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Radiative Corrections to the Total Cross Section for Annihilation of a Pair into Photons

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We calculate the total cross section to order e^6 for annihilation of an electron-positron pair into photons. Comparison with experiments at CERN is discussed and some consideration about the behavior of cross section for electrodynamical processes at high energy is outlined. Some details of calculation are sketched in the Appendix.

1. INTRODUCTION

IN the last ten years, since the introduction of the renormalization techniques in the S-matrix theory, calculations of scattering cross sections for electrodynamical processes to orders higher than e^4 have been carried out by many authors. As is well known, in these calculations some trouble can be given by the zero mass of the photon that causes the appearance of the so-called infrared divergences. In order to remove them, the consideration is made that a scattering process (in a broad sense) can never be considered as purely elastic. In fact, there is always an inelastic contribution due to the emission of some supplementary real photons of low energy (a single photon in the e^6 approximation considered). From an experimental point of view these photons are not detected if their energy is less than a certain critical value ΔE, the resolving power of the experiment. However, this collateral process must not be omitted, even if not experimentally distinguishable from the elastic process. So we add to the virtual photon corrections the cross section of the process corresponding to the additive emission of a real photon whose energy is under the threshold of detection ΔE. This makes possible the complete elimination of the infrared divergences.

Usually, the procedure followed in the calculation is to consider the inelastic contribution to a scattering process from real soft photons, with energy less than a given quantity $m\Delta \ll m$ (in a particular reference frame). This condition implies an isotropic photon emission, leading to a great simplification in the work. But, while this statement is sufficient to avoid the infrared divergence, it is quite unrealistic. In fact, we cannot arrange an experiment where only soft photons contribute to the inelastic part because the resolving power ΔE of the experimental device does not fulfill the condition $\Delta E \ll m$ and generally has an angular dependence.

In this way, in order to compare the theoretical calculations with the experimental results one ought to have taken into account the contribution of the real photons (ideally not detected) whose energies range from $m\Delta \ll m$ to ΔE. These are usually called "hard" photons. A very instructive example on the way to perform these calculations, concerning soft and hard real photons, has been given by Tsai for some electron-electron scattering experiments arranged at Stanford.[1] A result independent of any experimental arrangement would be achieved, in that case, by allowing the photons' energies to reach the maximum value given by the conservation laws. This leads to the differential cross section for e^-e^- scattering with and without emission of a photon.

A further step can be represented by the knowledge of the integrated correction, which eliminates problems of angular dependence. This is not possible for processes like e^-e^- and e^+e^- scattering owing to the long range of the Coulomb field, but it can be accomplished for Compton scattering and for annihilation of the e^+e^- pair into a different pair of particles, for instance a $\mu^+\mu^-$ pair or two quanta (or more). In the present paper we are concerned with the calculation of the corrections to the annihilation into two photons due to the virtual photons and to the annihilation into three photons, whose energy is not limited by experimental considerations.

It is necessary to underline the precise meaning of the total correction we have calculated. By "radiative corrections" to a process one usually means the higher-order contributions due to the virtual photons inserted in the "skeleton" graphs of the process. Moreover we must add, owing to the infrared divergence, the cross section for emission of another real photon. If the range of the energy of the supplementary photon attains its maximum value, the inelastic part contribution becomes very large, of the same order of the radiative corrections or greater, as we will discuss in the final part. In so doing the usual distinction between the radiative corrections due to virtual photons and the related inelastic part is lost. In the following we will mean by total correction the sum of these two contributions (it is a radiative correction in a broader sense). In

[1] Yung Su Tsai, Phys. Rev. 120, 269 (1960).

the same frame of ideas, a corresponding experiment of this type is a bad measure of the radiative corrections (in the usual meaning) because there is no sharp distinction between final states with two or three photons.

Our work permits a knowledge of the high-energy behavior of the over-all radiative correction (in the broadest sense), integrated over all the variables, to the total cross section for the process considered. In other words, we would like to study what are the suggestions that an e^6 perturbative calculation offers on the asymptotic behavior of the total cross section for the e^+e^- annihilation into photons and possibly for other electrodynamical processes (where a total cross section can be defined). Practically it is to determine the power, first or second, of $\ln(E/m)$ in the expansion in α of the correction. In addition, our calculation allows, in the laboratory system energy range from 1 up to 10 BeV, a comparison between the e^6 predictions of the standard theory and the results of the high-energy experiment of pair annihilation, recently performed at CERN.[2]

This process has the advantage of being very favorable for a lowering of the limits of validity of quantum electrodynamics (Q.E.D.) at small distances, as pointed out by Andreassi, Budini, and Reina[3]; furthermore, it is free from any natural breakdown[4] of Q.E.D., as long as one does not take into account graphs of higher orders in α (e.g., to order e^8 we could have vacuum-polarization loops of strongly interacting particles). Another process whose final products are three photons is, to order e^6, $e^+e^- \to \pi^0\gamma$.[5,6] In the laboratory system the threshold of the process is about 70 GeV.

We explain now the scheme of the calculation: We start with the differential cross section for three photons annihilation of a free pair and perform the fivefold integration over the five independent variables.[7] This expression diverges in the limit of low energy of the photons. To avoid it, we assume a small rest mass Λ for the photon, and the final result has the expression

$$\sigma(\gamma)=A_3(\gamma)+f_3(\gamma)\ln(\Lambda/m). \qquad (1.1)$$

[Note the relativistic invariance of Eq. (1.1) because of the invariant meaning of Λ.]

In order to avoid considerable trouble in the calculations, we have proceeded in the following way: We have divided the emitted photons into hard and soft, by denoting as "hard" or "soft" the photons with energies larger or smaller than $m\Delta$ (in the laboratory system). We then get

$$\sigma_3(\gamma, >\Delta)=\Sigma_3(\gamma)+f_3\ln 2\Delta, \qquad (1.1')$$

$$\sigma_3(\gamma, <\Delta)=S_3(\gamma)+f_3\ln(\Lambda/2m\Delta). \qquad (1.1'')$$

The recombination of Eqs. (1.1′) and (1.1″) leads back to Eq. (1.1). Subsequently, we attribute the same photon mass Λ to the virtual photons for the radiative correction to a pair annihilation, leading to a divergence at low energies. The cross section for two quanta annihilation, corrected by virtual photons only, has the expression:

$$\sigma_2=S_2(\gamma)+f_2\ln(\Lambda/m). \qquad (1.2)$$

It is verified that $f_2=-f_3$, so that we can combine Eqs. (1.2) and (1.1″)

$$\sigma_2(\gamma,<\Delta)=\sigma_D+\Sigma_2(\gamma)-f_3\ln 2\Delta. \qquad (1.2')$$

σ_0 is the Born approximation result. In this way we are left with the Δ dependence only (in the laboratory system). Some physical meaning could be attributed to Eq. (1.1′) and (1.2′) separately (in the laboratory system). In order to obtain a Δ-independent, Lorentz-invariant expression, we have to add Eqs. (1.2′) and (1.1′), with the result

$$\sigma_{\text{tot}}(\gamma)=\sigma_2(\gamma,<\Delta)+\sigma_3(\gamma_1>\Delta)$$
$$=\sigma_D+\Sigma_2(\gamma)+\Sigma_3(\gamma). \qquad (1.3)$$

This final expression is physically meaningful and represents, as already discussed, the total cross section for annihilation of the e^+e^- pair into two and three photons, taking into account the virtual photons corrections up to order e^6. No meaning can be attributed to $\Sigma_2(\gamma)$ and $\Sigma_3(\gamma)$ separately.

2. TOTAL CROSS SECTION FOR THREE QUANTUM ANNIHILATION

The diagrams for this process, as is well known, are of the general type shown in Fig. 1, and the other five are obtained from this by taking all the other permutations of the three photons. We see that when the energy of one of the photons inserted in an external electron line of each graph is allowed to go to zero (which is possible for the kinematics: see Appendix I) the electron

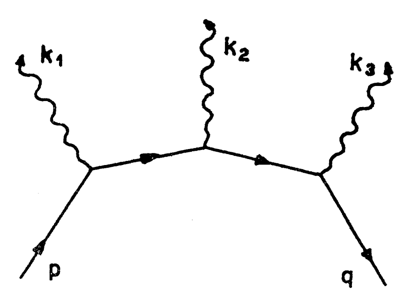

FIG. 1. Standard Feynman graph for three-quantum annihilation.

