s)$ along the real axis simply reflects the existence of this cut. The poles for real s are physically connected with resonances^{3,4} and should lie in the upper half l plane. Therefore, we conclude that the lower half l plane (at least for $\text{Re}l > -1/2$) is free of singularities, for real $s < s_0$.

Now let us suppose that s is real, but $s > s_0$. The denominator of (83) becomes

$$|\text{Im} \alpha \Lambda(g) + w_s(g)| = |2 \text{Re}(l+1/2) \text{Im} l \Lambda(g) + w_s(g)|.$$

We have $\text{Re}(l+1/2) > 0$, and $w_s(g) \leq 0$, $\Lambda(g) > 0$, so this denominator can vanish only for $\text{Im} l > 0$ —exactly as for the real potential. It is interesting to notice that, if one crosses the inelastic branch cut, so that $w_s(g)$

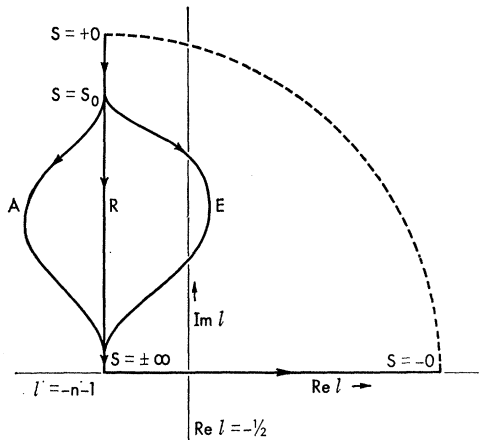


FIG. 1. Regge trajectories for emissive (E), absorptive (A), and real (R) Coulomb potentials. s_0 is the threshold for inelastic processes.

changes sign, singularities may occur for $\text{Im}l > 0$, $\text{Re}l < -1/2$.

So far, the qualitative behavior of Regge poles is the same for energy-dependent absorptive potentials as for real potentials. There is one further aspect which is worthy of note, and which points out the significance of the absorptivity condition. We study the Regge trajectories for real s with the soluble example of an energy-dependent absorptive Coulomb potential.

The Coulomb potential is perhaps not such a good illustration, because of its long range, but it is exactly soluble. Singh¹¹ and Ohme¹² have studied the Regge trajectories for real Coulomb potentials.

First we must understand the presumed path of pole trajectories for real potentials.^{3,4} In the l plane, the pole begins on the negative real axis at $s = -\infty$, and moves to the right as s increases, staying on the real axis for $s < 0$. If there are bound states, the trajectory stays on the real axis for a while in the physical region ($\text{Re}l \geq 0$), but, when s becomes positive, the trajectory curves into $\text{Im}l > 0$. Finally, as s increases, the trajectory crosses the line $\text{Re}l = -1/2$ again at some positive value of $\text{Im}l$.

Let us consider the potential

$$V(r,s) = g(s)/r, \quad (85)$$

where $g(s)$ is analytic in the s plane cut from s_0 to ∞ . the well-known solution to the Coulomb problem²⁰ allows us to write the S matrix easily:

$$e^{2i\delta} = \Gamma(l+1+ig(s)(2k)^{-1})/\Gamma(l+1-ig(s)(2k)^{-1}). \quad (86)$$

Now $\Gamma(z)$ is analytic for all z except for poles at $z=0, -1, -2, \dots$ so the poles of the S matrix occur at

$$l = -n-1-ig(s)(2k)^{-1}; \quad n=0, 1, 2, \dots \quad (87)$$

since $\Gamma(z)$ never vanishes. Let us suppose $\text{Re}g(s) < 0$, $s < s_0$. (This is automatically true for the energy-dependent part of absorptive potentials.) Then for all negative s ($k=i|k|$), the poles begin at $l = -n-1$, and stay on the real l axis, giving an infinite number of bound states for $l=0, 1, 2, \dots$, and leaving the real axis at $l = +\infty$ for $s = -0$. It then jumps discontinuously, so that at $s = +0$, $l = -n-1+i$. As s increases, the trajectory can behave three different ways, depending on whether $\text{Im}g(s)$ is less than, greater than, or equal to zero. Figure 1 shows the possible paths. The path labeled R corresponds to $g(s)$ real and constant; the path A corresponds to an absorptive potential [$\text{Im}g(s) \leq 0$]; and the path E corresponds to an emissive potential [$\text{Im}g(s) \geq 0$]. The path E may intrude into the physical region; absorptive potentials, however, display no such unphysical behavior.

In summary, the qualitative behavior of Regge poles for complex absorptive potentials in the physical regions of l and s is perfectly acceptable, while emissive potentials may behave anomalously. This is to be expected, in view of the fact that a Mandelstam representation exists for absorptive potentials, but not necessarily for emissive potentials. It is possible that absorptive potentials may give rise to additional singularities in unphysical regions, e.g., $\text{Re}l < -1/2$, or on the unphysical s sheet.

ACKNOWLEDGMENTS

We are grateful for helpful discussions with Professor C. Zemach. We would also like to thank Professor G. F. Chew for allowing us to see some unpublished work of S. Mandelstam.

²⁰ See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, New York, 1955), 2nd ed., p. 119.