Decay of Heavy Bosons*

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(Received May 31, 1962)

The properties of the decays of all heavy bosons of zero strangeness, angular momenta $\leqslant 2$, and isospin $\leqslant 1$ are analyzed. First, we describe the symmetry characteristics of N-pion states (for $N\leqslant 5$) with respect to permutations in isospin and momentum space. Secondly, we give the strong, first-order (with emission of a real photon), and second-order electromagnetic decays together with the nonrelativistic and extreme relativistic barrier factors for each initial spin, parity, and G parity. These barrier effects (together with the amount of phase space available) are important in determining relative decay rates. Finally, we present the matrix elements for strong and electromagnetic decays into 4 or fewer pions. All results are summarized in tabular form.

I. INTRODUCTION

In the past few years unstable bosons of zero strangeness have been both predicted and found experimentally. The existence of the ρ, ω, and η is now quite firmly established, and the quantum numbers (i.e., spin, parity, isospin) of the first two have also been determined.[1,2] Further resonances in the pion system are at present being sought and sometimes found,[3] and it is not unreasonable to anticipate that still more will appear as the search continues.

The arguments by means of which quantum numbers are assigned involve considerations of branching ratios for observed decay modes as well as (in the case of three-pion decays) the correlations displayed in a Dalitz plot. For the former, a table of transitions which can occur in each case is indispensable; parts of such a table have appeared in the literature,[4] but to our knowledge, no complete list has been published. Equally important, however, in making semi-quantitative estimates of the relative frequency of various allowed modes is to know the "barrier factor" associated with each one. These factors are only indirectly related to the angular momentum of the decaying boson, but they can inhibit a particular decay mode in the same way that an angular momentum barrier does. The principal motivation for this paper is to present fairly complete tables which include these barrier factors in each case.

The backbone of this article is contained in a series of tables related to final states of pions and photons. Since the main objective is to make these tables useful for experimental physicists, we have endeavored to minimize the amount of discussion having to do with derivations, but to be rather explicit with those explanations which tend to increase the usefulness of the material.

In Sec. II are presented two tables which give the symmetry characteristics of various pion states with respect to permutations of the isospin coordinates and the space coordinates. The first of these has been dealt with in very complete and detailed fashion by Pais.[5] These tables are used to determine the barrier factors for each decay mode, as well as the number of linearly independent matrix elements.

The tables intended to be most directly related to experimental observations of the branching ratios of pion resonances are contained in Sec. III together with sufficient explanatory material to enable them to be used; a discussion of their contents is included.

Finally, tables of the explicit matrix elements for decays (strong or electromagnetic) into $N\leqslant 4$ pion modes appear in Sec. IV.

II. π-MESON STATES

We list in Table I the names of the irreducible representations of the permutation group[5,6] belonging

Table I. The irreducible representations of the symmetric group associated with each isospin function for N pions, $2\leqslant N\leqslant 5$.

N	T	Representations
2	0	(2)
	1	(1^2)
	2	(2)
3	0	(1^3)
	1	(3); (2,1)
	2	(2,1)
	3	(3)
4	0	(4); (2^2)
	1	(3,1); (2,1^2)
	2	(4); (3,1); (2^2)
	3	(3,1)
	4	(4)
5	0	(3,1^2)
	1	(5); (4,1); (3,2); (2^2,1)
	2	(4,1); (3,2); (3,1^2)
	3	(5); (4,1); (3,2)
	4	(4,1)
	5	(5)

* Partially supported by the U. S. Atomic Energy Commission.

[1] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters 6, 628 (1961) where earlier references for the ρ-meson can be found.

[2] M. L. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, Phys. Rev. 125, 687 (1962), where earlier references for the ω-meson can be found.

[3] R. Barloutaud, J. Heughebaert, A. Leveque, J. Meyer, and R. Omnes, Phys. Rev. Letters 8, 32 (1962); B. Sechi Zorn, *ibid.* 8, 282 (1962); see also reference 1.

[4] T. D. Lee and C. N. Yang, Nuovo cimento 3, 749 (1956); B. T. Feld, Phys. Rev. Letters 8, 181 (1962).

[5] A. Pais, Ann. Phys. (New York) 9, 548 (1960).

[6] See for example, M. Hamermesh, *Group Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).