[2] F. Fabiani, M. Fidecaro, G. Finocchiaro, G. Giacomelli, D. Harting, N. H. Lipman, and G. Torelli (to be published).

[3] G. Andreassi, P. Budini, and I. Reina, Nuovo cimento 12, 488 (1959).

[4] We mean by natural breakdown the involvement of electromagnetically interacting particles other than photons, electrons, and μ mesons.

[5] G. Furlan, Nuovo cimento 19, 840 (1961).

[6] N. Cabibbo and R. Gatto, Phys. Rev. 124, 1577 (1961).

[7] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), pp. 272–273.

propagator originating (or terminating) in its vertex is divergent. This kind of divergence, the so-called infrared divergence, will be removed later. For the present we shall avoid it by forbidding the energies of all photons to reach values smaller than a fixed minimum $m\Delta$.

The cross section is[7]

$$\sigma=\frac{\alpha r_0^2}{16\pi^2}\int\frac{\delta^4(k_1+k_2+k_3-p-q)}{[(pq)^2-m^4]^{\frac{1}{2}}\omega_1\omega_2\omega_3}Xd^3k_1d^3k_2d^3k_3, \qquad (2.1)$$

where k_1, k_2, k_3; ω_1, ω_2, ω_3 are the four-momenta and the energies, respectively, of the photons, p and q are electron and positron four-momenta, respectively; X is the absolute square of the matrix element summed over the polarizations of the photons, and averaged over the electron and positron spins. (Note the complete symmetry in k_1, k_2, k_3.) The quantity X can be obtained from the trace for double Compton effect,[8] by applying the substitution law and changing the sign because of the replacement of the projection operator for the outgoing electron in the Compton case with the projection operator for the ingoing positron.

The result, can be expressed through the invariants defined by

$$\begin{aligned} m^2x_1&=-(pk_1), & m^2x_1'&=-(qk_1),\\ m^2x_2&=-(pk_2), & m^2x_2'&=-(qk_2), \qquad (2.2)\\ m^2x_3&=-(pk_3), & m^2x_3'&=-(qk_3),\end{aligned}$$

and reads

$$\begin{aligned}-X=2(ab-c)[(a+b)(x+2)-(ab-c)-8]\\ -2x(a^2+b^2)-8c+4(x/AB)\{(A+B)(x+1)\\ -(aA+bB)[2+z(1-x)/x]+x^2(1-z)+2z\}\\ -2\rho[ab+c(1-x)], \qquad (2.3)\end{aligned}$$

where

$$\begin{aligned} a&=\textstyle\sum_i(1/x_i), & b&=\textstyle\sum_i(1/x_i'), & c&=\textstyle\sum_i(1/x_ix_i'),\\ x&=\textstyle\sum_i x_i, & & & z&=\textstyle\sum_i x_ix_i',\\ A&=x_1x_2x_3, & B&=x_1'x_2'x_3', & \rho&=\textstyle\sum_i(x_i/x_i'+x_i'/x_i).\end{aligned}$$

By integrating over the δ function, we can reduce the differential

$$d^3k_1d^3k_2d^3k_3$$

to

$$d^3k_1d\Omega_2[\omega_2^2d\omega_2/d(\omega_1+\omega_2+\omega_3)],$$

where $d\Omega_2$ is the element of solid angle relative to photon k_2. Performing the differentiation and substituting in Eq. (2.1), we obtain

$$\sigma=\alpha r_0^2\int\frac{X\omega_1\omega_2d\omega_1}{m[(pq)^2-m^4]^{\frac{1}{2}}[\gamma_+(1-\beta_+\cos\theta_2)+\gamma_-(1-\beta_-\cos\theta_2')-(\omega_1/m)(1-\cos\theta_{12})}\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi} \qquad (2.4)$$

where $d\Omega_1$ is the solid angle of photon k_1. Thus, we are left with five variables of integration. Here, γ_+, γ_- are the energies of the positron and electron, respectively; β_+, β_- are the corresponding velocities.

For the angular part, we have

$$d\Omega_1d\Omega_2=\sin\theta_1d\theta_1d\varphi_1\sin\theta_2d\theta_2d\varphi_2,$$

where the angles refer to the polar system of coordinates in which the angles θ_i are the angles between the directions of the positron and the photon k_i, and the angle φ_i is the azimuth of the photon k_i, reckoned from the plane p,q. We have also introduced the angle θ_{ij} between the direction of the photons k_i and k_j. We will perform our calculation in the laboratory system in which

$$\beta_-=0,\quad \gamma_-=1,\quad \beta_+=\beta,\quad \gamma_+=\gamma,$$
$$x_i=\omega_i/m,\quad \textstyle\sum_i\omega_i=m(\gamma+1),$$

and the cross section simplifies to

$$\sigma=\alpha r_0^2\int\frac{X\omega_1\omega_2d\omega_1}{\beta\gamma m^3[1+\gamma(1-\beta\cos\theta_2)-(\omega_1/m)(1-\cos\theta_{12})]}\times\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}.$$

Here, of course, the expression for X is also simplified.

[8] F. Mandl and T. H. R. Skyrme, Proc. Roy. Soc. (London) **A215**, 497 (1952).

Solving for a, b, etc., in (2.3) we see that we have to integrate a very large number of terms. This number, however, can be considerably reduced with the following observations:

We are concerned with the total cross section. Now it is obvious that as long as only the total cross section is required, functions which are obtained by interchanging k_1, k_2, k_3, on account of the indistinguishability of the three photons, will yield, upon integration, the same contribution. For instance, it is clear that we can replace

$$\int\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)d(1,2,3) \quad\text{with}\quad \int\frac{3}{k_1^2}d(1,2,3).$$

In this way, choosing the simplest terms, we not only obtain a great reduction of terms, but can also avoid the most complicated integrations. We choose the following terms:

$$\phi_1=\frac{1}{x_1'x_2^{2'}},\qquad \phi_2=\frac{1}{x_1x_2x_2'},\qquad \phi_3=\frac{1}{x_1'x_2x_3},$$

$$\phi_4=\frac{1}{x_1'x_2x_2'},\qquad \phi_5=\frac{1}{x_1x_2'^2},\qquad \phi_6=\frac{1}{x_1x_2'x_3'},$$

$$\phi_7=\frac{1}{x_1x_2'},\qquad \phi_8=\frac{1}{x_1^2x_2'^2},\qquad \phi_9=\frac{1}{x_1x_1'x_2x_2'},$$

$$\phi_{10}=\frac{1}{x_1'x_2x_2'x_3},\quad \phi_{11}=\frac{1}{x_1x_2'^2x_3},\quad \phi_{12}=\frac{1}{f_1x_2x_2'},$$
$$\phi_{13}=\frac{1}{x_1'x_2^2x_3'},\quad \phi_{14}=\frac{1}{x_2x_2'},\quad \phi_{15}=\frac{1}{x_1'x_2'},$$
$$\phi_{16}=\frac{1}{f_2x_1x_2'},\quad \phi_{17}=\frac{1}{f_1x_2'x_3},\quad \phi_{18}=\frac{f_1}{x_1x_2'},$$
$$\phi_{19}=\frac{1}{x_1x_2},\quad \phi_{20}=\frac{x_3'}{x_3x_1x_2'},\quad \phi_{21}=\frac{1}{x_2^2},$$
$$\phi_{22}=\frac{f_1}{x_2x_2'},\quad \phi_{23}=\frac{1}{x_2'^2}.$$

The decomposition of ϕ's results in three types of integrals: W's, Z's, and V's. The integration of W's over θ_2 and $\varphi=\varphi_1-\varphi_2$ is straightforward. (For full understanding of this point we refer to Appendix I.) The result is

$$\sigma_3=\frac{\alpha r_0^2}{4\pi\beta\gamma}\{[(2x^2-x-2)\eta+2(x+2)]$$
$$\times 2\pi(W_1+W_2)-4\pi W_3-2\pi(x^3-3x+1)\eta W_4$$
$$+4\pi(x-1)\eta W_5-4\pi(\eta+2)W_6-2\pi x\eta(W_7+W_8)$$
$$-x(Z_1+Z_2)+8Z_3-2(x-1)Z_4+2(Z_5+Z_6)$$
$$+Z_8+Z_9+(x-3/2)(Z_7+Z_{10})-(x+2)(V_1+V_5)$$
$$-(2x^2-x-2)(V_2+V_4)+V_8-2(x-1)V_{10}$$
$$+[(x-1)^3+3(x-1)^2-1]V_9+2V_{14}+V_{21}+V_{23}$$
$$+x(V_{12}+V_{22})\},\quad (2.6)$$

where

$$x=\gamma+1,\quad \eta=\int_{-1}^{+1}\frac{1}{f_2}d\cos\theta_2=\frac{2}{\beta\gamma}\ln[\gamma(1+\beta)],$$

and

$$W_1=\int\frac{1}{f_1l}dy_2,\quad W_2=\int\frac{1}{l}dy_2,\quad W_3=\int\frac{1}{x_1l}dy_2,$$
$$W_4=\int\frac{1}{f_1x_1l}dy_2,\quad W_5=\int\frac{1}{f_1(x-x_1)l}dy_2,\quad W_6=\int\frac{x_1}{l}dy_2,$$
$$W_7=\int\frac{f_1x_1}{l}dy_2,\quad W_8=\int\frac{x_1}{f_1l}dy_2;$$

$$Z_1=\int\vartheta_0\frac{1}{f_1(x-x_1)}dy_2,\quad Z_2=\int\vartheta_1\frac{1}{x-f_1x_1}dy_2,\quad (2.7)$$
$$Z_3=\int\vartheta_0\frac{1}{f_1}dy_2,\quad Z_4=\int\vartheta_2\frac{1}{f_1(x-x_1)^2}dy_2,$$
$$Z_5=\int\vartheta_0\frac{1}{f_1^2x_1(x-x_1)}dy_2,\quad Z_6=\int\vartheta_1\frac{1}{x_1(x-f_1x_1)}dy_2,$$
$$Z_7=\int\vartheta_1\frac{1}{f_1}dy_2,\quad Z_8=\int\vartheta_0\frac{1}{f_1^2}dy_2,$$
$$Z_9=\int\vartheta_1f_1dy_2,\quad Z_{10}=\int\vartheta_0dy_2;$$

where

$$l=x\left\{1-x_1\left[1-\frac{\beta\gamma}{x}\cos\theta_1\right]\right\},$$
$$f_i=\gamma(1-\beta\cos\theta_i),$$
$$dy_2=dk_1d\cos\theta_1,$$

and

$$\vartheta_0=\int_{-1}^{+1}d\cos\theta_2\int_0^{2\pi}\frac{k_2}{l}d\varphi,\quad \vartheta_1=\int_{-1}^{+1}\frac{d\cos\theta_2}{f_2}\int_0^{2\pi}\frac{k_2}{l}d\varphi,\quad (2.8)$$
$$\vartheta_2=\int_{-1}^{+1}\frac{d\cos\vartheta_2}{f_2}\int_0^{2\pi}\frac{R_2(X-k_1)}{R_1+k_2-x}\frac{d\varphi}{l}.$$

θ_i ranges from 0 to π; k_1 ranges from Δ to $1/(p+r\cos\theta_1)$ for the W's and Z's, where

$$p=\frac{\gamma+1-2\Delta}{(1-2\Delta)(\gamma+1)+2\Delta^2},\quad r=-\frac{\beta\gamma}{(1-2\Delta)(\gamma+1)+2\Delta^2}.$$

The V's have not been given explicitly here because V_i is the (fivefold) integral of ϕ_i when k_1 takes up values greater than $1/(p+r\cos\theta_1)$, as is explained in Appendix I.

The other two integrations are lengthy but can be carried out exactly except for a few integrals in the last variable θ_i.

The Δ-dependent part of the result (which gives the logarithmic infrared divergence) comes only from completely integrated terms and can therefore be evaluated exactly.

The single contributions are (in the limit $\Delta\to 0$):

$$(W_1+W_2)_\Delta=\eta\ln 2\Delta,\quad (W_3)_\Delta=\frac{4}{\gamma+1}\ln\Delta,\quad (W_4)_\Delta=\frac{2\eta}{\gamma+1}\ln 2\Delta,$$
$$(W_5)_\Delta=\frac{2}{\gamma+1}\ln 2\Delta,\quad (W_6)_\Delta=\ln 2\Delta,\quad (W_7+W_8)_\Delta=[(\gamma+1)\eta-2]\ln 2\Delta,\quad (2.9)$$
$$(Z)_\Delta=-\frac{4\eta}{\gamma+1}\ln 2\Delta.$$

The total contribution to the Z integrals is given only by $\mathcal{I}_0$ and $\mathcal{I}_1$. There is no infrared contribution from the V's. We obtain

$$\sigma_3=\frac{\alpha r^2}{2\beta\gamma}\Bigg\{\frac{2}{\beta\gamma}\Bigg\langle\left(\gamma+\frac{1}{\gamma+1}\right)\frac{1}{\beta\gamma}\ln[\gamma(1+\beta)]\mathcal{L}_2\left(\frac{2\beta}{1+\beta}\right)-(\gamma+\tfrac{1}{2})\gamma(1-\beta)\mathcal{L}_2\left(-\frac{2\beta}{1-\beta}\right)$$
$$-\left[2(\gamma^2+\gamma-2)\frac{1}{\beta\gamma}\ln[\gamma(1+\beta)]+2\gamma^2+\gamma+\tfrac{4}{3}+\beta\gamma(2\gamma+1)\right]\mathcal{L}_2(-\gamma+\beta\gamma)+(2\gamma+1)\gamma(1+\beta)$$
$$\times\mathcal{L}_2(1-\gamma-\beta\gamma)-\left(\frac{\gamma^2}{2}-\frac{\gamma}{2}-1-\frac{1}{\gamma+1}\right)\frac{1}{\beta\gamma}\ln^3[\gamma(1+\beta)]+\frac{2}{\beta\gamma(\gamma+1)}(\gamma^3+3\gamma-1)\ln^2[\gamma(1+\beta)]$$
$$\times\ln[2(\gamma+1)]-\left[\frac{1}{\beta\gamma}\left(\frac{3}{2}\gamma^2+\frac{22}{3}\gamma+\frac{1}{2}+\frac{10}{3(\gamma+1)}\right)-\gamma+\tfrac{4}{3}-\frac{2}{\gamma+1}\right]\ln^2[\gamma(1+\beta)]$$
$$-\left(3\gamma+14/3-\frac{4}{\gamma+1}\right)\ln[\gamma(1+\beta)]\ln[2(\gamma+1)]+[(2\gamma+1)\gamma(1+\beta)+2\gamma+2/3]\ln[\gamma(1+\beta)]\ln[(\gamma-\beta\gamma+1)]$$
$$-\left[\gamma+6-\frac{4}{3(\gamma+1)}-\tfrac{4}{3}\pi^2+\frac{\pi^2}{6\beta\gamma}(\gamma^2+\gamma-2)\right]\ln[\gamma(1+\beta)]\Bigg\rangle+\left(2+\frac{4}{\gamma+1}\right)\ln 2[(\gamma+1)]$$
$$-\frac{\pi^2}{3}\left[\tfrac{5}{2}+2\frac{\gamma-1}{(\gamma+1)^2}+\frac{\gamma(1-\beta)}{\beta}+\frac{1}{2\beta}+\frac{2}{3\beta\gamma}\right]+\frac{16}{3(\gamma+1)}+2+\int_{-1}^{+1}F_1(\theta_1)d\cos\theta_1$$
$$+\frac{4}{\beta(\gamma+1)}\Bigg\langle-\left(\gamma+4+\frac{1}{\gamma}\right)\frac{1}{\beta}\ln^2[\gamma(1+\beta)]+\frac{2\gamma^2+7\gamma+1}{\gamma}\ln[\gamma(1+\beta)]-\beta(\gamma+3)\Bigg\rangle\ln 2\Delta\Bigg\}=\Sigma_3(\gamma)+f_3(\gamma)\ln 2\Delta \quad (2.10)$$