TABLE II. The irreducible representations of the symmetric group associated with each spatial function for N pions ($2 \leqslant N \leqslant 5$), for both parities and for $J \leqslant 2$. The barrier factor L is the power of the momentum that appears in the wave function in the nonrelativistic limit; Λ is the same factor for the extreme relativistic case. The exponent outside a group entry is the number of times the particular representation appears. The dots refer to classes not written down, for reasons explained in the text.

N	J^P	L	Λ	Representations
2	0^+	0	0	(2)
	1^-	1	1	(1^2)
	2^+	2	2	(2)
3	0^-	0	0	(3)
		2	1	$(2,1)$
		4	2	$(3);\ (2,1)$
		6	3	$(3);\ (2,1)^2;\ (1^3)$
	0^+			forbidden
	1^-	2	2	(1^3)
		4	3	$(2,1)$
		6	4	$(2,1);\ (1^3)$
		8	5	$(3);\ (2,1)^2;\ (1^3)$
	1^+	1	1	$(2,1)$
		3	2	$(3);\ (2,1);\ (1^3)$
	2^-	2	2	$(3);\ (2,1)$
		4	3	$(3);\ (2,1)^2;\ (1^3)$
	2^+	3	3	$(2,1)$
		5	4	$(3);\ (2,1);\ (1^3)$
4	0^+	0	0	(4)
		2	1	$(3,1)$
		2	2	(2^2)
		4	2	$(4);\ (3,1);\ (2^2)$
		4	3	$(3,1);\ (2,1^2)$
		4	4	$(4);\ (2^2)$
		6	3	$\cdots;\ (2,1^2)^2$
		6	4	$\cdots;\ (1^4)^2$
		6	5	$\cdots$
		6	6	$\cdots;\ (1^4)$
	0^-	3	3	(1^4)
		5	4	$(2,1^2)$
		5	5	(2^2)
		7	5	$(2^2);\ \cdots$
		7	6	$(3,1);\ \cdots$
		7	7	$\cdots$
		9	6	$(3,1)^2;\ \cdots$
		9	7	$(4)^2;\ \cdots$
		9	8	$\cdots$
		9	9	$(4);\ \cdots$
	1^+	2	2	$(2,1^2)$
		4	3	$(3,1);\ (2^2);\ \cdots$
		4	4	$(3,1);\ \cdots$
		6	4	$\cdots$
		6	5	$(4);\ \cdots$
		6	6	$\cdots$
		8	5	$(4)^2;\ \cdots$
4	1^-	1	1	$(3,1)$
		3	2	$(4);\ \cdots;\ (2^2);\ (2,1^2)$
		3	3	$\cdots;\ (2,1^2)$
	2^+	2	2	$(4);\ (3,1);\ (2^2)$
		4	3	$\cdots;\ (2,1^2)^2$
		4	4	$\cdots;\ (2,1^2);\ \cdots$
	2^-	3	3	$(3,1);\ (2^2);\ (2,1^2)$
		5	4	$(4);\ \cdots$
		5	5	$(4);\ \cdots$
5	0^-	0	0	(5)
		2	1	$(4,1)$
		2	2	$(3,2)$
		4	2	$(5);\ (4,1);\ (3,2)$
		4	3	$(4,1);\ (3,2);\ (3,1^2);\ (2^2,1)$
		4	4	$(4,1);\ (3,2);\ (2^2,1)$
		6	3	$\cdots;\ (3,1^2)^2;\ (2^2,1)$
		6	4	$\cdots;\ (2,1^3)^3$
		6	5	$\cdots;\ (2,1^3)^3$
		6	6	$\cdots;\ (2,1^3)^2$
	0^+	3	3	$(2,1^3)$
		5	4	$(3,1^2);\ (2^2,1);\ \cdots$
		5	5	$(3,2);\ (3,1^2);\ (2^2,1);\ \cdots$
		7	5	$(3,2);\ \cdots$
		7	6	$(4,1)^2;\ \cdots$
		7	7	$(4,1);\ \cdots$
		9	6	$(4,1)^3;\ \cdots$
		9	7	$(5)^3;\ \cdots$
		9	8	$(5)^3;\ \cdots$
		9	9	$(5)^2;\ \cdots$
	1^-	2	2	$(3,1^2)$
		4	3	$(4,1);\ (3,2);\ \cdots;\ (2^2,1);\ \cdots$
		4	4	$(4,1);\ (3,2);\ \cdots;\ (2^2,1);\ \cdots$
		6	4	$\cdots$
		6	5	$(5);\ \cdots$
		6	6	$\cdots$
		8	5	$(5)^2;\ \cdots$
	1^+	1	1	$(4,1)$
		3	2	$(5);\ \cdots;\ (3,2);\ (3,1^2)$
		3	3	$\cdots;\ (3,2);\ (3,1^2);\ (2^2,1)$
		5	3	$\cdots;\ (2^2,1)$
	2^-	2	2	$(5);\ (4,1);\ (3,2)$
		4	3	$\cdots;\ (3,1^2)^2;\ (2^2,1)$
		4	4	$\cdots;\ (3,1^2)^2;\ (2^2,1)^2;\ \cdots$
	2^+	3	3	$(4,1);\ (3,2);\ (3,1^2);\ (2^2,1)$
		5	4	$(5);\ \cdots$
		5	5	$(5);\ \cdots$