(terms going to zero in the limit $\Delta\to 0$ are omitted), where

$$F_1(\theta_1)=\frac{8}{a}\left\{\mathcal{L}_2\left[1-\frac{2(\gamma+1)a}{(1+a)^2}\right]-\mathcal{L}_2\left(1-\frac{\gamma+\beta\gamma+1}{1+a}\right)-\mathcal{L}_2\left(1-\frac{\gamma-\beta\gamma+1}{1+a}\right)\right\}$$
$$-\frac{1}{\beta\gamma}\left(\gamma+\frac{1}{\gamma+1}\right)\frac{1}{a}\left\{\mathcal{L}_2\left(\frac{2\beta\gamma}{\beta\gamma-\gamma+a}\right)-\mathcal{L}_2\left(\frac{2\beta\gamma a}{\gamma(1+\beta)a-1}\right)+(\ln a)[\ln(\beta\gamma-\gamma+a)-\ln(\beta\gamma+\gamma-a)]\right\},$$
$$a=\gamma(1-\beta\cos\theta_1)$$

(all the variables are evaluated in the laboratory system), and we define the Spence function $\mathcal{L}_2(x)$ as

$$\mathcal{L}_2(x)=-\int_0^x \ln|1-t|\frac{dt}{t},$$

Equation (2.10) in the nonrelativistic (NR) limit gives the result:

$$\sigma_3^{\mathrm{NR}}=\tfrac{4}{3}(\alpha r_0^2/\beta)(\pi^2-9), \quad (2.11)$$

in accord with that evaluated directly by Ore and Powell.[9] (Note the disappearance of the $\ln\Delta$ terms.)

Of greater interest, in view of future considerations, is the possibility of obtaining a closed expression for the extreme relativistic (ER) limit. If we neglect terms of the order $(1/\gamma^2)\ln^3(2\gamma)$ (or smaller) the cross section takes the form:

$$\sigma_3^{\mathrm{ER}}=\frac{\alpha r_0^2}{2\gamma}\{3\ln^3 2\gamma-7\ln^2 2\gamma+(4-\tfrac{1}{3}\pi^2)\ln 2\gamma-\pi^2+6\}$$
$$-\frac{\alpha r_0^2}{2\gamma}[\ln^2 2\gamma-2\ln 2\gamma+1]4\ln 2\Delta. \quad (2.12)$$

(In both the ER and NR limits the integral $\int_{-1}^{+1}F_1(\theta_1)\times d\cos\theta_1$ does not contribute.)

Formula (2.10) can be interpreted as the total cross section for annihilation of a pair into three photons, each of energy greater than $m\Delta$, when Δ can be neglected compared to 1. Moreover, Δ has to assume values that, however small, shall guarantee the emission of photons still really observable and not too soft.

[9] A. Ore and J. L. Powell, Phys. Rev. 75, 1696 (1949).

3. TOTAL CROSS SECTION FOR TWO-PHOTON ANNIHILATION TO ORDER e^6

We evaluate now the total cross section for two-quantum annihilation to order e^6. This process is described to this order by the Feynman graphs shown in Fig. 2. The corresponding differential cross section has been calculated by Brown and Harris.[10] It contains the physically meaningless photon mass Λ that is eliminated by adding to it the differential cross section for the annihilation of a pair into three photons, one of which is soft, and whose greatest energy is assumed to be $\omega_{\max} \ll m$ (i.e., $\Delta \ll 1$). Because this condition is not covariant, all the calculations are understood to be carried out in the laboratory system (for details we refer to the quoted article by Brown and Harris[10]).

We are interested in the total cross section. The integration of the differential cross section can be performed exactly except for two terms of the same type as the ones we have met in Eq. (2.10). The final result is

$$
\begin{aligned}
\sigma_2+\sigma_{3\,\mathrm{soft}}=\sigma_D+\frac{\alpha r_0^2}{2\beta\gamma}\biggl\{&\frac{1}{\beta\gamma}\biggl\langle 2\biggl[\biggl(3\gamma^2+7\gamma-6+\frac{4}{\gamma+1}\biggr)\frac{1}{\beta\gamma}\,[\ln\gamma(1+\beta)]-2\gamma+1+\frac{4}{\gamma+1}\biggr]\mathcal{L}_2(\gamma+\beta\gamma+1)\\
&+8\biggl[\biggl(\gamma+3-\frac{2}{\gamma+1}\biggr)\frac{1}{\beta}\ln[\gamma(1+\beta)]-\gamma-2+\frac{2}{\gamma+1}\biggr]\mathcal{L}_2(1-\gamma-\beta\gamma)+\frac{1}{2}\biggl[3(\gamma+1)-\frac{2}{\gamma+1}\\
&+\frac{1}{\beta\gamma}\biggl(3\gamma^2+11\gamma-6-\frac{8}{\gamma+1}\biggr)\biggr]\ln^3[\gamma(1+\beta)]-\frac{1}{\beta\gamma}\biggl[5\gamma^2+11\gamma-10+\frac{6}{\gamma+1}\biggr]\ln^2[\gamma(1+\beta)]\ln2[(\gamma+1)]\\
&+\frac{4}{\beta\gamma}\biggl[\gamma^2+2\gamma-2+\frac{1}{\gamma+1}\biggr]\ln^2[\gamma(1+\beta)]\ln[(\gamma+\beta\gamma+1)]-\frac{1}{2}\biggl[4\gamma+9-\frac{8}{\gamma+1}-\frac{1}{2\gamma+1}-\frac{1}{(2\gamma+1)^2}\\
&-\frac{1}{\beta\gamma}\biggl(5\gamma^2+13\gamma-2+\frac{8}{\gamma+1}\biggr)\biggr]\ln^2[\gamma(1+\beta)]+\biggl[2\gamma+10-\frac{10}{\gamma+1}-\frac{1}{2\gamma+1}-\frac{1}{(2\gamma+1)^2}\biggr]\ln[\gamma(1+\beta)]\\
&\times\ln[2(\gamma+1)]-\biggl[2\gamma-2-\frac{4}{\gamma+1}+\frac{1}{2\gamma+1}+\frac{1}{(2\gamma+1)^2}\biggr]\ln\gamma[(1+\beta)]\ln[(\gamma+\beta\gamma+1)]\\
&-\biggl[\frac{\pi^2}{2}\biggl(\gamma+1-\frac{1}{\beta\gamma}\biggl\{\gamma^2+5\gamma-2+\frac{4}{\gamma+1}\biggr\}\biggr)+7\gamma+\tfrac{3}{2}-\frac{3}{2(2\gamma+1)}\biggr]\ln[\gamma(1+\beta)]\biggr\rangle+\biggl(1-\frac{4}{\gamma+1}-\frac{1}{2\gamma+1}\biggr)\\
&\times\ln[2(\gamma+1)]+\frac{\pi^2}{2}\biggl[1+\frac{1}{\gamma+1}-\frac{1}{\beta\gamma}\biggl(2\gamma+9-\frac{4}{\gamma+1}\biggr)\biggr]-\frac{2(\gamma+3)}{\gamma+1}+\int_{-1}^{+1}F_2(\theta_1)d\cos\theta_1\\
&-\frac{4}{\beta(\gamma+1)}\biggl\langle-\biggl(\gamma+4+\frac{1}{\gamma}\biggr)\frac{1}{\beta}\ln^2[\gamma(1+\beta)]+\frac{2\gamma^2+7\gamma+1}{\gamma}\ln[\gamma(1+\beta)]-\beta(\gamma+3)\biggr\rangle\ln2\Delta\biggr\}\\
&=\sigma_D+\Sigma_2(\gamma)-f_3(\gamma)\ln2\Delta,
\end{aligned}
$$

where σ_D is the well-known Dirac cross section and

$$
F_2(\theta_1)=\biggl[2(\gamma+3)\frac{1}{1+a}-\biggl(4\gamma+6-\frac{3}{\gamma+1}\biggr)\frac{1}{a}\biggr]\times\mathcal{L}_2\biggl[1-\frac{2(\gamma+1)}{1+a}\biggr],
$$

$\sigma_{3\,\mathrm{soft}}$ consists of a divergent part (in $\ln2\Delta$) as well as a finite part; the latter includes the finite terms arising from integration over the energy of the soft photon ($0\leq\omega\leq m\Delta$), which has not been taken into account for σ_3 (see Appendix I). It reads (in the laboratory system):

$$
\sigma_{3\,(\mathrm{soft,finite})}=\sigma_D\frac{\alpha}{\pi}\biggl\{1-\frac{1}{\beta}\ln\gamma(1+\beta)+\frac{2}{\beta}\biggl[\ln[\gamma(1+\beta)]-\tfrac{1}{2}\ln^2[\gamma(1+\beta)]+\frac{\pi^2}{4}+\mathcal{L}_2(1-\gamma-\beta\gamma)-\mathcal{L}_2(1+\gamma+\beta\gamma)\biggr]\biggr\}.
$$

In the NR limit (3.1) takes the form

$$
\sigma_2^{\mathrm{NR}}=\sigma_D^{\mathrm{NR}}\biggl[1+\frac{\pi\alpha}{\beta}-\frac{\alpha}{\pi}\biggl(5-\frac{\pi^2}{4}\biggr)\biggr]. \tag{3.2}
$$

[10] I. Harris and L. M. Brown, Phys. Rev. **105**, 1656 (1957).

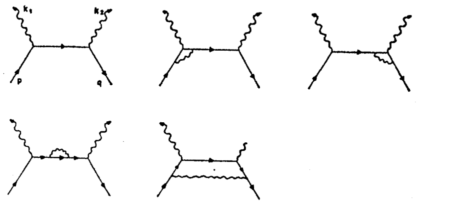

FIG. 2. Standard Feynman graphs for two-quantum annihilation to order e^4.

Also in this expression, ln2Δ terms do not contribute. In addition, there is a divergent term that following Brown and Harris[10] could be eliminated by taking exactly into account the Coulomb interaction at low energy.

With the same approximation as for σ_3 we derive the ER limit for σ_2; here, however, the situation is more complicated because now the integral of $F_2(\theta_1)$ is not zero in this limit (the explicit calculations are reported in Appendix II).

$$\sigma_2^{\mathrm{ER}}=\sigma_D+\frac{\alpha r_0^2}{2\gamma}\left\{-\frac{8}{3}\ln^3 2\gamma+\frac{13}{3}\ln^2 2\gamma+(\pi^2-6)\ln 2\gamma -\frac{\pi^2}{2}-2+4[\ln^2 2\gamma-2\ln 2\gamma+1]\ln 2\Delta\right\}. \quad (3.3)$$

We see that the divergent part in Eqs. (3.1) or (3.3) differs from the divergent part of Eqs. (2.10) or (2.12) only by the over-all sign. So the total contribution to order α^3 turns out to be independent of Δ.

4. TOTAL CORRECTION

Now we are in a position to give a quite general expression that leads to the total cross section for a pair annihilation into photons to the order e^6, i.e., into two and three photons. This is accomplished by adding $\sigma_2(e^6)$ and $\sigma_3(e^6)$. In fact, in this sum the Δ-dependent terms which are related in some way to the particular experimental procedure cancel each other.

We now wish to obtain the amount by which this sum exceeds the Dirac cross section σ_D. If we write

$$\sigma_{\mathrm{tot}}=\lim_{\Delta\to 0}(\sigma_D+\sigma_{2c}+\sigma_3)=\sigma_D(1+\delta),\quad \sigma_{2c}=\sigma_2(e^6)-\sigma_D,$$

the total correction reads

$$\delta=\lim_{\Delta\to 0}\frac{\sigma_2+\sigma_3}{\sigma_D}. \quad (4.1)$$

In the NR limit the correction is

$$\delta=\frac{\alpha\pi}{\beta}+\frac{\alpha}{\pi}\left(\frac{19}{12}\pi^2-17\right). \quad (4.2)$$

The $\pi\alpha/\beta$ divergent term could be avoided if we defined instead of the correction to σ_D, that to σ_c, i.e., to the cross section to e^4 order corrected for the Coulomb interaction between the initial particles at low momenta.[10]

The closed expression for the ER limit is very interesting:

$$\delta=\frac{\alpha}{12\pi}\left[2\ln^2 2\gamma-\ln 2\gamma+(4\pi^2-13)-\frac{5\pi^2-11}{\ln 2\gamma-1}\right], \quad (4.3)$$

where, as the ER limit for the Dirac cross section, we have taken

$$\sigma_D^{\mathrm{ER}}=\pi r_0^2(1/\gamma)(\ln 2\gamma-1).$$

TABLE I. Theoretical corrections for several values of the energy $\delta_T^{\mathrm{exact}}$ and δ_T^{ER} are total corrections; for G_2 and G_3 see Eqs. (5.1).

γ_L	$\gamma_{\mathrm{c.m.}}$	$\delta_T^{\mathrm{exact}}$	δ_T^{ER}	G_2	G_3
2	1.225	0.0219		0.0140	0.0079
20	3.24	−0.0011		−0.0185	0.0174
2×10^2	10	0.0136	0.0264	−0.0761	0.897
2×10^3	31.6	0.0286	0.0291	−0.1672	0.1958
2×10^4	10^2	0.0457	0.0457	−0.2916	0.3373
10^6	7.07×10^2	0.0832	0.0832	−0.5783	0.6615
10^8	7.07×10^3	0.1424	0.1424	−1.0377	1.1801

5. NUMERICAL RESULTS

We give now some numerical results and discuss them in the light of the recent experiment performed at Geneva.[2] In Table I we give the total correction for some values of the energy in a range from 1 MeV up to 5×10^4 BeV. More precisely, columns III and IV contain the values of δ_T calculated from the exact, complete formula (4.1) and from the ER formula (4.3) respectively. As is seen, the agreement is very good for high energies while the difference begins to be sensitive at $\simeq 10^2$ MeV. The most important terms left out in calculating Eq. (4.3) are of the type $n\propto(1/\gamma)\ln^2 2\gamma$. With $n\sim 1.4$ it is possible to lower the range of validity of our ER expression down to $\gamma\simeq 200$. Moreover we have considered in columns V and VI the quantities

$$G_2=\Sigma_2(\gamma)/\sigma_D,\quad G_3=\Sigma_3(\gamma)/\sigma_D. \quad (5.1)$$

Their values are purely indicative and physically meaningless because only σ_2 and σ_3 are observable [$\Sigma_2(\gamma)+\sigma_D$, $\Sigma_3(\gamma)$ may possibly coincide with σ_2 and σ_3 for the specific value $\Delta=\frac{1}{2}m$]. We want merely to show how the large compensation between G_3 (contribution of hard real photons) and G_2 (contribution of virtual and soft real photons) gives a reasonable value of the total correction. (It is to be pointed out again that G_2 and G_3, as well as σ_2 and σ_3, are not invariant quantities and the reported values refer to our laboratory system calculation.) We see that G_3 is always positive and increases with the energy more quickly than G_2, which is negative (but at very low energies), so that δ_T passes through zero (at about 12 MeV) becomes positive and increases with the energy as $\ln^2(2\gamma)$ ($\gamma=E/m$). To clarify this

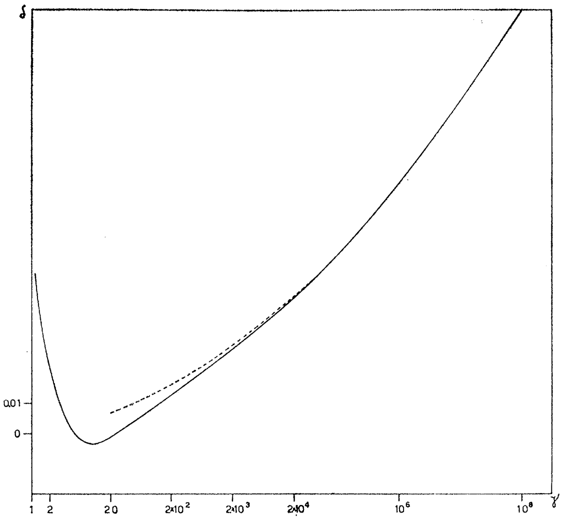

FIG. 3. Total correction as a function of $\ln(2\gamma)$. The solid curve refers to the exact formula; the dashed one to the ER limit.

point we plot in Fig. 3 the total correction as a function of $\ln(2\gamma)$.

Now for the Geneva experiment's results. As already discussed, in this experiment one measures at energies in the range from 2 up to 10 BeV, the total cross section for the annihilation of the e^+e^- pair into photons (two, three, and possibly more).