to each isospin function for two to five mesons.[7] In this table, the symbol (3) stands for a function completely symmetric in the isospin variables, (1^3) for one completely antisymmetric, and so on.[6] It will not be necessary to have any detailed understanding of the relationship between the symbols and the representations for which they stand in order to use the table.

Table II refers to the effects of permutations of the

[7] A very explicit prescription for the construction of this table is given by Pais (reference 5), who also has tabulated the entries for $T=0$ and 1. We require the isospin values up to $T=3$, since we consider initial states with $T=0$ or 1 and processes for which $\Delta T \leqslant 2$.

pion momentum coordinates.[8] For each spin $J \leqslant 2$ and both parities P, the possible values of the barrier factors are given. Two such factors are listed for each entry. The one which we call L is the number of powers of the momentum which appear in the pion wave function in the nonrelativistic limit; the one called Λ is the same number in the extreme relativistic limit. The reason for the distinction can be seen upon consulting some of the explicit matrix elements to be given in Tables V–VIII: a factor such as $(\omega_1-\omega_2)$ contributes $L=2$ and $\Lambda=1$, while $(p_1-p_2)\cdot(p_3-p_4)$ contributes $L=\Lambda=2$.[9]

In every case, the symmetries allow pion wave functions to be multiplied by powers of $|\sum_{i=1}^N \omega_i|^\lambda$, where the ω's are energies in the c.m. system and λ is an integer. Since energy conservation ensures that this will multiply matrix elements by constants in cases where pions alone form the final state, we have omitted all such factors from consideration in Table II. When the table is used for final states in which particles other than pions are involved, these additional factors should be included.

For every $[L,\Lambda]$, Table II gives the irreducible representations (classes) which occur. Since not all of them are needed, many are indicated only by dots. For $N \geqslant 4$, the philosophy has been to omit the entry for a class if it satisfies (a) plus either (b) or (c) of the following criteria: (a) It is not useful for constructing entries for higher $[L,\Lambda]$; (b) it cannot be combined with a pion isospin function; (c) the class has already been encountered for lower values of *both* L and Λ.

The method of construction of such a table need not be understood in order to make use of it, and we will go into no details. We only will mention that the table for $J=0$, $P=(-1)^N$ is really a listing of the symmetry properties of Lorentz invariants made up of the pion momenta. Thus, for all other J^P, one need work out the transformation properties of only the lowest $[L,\Lambda]$. A table of group multiplication then provides the entries for all higher $[L,\Lambda]$.

In order to construct functions that are symmetric in isospin and momentum space, identical classes from Tables I and II must be combined. Each time that a representation is listed, an independent wave function (and therefore matrix element for which this is the final state) can be formed. As an example of this construction, consider the problem of making up a five-pion state with $T=1$ and $J^P=1^-$. Since the class $(3,1^2)$ required for $[L,\Lambda]=[2,2]$ is not shown under $T=1$ for five pions in Table I, there is no such state; i.e., no wave function which contains just two powers of the particle momenta.

In each of the cases $[L,\Lambda]=[4,3]$ and $[4,4]$, there is one state with isospin symmetry (4,1), one with (3,2), and one with $(2^2,1)$. Whether or not the two sets of matrix elements should be considered to compete with each other is a separate question to be discussed briefly in Sec. IIIB. In order to find a state with T-symmetry (5), one must go to $[L,\Lambda]=[6,5]$ and $[8,5]$.

[8] By reinterpreting J as the orbital contribution to the angular momentum and relabeling the parity if necessary, one can use Table II in combination with a similar one for spin coordinates to give corresponding information for other identical particles.

[9] $\mathbf{p}_j$ is the three-momentum and p_j is the four-momentum of the jth pion.

TABLE III. Decays into pions and photons of a $T=0$ boson with various J^{PG}. After each configuration of pions alone appears the nonrelativistic barrier factor L followed by the fully relativistic barrier factor Λ. For pions accompanied by gammas, L is given together with the multipolarity of the photon. Missing entries (within the boundaries explained in the text) are forbidden. $(+-0)$ stands for $(\pi^+\pi^-\pi^0)$, and an entry written as $(\pi\pi\pi)$ stands for all pion charge contributions that give the correct total charge.

J^{PG}	C	Strong	Order e	Order e^2
0^{++}	+	$(\pi\pi)0, 0$ $(\pi\pi\pi\pi)0, 0$	$(+-)1, \gamma 1^-$ $(+-0)2, \gamma 1^-$	$(\pi\pi)0, 0$ $\gamma\gamma$ $(0), \gamma\gamma$
0^{+-}	−	$(++--0)5, 4$ $(+-000)5, 4$	$(\pi\pi)2, \gamma 2^+$ $(+-0)1, \gamma 1^+$ $(000)3, \gamma 1^+$	$(++--)2, 1$ $(+-00)2, 1$ $(+-)1, \gamma\gamma$
0^{-+}	+	$(++--)5, 5$ $(+-00)5, 5$ $(0000)9, 7$	$(+-)1, \gamma 1^+$ $(+-0)2, \gamma 1^+$	$(\pi\pi\pi)0, 0$ $\gamma\gamma$ $(0), \gamma\gamma$
0^{--}	−	$(+-0)6, 3$ $(++--0)4, 3$ $(+-000)4, 3$	$(\pi\pi)2, \gamma 2^-$ $(+-0)1, \gamma 1^-$ $(000)3, \gamma 1^-$	$(+-0)2, 1$ $(+-)1, \gamma\gamma$ $(0e^+e^-)1, 1$
1^{++}	+	$(++--)4, 3$ $(+-00)4, 3$ $(0000)6, 5$	$(+-)1, \gamma 1^-$ $(+-0)2, \gamma 1^-$	$(+-0)1, 1$ $(000)3, 2$ $(0), \gamma\gamma$
1^{+-}	−	$(+-0)3, 2$ $(++--0)3, 2$ $(+-000)3, 2$	$(0), \gamma 1^-$ $(\pi\pi)0, \gamma 1^+$ $(\pi\pi\pi)0, \gamma 1^-$	$(+-0)1, 1$ $(+-)1, \gamma\gamma$
1^{-+}	+	$(\pi\pi\pi\pi)3, 2$	$(+-)1, \gamma 1^+$ $(+-0)2, \gamma 1^+$	$(+-0)4, 3$ $(000)8, 5$ $(0), \gamma\gamma$
1^{--}	−	$(+-0)2, 2$ $(++--0)2, 2$ $(+-000)2, 2$	$(0), \gamma 1^+$ $(\pi\pi)0, \gamma 1^-$ $(\pi\pi\pi)0, \gamma 1^+$	$(+-)1, 1$ $(+-0)2, 2$ $(+-)1, \gamma\gamma$ $(e^+e^-)0, 0$
2^{++}	+	$(\pi\pi)2, 2$ $(\pi\pi\pi\pi)2, 2$	$(+-)1, \gamma 1^-$ $(+-0)2, \gamma 1^-$	$(\pi\pi)2, 2$ $(+-0)3, 3$ $(000)5, 4$ $\gamma\gamma$ $(0), \gamma\gamma$
2^{+-}	−	$(+-0)5, 4$ $(++--0)3, 3$ $(+-000)3, 3$	$(0), \gamma 2^-$ $(\pi\pi)0, \gamma 2^+$ $(\pi\pi\pi)0, \gamma 2^-$ $(+-0)1, \gamma 1^+$	$(+-0)3, 3$ $(+-)1, \gamma\gamma$
2^{-+}	+	$(++--)3, 3$ $(+-00)3, 3$ $(0000)5, 4$	$(+-)1, \gamma 1^+$ $(+-0)2, \gamma 1^+$	$(\pi\pi\pi)2, 2$ $\gamma\gamma$ $(0), \gamma\gamma$
2^{--}	−	$(+-0)4, 3$ $(++--0)4, 3$ $(+-000)4, 3$	$(0), \gamma 2^+$ $(\pi\pi)0, \gamma 2^-$ $(\pi\pi\pi)0, \gamma 2^+$ $(+-0)1, \gamma 1^-$	$(+-0)2, 2$ $(+-)1, \gamma\gamma$