In Table II we report the experimental values of the total cross section $\sigma_{\exp}$,[2] the Dirac cross section σ_D, and the cross section with radiative corrections σ_{theor}. Figure 4 shows a comparison between σ_{theor} (dashed curve), σ_D (solid curve) and $\sigma_{\exp}$ (after reference 2).

Experimental data available until now do not allow any distinction between the Dirac cross section and the cross section with the first radiative correction because both fall within the experimental errors. So such an experiment is not a good measure of the radiative corrections. On the other hand, the correspondence between the predictions of the local Q.E.D. and the experiment makes possible a lowering of the present limits of validity of the theory. It has been observed[3] that the distance reached with such an annihilation experiment is strongly dependent on the modification used for the electron propagator. In fact, some care is to be taken in introducing a minimal length in the electron propagator, in order not to destroy the gauge invariance of the theory.[11] The simplest (non-gauge-invariant) approach of Drell[12] has as a consequence that by using different "gauges," e.g., in the sum over the photons' polarizations, different results are obtained. The current procedure requires a simultaneous modification of the electron propagator and of the vertex function (Ward identity), so that the over-all deviation from the local theory depends essentially on two form factors. Another approach could be the use of the intermediate auxiliary fields. We do not pursue this matter here; it is sufficient to report that, in the most favorable case, the distance tested by the experiment of Fabiani *et al.* seems to be of the order of $(0.2–0.4)\times 10^{-13}$ cm.

6. CONCLUDING REMARKS

The main results of our work are represented by the total cross section $\sigma_3(\gamma, >\Delta)$, Eqs. (2.10) or (2.12), and by the total correction δ_T, Eqs. (4.1) or (4.3), whose meaning has been discussed in the text.

It is possible to use $\sigma_3(\gamma, >\Delta)$ independently and not solely as an intermediate step in arriving at δ_T. Its expression, which is dependent on Δ, has been obtained for $\Delta \ll 1$, but, owing to the rather slow variation of the logarithm, it can be taken valid up to Δ values $\lesssim \frac{1}{2}$, if someone is interested in an evaluation of orders of magnitude. Moreover Eq. (4.3) holds in the laboratory system only, although its expression can be also obtained in the c.m. system, as we will explain in the following.

Let us discuss the total (Lorentz-invariant) correction δ_T. The really interesting point is the $\ln^2(2\gamma)$ high-energy dependence. In order to fully analyze the meaning of such a result it is convenient to summarize the conclusions contained in the previous work on the high-energy behavior of the radiative corrections. In this connection, a rather important theorem has been established by Eriksson and Petermann.[13,14] They considered the behavior of the radiative corrections to the differential cross section, for large values of the momentum transfer (i.e., $q^2 \gg m^2$) in the c.m. system, by taking into account soft photons only. Then they succeeded in showing that under these conditions, the correction behaves as $(\alpha/\pi)\ln(q^2/m^2)$, or better still, that the series of the radiative corrections, in the usual sense, is an expansion in terms of a parameter $(\alpha/\pi)\times\ln(q^2/m^2)$. Actually, their result must be understood in the sense that, for $q^2 \gg m^2$ in c.m., the correction to the order e^6 can be put in the form:

$$\delta = \frac{\alpha}{\pi}\left[c_1 \ln\frac{E}{\Delta E}\ln\frac{q^2}{m^2} + c_2 \ln\frac{q^2}{m^2} + c_3\right], \qquad (6.1)$$

TABLE II. Experimental total cross section $\sigma_{\exp}$, Dirac cross section σ_D, and theoretical total cross section σ_{theor}, to order e^6. Cross sections are measured in mb.

GeV	$\sigma_{\exp}$	σ_D	σ_{theor}
1.94	0.522±0.015	0.523	0.538
5.80	0.197±0.06	0.199	0.206
7.71	0.156±0.06	0.154	0.160
9.64	0.127±0.07	0.126	0.131

[11] P. Budini and T. Weber, Nuovo cimento **22**, 1321 (1961).
[12] S. D. Drell, Ann. Phys. (New York) 4, 75 (1958).

[13] K. E. Eriksson and A. Petermann, Phys. Rev. Letters **5**, 446 (1960).
[14] K. E. Eriksson, Nuovo cimento **19**, 1044 (1961).

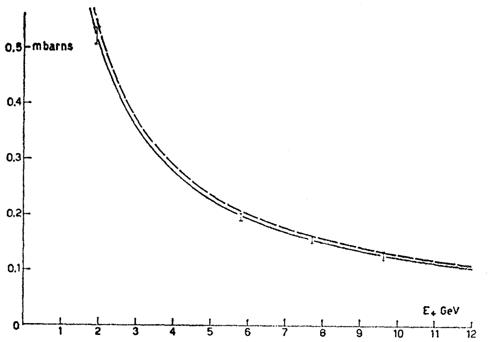

FIG. 4. A comparison between the experimental points and the Dirac cross section (solid curve) and the corrected one (dashed curve).

where c_1, c_2, c_3 are numerical, possibly angle-dependent, constants. This result (which practically gives the classical Schwinger's expression of the correction for the elastic scattering in an external field) seems to be confirmed by current calculations for different processes in the c.m. system. The subsequent addition of the contribution of hard photons does not change this conclusion. (Possibly terms like $\ln^2(E/\Delta E)$ can appear, but taking into account the fact that for the high-energy experiments $\Delta E/E$ is several units, these quantities are not too important.) This situation holds, without any doubt, also for the $e^+ + e^- \to 2\gamma$ differential cross section. To arrive at our δ_T, one has to take into account all the hard photons and then to integrate over the angles. At first sight, from the Eriksson and Petermann theorem, one could expect that the total correction also contains a first power of $\ln 2\gamma$, so that our expression (4.3) for δ_T can seem to be rather surprising. The appearance of the $\ln^2 2\gamma$ terms can be completely understood by following the discussion given in a recent paper by Andreassi, Budini, and Furlan.[15] The point is, that to get the total cross section, we need the expression for the radiative corrections (in the usual sense, i.e., with soft photons only) also for small values of the momentum transfer ($q^2 \gtrsim m^2$). Now it is possible to show that in this region, where the Eriksson and Petermann theorem does not hold, the correction cannot be put in the form of Eq. (6.1), and more precisely that (in the c.m. system):

$$\delta = (\alpha/\pi)[d_1 \ln(E/\Delta E) \ln 2\gamma + d_2 \ln^2 2\gamma + d_3]. \quad (6.2)$$

In addition, the $\ln^2 2\gamma$ terms can be very clearly ascribed to the contribution of soft photons to the correction. (For the $e^+ + e^- \to 2\gamma$ process considered, this soft-photon contribution is isotropic). Thus, the $\ln^2 2\gamma$ term remains present. Also after the angular integration and following reference 15 it is easily seen that, in the ER limit, the cross section corrected for virtual and soft photons is

$$\sigma_{2c}(\gamma, >\Delta) = \sigma_D \left\{ 1 + \frac{\alpha}{12\pi} \left[24(2\ln 2\gamma - 1)\ln(\Delta/\gamma) + 8\ln^2 2\gamma + 22\ln 2\gamma + 4\pi^2 - 25 + \frac{3\pi^2 - 25}{2\ln 2\gamma - 1} \right] \right\}. \quad (6.3)$$

This formula, makes clear the continued presence of the $\ln^2 2\gamma$ term (in the c.m. system).

Analogously we can calculate the cross section (1.1′) in the c.m. system. The result is

$$\sigma_3(\gamma, >\Delta) = \sigma_D \frac{\alpha}{\pi} \left[2(2\ln 2\gamma - 1)\ln(\gamma/\Delta) - 2\ln 2\gamma + 1 + \frac{9 - 2\pi^2}{6(2\ln 2\gamma - 1)} \right]. \quad (6.4)$$

This part does not contain $\ln^2 2\gamma$ terms (as expected), so that by adding it to Eq. (6.3) to obtain the total correction, the high-energy behavior of σ_{2c} is not changed. These considerations are sufficient to explain the $\ln^2 2\gamma$ term and to ascribe it to the soft inelastic part at small angles, where there is no compensation between the virtual and soft photon contribution.