TABLE IV. Decays into pions and photons of a $T=1$ boson with various J^{PG}. See Table III for explanations of symbols.

J^{PG}	C	Q	Strong	Order e	Order e^2
0^{++}	$-$	0	$(++--)2, 1$ $(+-00)2, 1$	$(\pi\pi)2, \gamma 2^+$ $(+-0)1, \gamma 1^+$ $(000)3, \gamma 1^+$ $(000)2, \gamma 2^-$	$(++--)2, 1$ $(+-00)2, 1$ $(+-)1, \gamma\gamma$
		1	$(\pi\pi\pi\pi)2, 1$	$(\pi\pi)1, \gamma 1^-$ $(\pi\pi\pi)1, \gamma 1^+$	$(\pi\pi)0, 0$ $(+), \gamma\gamma$
0^{+-}	$+$	0	$(++--0)5, 4$ $(+-000)5, 5$ $(00000)9, 7$	$(+-)1, \gamma 1^-$ $(+-0)1, \gamma 1^+$	$(\pi\pi)0, 0$ $\gamma\gamma$ $(0)\gamma\gamma$
		1	$(+++--)5, 5$ $(++-00)5, 4$ $(+0000)7, 6$	See 0^{++}	See 0^{++}
0^{-+}	$-$	0	$(++--)7, 6$ $(+-00)5, 4$	$(\pi\pi)2, \gamma 2^-$ $(+-0)1, \gamma 1^-$ $(000)3, \gamma 1^-$ $(000)2, \gamma 2^+$	$(+-0)2, 1$ $(+-)1, \gamma\gamma$ $(0e^+e^-)1, 1$
		1	$(++-0)5, 4$ $(+000)7, 6$	$(\pi\pi)1, \gamma 1^+$ $(\pi\pi\pi)1, \gamma 1^-$	$(\pi\pi\pi)0, 0$ $(+), \gamma\gamma$ $(+e^+e^-)1, 1$
0^{--}	$+$	0	$(\pi\pi\pi)0, 0$ $(\pi\pi\pi\pi\pi)0, 0$	$(+-)1, \gamma 1^+$ $(+-0)1, \gamma 1^-$	$(\pi\pi\pi)0, 0$ $\gamma\gamma$ $(0), \gamma\gamma$
		1	$(\pi\pi\pi)0, 0$ $(\pi\pi\pi\pi\pi)0, 0$	See 0^{-+}	See 0^{-+}
1^{++}	$-$	0	$(++--)4, 3$ $(+-00)2, 2$	$(0), \gamma 1^-$ $(\pi\pi)0, \gamma 1^+$ $(\pi\pi\pi)0, \gamma 1^-$	$(+-0)1, 1$ $(+-)1, \gamma\gamma$
		1	$(++-0)2, 2$ $(+000)4, 3$	$(+), \gamma 1^-$ $(\pi\pi)0, \gamma 1^+$ $(\pi\pi\pi)0, \gamma 1^-$	$(\pi\pi\pi)1, 1$ $(+), \gamma\gamma$
1^{+-}	$+$	0	$(+-0)1, 1$ $(000)3, 2$ $(++--0)1, 1$ $(+-000)1, 1$ $(00000)3, 2$	$(+-)1, \gamma 1^-$ $(+-0)1, \gamma 1^+$	$(+-0)1, 1$ $(000)3, 2$ $(0), \gamma\gamma$
		1	$(\pi\pi\pi)1, 1$ $(\pi\pi\pi\pi\pi)1, 1$	See 1^{++}	See 1^{++}
1^{-+}	$-$	0	$(+-)1, 1$ $(++--)1, 1$ $(+-00)1, 1$	$(0), \gamma 1^+$ $(\pi\pi)0, \gamma 1^-$ $(\pi\pi\pi)0, \gamma 1^+$	$(+-)1, 1$ $(+-0)2, 2$ $(+-)1, \gamma\gamma$ $(e^+e^-)0, 0$
		1	$(\pi\pi)1, 1$ $(\pi\pi\pi\pi)1, 1$	$(+), \gamma 1^+$ $(\pi\pi)0, \gamma 1^-$ $(\pi\pi\pi)0, \gamma 1^+$	$(\pi\pi)1, 1$ $(\pi\pi\pi)4, 3$ $(+), \gamma\gamma$
1^{--}	$+$	0	$(+-0)4, 3$ $(000)8, 5$ $(++--0)4, 3$ $(+-000)4, 3$ $(00000)6, 5$	$(+-)1, \gamma 1^+$ $(+-0)1, \gamma 1^-$	$(+-0)4, 3$ $(000)8, 5$ $(0), \gamma\gamma$
		1	$(\pi\pi\pi)4, 3$ $(\pi\pi\pi\pi\pi)4, 3$	See 1^{-+}	See 1^{-+}
2^{++}	$-$	0	$(++--)2, 2$ $(+-00)2, 2$	$(0), \gamma 2^-$ $(\pi\pi)0, \gamma 2^+$ $(\pi\pi\pi)0, \gamma 2^-$ $(+-0)1, \gamma 1^+$	$(+-0)3, 3$ $(+-)1, \gamma\gamma$
		1	$(\pi\pi\pi\pi)2, 2$	$(+), \gamma 2^-$ $(\pi\pi)0, \gamma 2^+$ $(\pi\pi)1, \gamma 1^-$ $(\pi\pi\pi)0, \gamma 2^-$ $(\pi\pi\pi)1, \gamma 1^+$	$(\pi\pi)2, 2$ $(\pi\pi\pi)3, 3$ $(+), \gamma\gamma$
2^{+-}	$+$	0	$(+-0)3, 3$ $(000)5, 4$ $(++--0)3, 3$ $(+-000)3, 3$ $(00000)5, 4$	$(+-)1, \gamma 1^-$ $(+-0)1, \gamma 1^+$	$(\pi\pi)2, 2$ $(+-0)3, 3$ $(000)5, 4$ $\gamma\gamma$ $(0), \gamma\gamma$
		1	$(\pi\pi\pi)3, 3$ $(\pi\pi\pi\pi\pi)3, 3$	See 2^{++}	See 2^{++}
2^{-+}	$-$	0	$(++--)3, 3$ $(+-00)3, 3$	$(0), \gamma 2^+$ $(\pi\pi)0, \gamma 2^-$ $(\pi\pi\pi)0, \gamma 2^+$ $(+-0)1, \gamma 1^-$	$(+-0)2, 2$ $(+-)1, \gamma\gamma$
		1	$(\pi\pi\pi\pi)3, 3$	$(+), \gamma 2^+$ $(\pi\pi)0, \gamma 2^-$ $(\pi\pi)1, \gamma 1^+$ $(\pi\pi\pi)0, \gamma 2^+$ $(\pi\pi\pi)1, \gamma 1^-$	$(\pi\pi\pi)2, 2$ $(+), \gamma\gamma$
2^{--}	$+$	0	$(\pi\pi\pi)2, 2$ $(\pi\pi\pi\pi\pi)2, 2$	$(+-)1, \gamma 1^+$ $(+-0)1, \gamma 1^-$	$(\pi\pi\pi)2, 2$ $\gamma\gamma$ $(0), \gamma\gamma$
		1	$(\pi\pi\pi)2, 2$ $(\pi\pi\pi\pi\pi)2, 2$	See 2^{-+}	See 2^{-+}