Looking now at the numerical results, one sees that, except at very high energies ($\gamma \gtrsim 10^6$), the correction is rather small. However, at energies higher than this, it is not reasonable to continue to neglect the α^2 terms of the correction, coming from the higher approximation.[16] It is better to specify, at this point, that these next higher contributions which we should take into account, introduce with increasing α an increasing number of photons in the final state because there are no threshold problems. (That is, at the order e^8 we ought to add the four-quantum annihilation and the virtual photon corrections for the two- and three-quantum annihilations.) This is so because the object of our calculation is a total cross section for annihilation into photons whose maximum number depends on the approximation used (that is, not only the integral cross section for annihilation into a fixed number of photons). In this way the total correction δ_T, as defined by us, can also be thought of as an expansion into virtual and real photon numbers (obviously in a figurative sense).

Radiative corrections, in the usual meaning, to the integral cross section for the $e^+ + e^- \to 2\gamma$ process involve rather virtual photons and real soft photons (not detected). However, the point to be stressed (see reference 15) is that the asymptotic behavior both of the integrated radiative correction and of our total correction is, in the c.m. system, completely determined by the soft-photon part, as we have discussed above, Eqs.

[15] G. Andreassi, P. Budini, and G. Furlan, Phys. Rev. Letters **8**, 184 (1962).

[16] If we assume, as an indication, that the next term is like $(\alpha/\pi)^2 \ln^4 2\gamma$, it could be of the order of several percent at $\gamma \geq 10^5$.

(6.3) and (6.4), in order to explain the $\ln^2 2\gamma$ appearance. Therefore we conclude that, whatever the distribution of the hard photons (in the c.m. system), their contribution has no influence on the more representative power of the logarithm. Our considerations are based on the knowledge of the terms up to e^6, but they are probably true at every order in α (in the c.m. system). In fact the Eriksson and Petermann theorem says that for every term of the perturbation series, there will be the compensation at large angles between the elastic (virtual photons) correction and the inelastic one (soft real photons); in the same way, if this compensation does not occur at small angles, after angular integrations the appearance of the $(\alpha/\pi)^n \ln^{2n}2\gamma$ terms will be allowed. Taking into account the hard photons, no matter how one does this, will successively modify the lower powers of the logarithm.

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APPENDIX I

It is easily seen from the kinematics that the ranges of integration of our variables are

$$0\leq\theta_1\leq\pi,\quad 0\leq\theta_2\leq\pi,\quad 0\leq\varphi_1\leq 2\pi,\quad 0\leq\varphi_2\leq 2\pi;$$

$$0\leq x_1\leq k_M=\frac{1}{1-[\beta\gamma/(\gamma+1)]\cos\theta_1}.$$

It is well known, however, that if we allow x_1 to approach the value 0, the so-called infrared divergence occurs. In order to avoid this fact, we increase the lower limit for x_1 by an arbitrary quantity Δ (satisfying the condition $\Delta\ll 1$; that is, $0<\omega_{1\min}\ll m$). In the following considerations we put, for simplicity, $m=1$, $x_i=\omega_i$.

We will now show that if we are to take into account the symmetry between the three photons, i.e., if we are to prevent photons x_2 and x_3 from reaching values $<\Delta$, we must also decrease the upper limit for x_1. We are

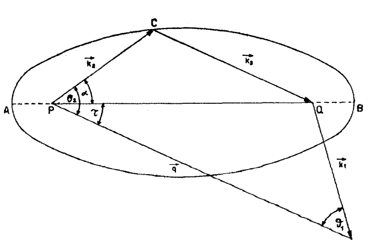

FIG. 5. Kinematical situation for k_1, θ_1 fixed.

concerned with the states for which all three photons have energies $\geq\Delta$. To obtain these states, we shall proceed as it follows. Let us fix a value for k_1. For a generic value of x_1 between Δ and k_M, we have

$$\omega_2+\omega_3=\text{const}=E-\omega_1,$$

so that the point C moves on an ellipsoid of revolution around the PQ axis and with foci P and Q (see Fig. 5).

The range AP (QB), which represents the minimum value for ω_2 (ω_3), is decreasing as x_1 increases to k_M (in particular it is zero for $x_1=k_M$). For x_1 greater than a certain value, ω_2 (ω_3) becomes smaller than Δ. Now we determine this value. Clearly,

$$PQ=(\mathbf{q}^2+\omega_1^2-2|\mathbf{q}|\omega_1\cos\theta_1)^{1/2}.$$

We see that $\omega_{2,3}$ is minimum (or maximum) when $C\equiv A$ (or $C\equiv B$). In this situation, we have

$$2\omega_{\min}+PQ=E-\omega_1,\quad \omega_{\min}=\tfrac{1}{2}(E-\omega_1-PQ).$$

Now we impose the conditions:

$$\omega_{2\min}\geq\Delta\quad (\text{i.e., } AP\geq\Delta),$$

and

$$\omega_{3\min}\geq\Delta\quad (\text{i.e., } BQ\geq\Delta).$$

It is clear that AP approaches zero as x_1 approaches its maximum value allowed by kinematics.

From the above inequality we have

$$E-\omega_1-(\mathbf{q}^2+\omega_1^2-2|\mathbf{q}|\omega_1\cos\theta_1)^{1/2}\geq 2\Delta.$$

The upper limit for x_1 is obtained by equating the two members in the above equation.

The result is

$$k(\Delta)=1(/p+r\cos\theta)_1,$$

where:

$$p=\frac{\gamma+1-2\Delta}{(\gamma+1)(1-2\Delta)+2\Delta^2},\quad r=\frac{-\beta\gamma}{(\gamma+1)(1-2\Delta)+2\Delta^2}.$$

[All states with $x_1\leq k(\Delta)$ and then having $\omega_2,\omega_3\geq\Delta$ are to be taken into account.] However, not all the states falling in the range $[k(\Delta),k_M]$ are to be discarded, since also for $x_1>k(\Delta)$ we have states with $\omega_2,\omega_3\geq\Delta$.

In fact, when C falls outside of the intersection of the ellipsoid with the spheres of radii Δ and centers P and Q, respectively, we have $\omega_2>\Delta$ and $\omega_3>\Delta$, i.e., such states are to be taken into account.

When k_1 ranges between $k(\Delta)$ and k_M it turns out to be convenient to perform a change of the angular variables determining the direction of $\mathbf{k}_2$. We choose as as the colatitude, the angle α between $\mathbf{k}_2$ and PQ and as the azimuth ψ, the angle between the planes $(\mathbf{k}_2,PQ)$ and $(PQ,\mathbf{k}_1)$. One finds

$$\cos\varphi=(\cos\alpha\sin\tau+\sin\alpha\cos\tau\cos\psi)/\sin\theta_2,$$

$$\cos\theta_2=\cos\alpha\cos\tau+\sin\alpha\sin\tau\cos\psi,$$

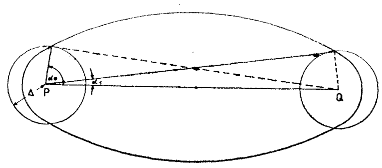

FIG. 6. Range of integration for α.

where τ is the angle between the direction of the positron and the axis PQ; whence

$$d\cos\theta_2 d\varphi = d\cos\alpha d\psi.$$

It is easily seen from symmetry that ψ ranges from zero to 2π. We now derive the minimum α_0 and the maximum α_1 of α (see Fig. 6). We choose the pole in P from the polar equation of the ellipse

$$\rho = b^2/a(1+\epsilon\cos\alpha),$$

where b and a are the semiminor and semimajor axis of the ellipse, and ϵ is the eccentricity. Their expressions are

$$a=\tfrac{1}{2}(x-x_1),\quad b^2=\tfrac{1}{2}l,\quad \epsilon=PQ/(x-x_1).$$

Noting that ρ has to go from Δ to $E-\omega_1-\Delta$, we have

$$\cos\alpha_0=(x-x_1)/PQ-l/\Delta PQ,$$
$$\cos\alpha_1=(x-x_1)/PQ-l/(x-x_1-\Delta)PQ.$$

Thus, our integrals I are of the type

$$I=\int_{-1}^{+1} d\cos\theta_1\int_{\Delta}^{k(\Delta)} dx_1\int_{-1}^{+1} d\cos\theta_2\int_0^{2\pi} d\varphi\int_0^{2\pi} d\varphi_2 \times F(\theta_1,x_1,\theta_2,\varphi,\varphi_2)+\int_{-1}^{+1} d\cos\theta_1\int_{K(\Delta)}^{K_M} dx_1\int_{\cos\alpha_0}^{\cos\alpha_1} d\cos\alpha \times\int_0^{2\pi} d\psi\int_0^{2\pi} d\varphi_2\, F(\theta_1,x_1,\alpha,\psi,\varphi_2),$$

where the first integral gives the terms called W and Z, and the second one gives the terms called V. In the text, care must be taken that the integrand for the V's should be the same Φ's as used in W's and Z's, and not other ones obtained from them by symmetry. At first sight, since Δ is allowed to go to zero, one might think that the second integral in I is useless for, in the limit $\Delta\to 0$, we recover all states. It is easy to see, however, that if the integrand is divergent for $x_1\to k_M$ we may obtain from the second integral a finite contribution which is not given by the first integral in the limit $\Delta\to 0$. The state of affairs is quite clear in the nonrelativistic limit to which we briefly refer (for the details and notation we refer to Jauch.[7] The total cross section is given by

$$\sigma_{NR}=\frac{4}{3}\frac{\alpha r_0^2}{\beta}\int_0^1 d\omega_1\int_0^1 d\omega_2 \times\left[\left(\frac{1-\omega_1}{\omega_2\omega_3}\right)^2+\left(\frac{1-\omega_2}{\omega_3\omega_1}\right)^2+\left(\frac{1-\omega_3}{\omega_1\omega_2}\right)^2\right],$$

where $\omega_1+\omega_2+\omega_3=2$.

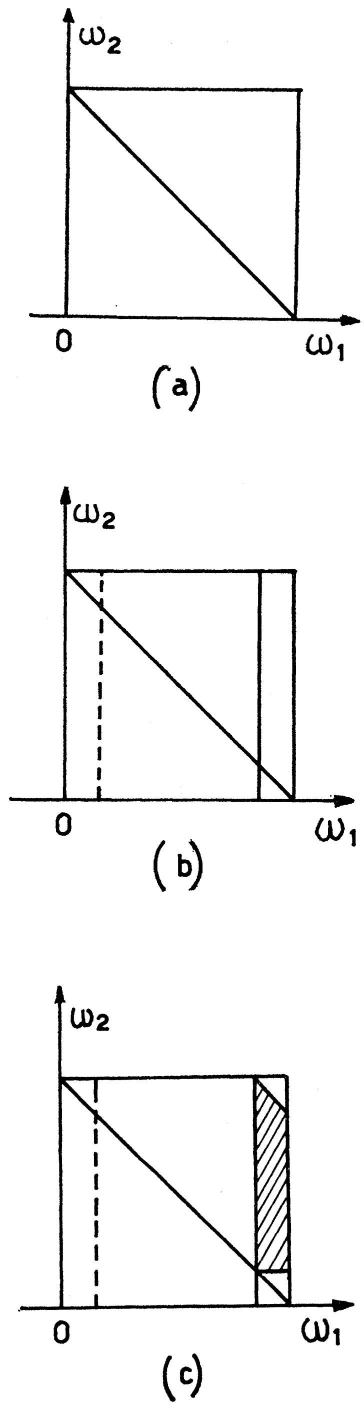

FIG. 7. Domains of integration in the NR limit.

The domain of integration is shown in Fig. 7. Here the variables of integration are all energies and, therefore, the situation is simpler. In order to satisfy the condition that the energy of the three photons does not become smaller than $m\Delta$ we can cut the domain of integration as in Fig. 7(b) (which corresponds to the cutoff in the general case), but in so doing we destroy the symmetry of the domain and, therefore, we do not treat all variables on the same footing. This is matched mathematically with the fact that a finite contribution from some functions diverging in the discarded area is lost in the limit $\Delta \rightarrow 0$. The symmetry is accomplished by adding the contribution from the shaded area in Fig. 7(c), which also contains allowed states.

The same problem would arise for the lower limit of integration for $\Delta \rightarrow 0$. We would have to take into account a finite contribution from integration over $(0,\Delta)$ just as we had from integration over $k(\Delta)$, k_M. However, these finite terms will be regained in the evaluation of the soft-photon part in the correction to the cross section for two-quantum annihilation (see Sec. 3); therefore, we will not consider them now.

APPENDIX II

Types of Integrals

We examine now the types of integrals which occur most frequently in the course of the calculations.

Integration over dx_1 of the W's is straightforward; it leads to integrals in $d\cos\theta_1$ of the type $\int[\ln(a\cos\theta_1+b)/(c\cos\theta_1+d)]d\cos\theta_1$ whose primitive is expressed in terms of the function $\mathcal{L}_2[f(\cos\theta_1)]$; see Mitchell.[17] More laborious are the integrations concerning the Z's; if we look at I, the integration over $d\varphi$ is

$$\int_0^{2\pi} \frac{d\varphi}{a+b\cos\varphi}, \quad (a>b)$$

where a, b are functions of $\cos\theta_1$, $\cos\theta_2$, and x_1. This integration results in a function of $\cos\theta_2$ of the form $[A\cos\theta_2+B\cos\theta_2+C]^{-1/2}$ (obviously A, B, and C are functions of $\cos\theta_1$ and x_1), whose subsequent integration over $d\cos\theta_2$ leads to a rather complicated function (of $\cos\theta_1$ and x_1) of the form

$$\frac{g(x_1\cos\theta_1)\ln|L-x_1\pm(M+x_1^2+Nx_1\cos\theta_1)^{1/2}|}{(M+x_1^2+Nx_1\cos\theta_1)^{\frac{1}{2}}};$$

here g is an algebraic function. This last integral in dx_1 reduces to a simpler form after rationalization of the root and gives, in addition to the usual functions, $F(\cos\theta_1)\mathcal{L}_2[f(\cos\theta_1)]$, where f and F are algebraic functions; the integration over $d\cos\theta_1$ is not always possible. In particular, for the types

$$\int\frac{\mathcal{L}_2(\cos\theta_1)}{(\cos\theta_1)^n}d\cos\theta_1, \quad \int\frac{\mathcal{L}_2(\cos\theta_1)}{(a\cos\theta_1+b)^n}d\cos\theta_1,$$

the integration cannot be performed analytically when $n=1$.

In the evaluation of the ER limit for the $\sigma_T(e^6)$ it is necessary to calculate the contribution of the integral $\int_{-1}^{+1}F_2(\cos\theta_1)d\cos\theta_1$ which happens not to be zero. For this purpose, changing the variable of integration a into v with the substitution $v=2x/(1+a)$, we split this integral into other of the type $\int\mathcal{L}_2(av+b)dv/v$.

This can be reduced to the transcendental function $\mathcal{L}_3(x)$ defined as $\mathcal{L}_3(x)=\int_0^x\mathcal{L}_2(t)dt/t$, as follows:

$$\begin{aligned}\int\mathcal{L}_2(ax+b)\frac{dx}{x} &= \ln|x|\ \ln|ax+b|\ \ln|1-ax-b| - \tfrac{1}{2}\ln|b|\ \ln^2|x| \\ &\quad +\ln|1-ax-b|\,\mathcal{L}_2(1-ax-b)+\ln|x| \\ &\quad \times\left[\mathcal{L}_2(ax+b)+\mathcal{L}_2\left(-\frac{ax}{b}\right)\right]+\ln\left|a+\frac{b-1}{x}\right| \\ &\quad \times\left[\mathcal{L}_2\left(\frac{b-1}{ax}+1\right)-\mathcal{L}_2\left(b\frac{b-1}{ax}+b\right)\right]-\mathcal{L}_3(-ax/b) \\ &\quad -\mathcal{L}_3(1-ax-b)-\mathcal{L}_3\left(\frac{b-1}{ax}+1\right)+\mathcal{L}_3\left(b\frac{b-1}{ax}+b\right);\end{aligned}$$

here a, b, and x are real numbers.

Properties analogous to those holding for $\mathcal{L}_2(x)$ can be easily obtained for $\mathcal{L}_3(x)$.[18]

[17] K. Mitchell, Phil. Mag. 40, 351 (1949), p. 301, Eq. (12).

[18] See reference 14, pp. 366, 367.