Only one other simple group property must be explained before the tables can be fully exploited. It is necessary to know which representations are contained in a given set of pion charges. The general method will be clear if explained for five pions.[10] (a) Arrange the partitions of 5 in "descending order"[6]: (5); (4,1); (3,2); $(3,1,1)\equiv(3,1^2)$; $(2,2,1)\equiv(2^2,1)$; $(2,1^3)$; (1^5). (b) To any pion specification attach a partition which tells how many particles are in each charge state. Thus, $\pi^+\pi^+\pi^+\pi^-\pi^-$ has three pions with $+$ charge, two with $-$ charge, and none with zero charge; we thus associate with it the partition (3,2,0) or (3,2). (c) From this charge specification, one can make linear combinations which transform as all the irreducible representations from the most symmetric up to and including

[10] See also A. Pais, reference 5, Sec. II(d).

the partition written in (b) above. In that example, we would be able to conclude that the state $(\pi^+\pi^+\pi^+\pi^-\pi^-)$ contains *at least once* the irreducible representations (5), (4,1), and (3,2), and not at all the representations $(3,1^2)$, $\cdots$.

To return to the example of a 1^-, $T=1$ state, suppose that we want to complete the enumeration of the $(\pi^+)^3(\pi^-)^2$ states. From the above argument, we see that there are two states with $[L,\Lambda]=[4,3]$ and two with $[4,4]$. On the other hand, the configuration $\pi^+(\pi^0)^4$ has only one state with $[4,3]$ and one with $[4,4]$. The mode $(\pi^0)^5$ occurs once with $[6,5]$ and twice in a (probably noncompeting) wave function corresponding to $[8,5]$.

Of course, the number of independent states which can be written according to the above discussion is identical with the number of independent matrix elements which exist for a decay from an initial boson with the given J^P and T to one of these pion states. Tables I and II used in this way prove very useful in indicating how many independent matrix elements must be considered.

More important for the moral we want to draw in this paper is the fact that the tables enable one to pick out the minimum barrier factors which can occur for each set of quantum numbers; this is what we do in the following section.

III. ALLOWED DECAY MODES

A. Tables

In this section we present a systematic set of tables for the possible decays of initial bosons with spins $\leqslant 2$ via strong and electromagnetic interactions. Table III refers to isospin $T=0$ and Table IV to $T=1$.

In both tables, all possible combinations of $J\leqslant 2$, P, and G are given in the first column, together with the charge conjugation quantum number $C=(-1)^TG$ for neutral bosons. The second column includes all possible N-pion decays for $N\leqslant 5$ which can occur through strong interactions. The following remarks are relevant for this column: (i) Symbols such as $(+-0)$, etc., refer to particular pion charges; a symbol such as $(\pi\pi\pi)$ implies that all combinations of charge consistent with charge conservation for the given number of pions are possible with the same $[L,\Lambda]$. (ii) The numbers which appear after each pion configuration are the *minimum* barrier factors $[L,\Lambda]$ in that order. (iii) Any missing configuration for $N\leqslant 5$ is strictly forbidden.

In the "e" column, final states involving one real photon and up to three pions are given. After each gamma appear the multipolarity and parity of the photon. For each pion charge specification, only the final states with the least possible value of $L_\pi+L_\gamma$ are listed.[11] Again, within these limits, the lack of an entry indicates a forbidden final state.

TABLE V. Strong interaction decay matrix elements for bosons with $T=0$ and $J\leqslant 1$ into $N\leqslant 4$ pions. Entries are relativistically correct in the rest frame of the boson, and are given only for the lowest possible L value. Relative signs for different charge configurations are arbitrary; magnitudes are not, unless explicitly indicated by constants a, b, $\cdots$. The symbols p_1, p_2, $\cdots(\mathbf{p}_1,\mathbf{p}_2\cdots)$ refer to the 4-momenta (3-momenta) of the pions *in the order* listed in column 2. Matrices for $(\pi^0)^3$ or $(\pi^0)^4$ are not shown unless they compete with the charged pion decay modes. The following symbols are used in the table:

$$U=\mathbf{p}_1\cdot\mathbf{p}_2\times\mathbf{p}_3-\mathbf{p}_2\cdot\mathbf{p}_3\times\mathbf{p}_4+\mathbf{p}_3\cdot\mathbf{p}_4\times\mathbf{p}_1-\mathbf{p}_4\cdot\mathbf{p}_1\times\mathbf{p}_2;$$
$$\mathbf{A}=\mathbf{p}_1\times\mathbf{p}_2+\mathbf{p}_2\times\mathbf{p}_3+\mathbf{p}_3\times\mathbf{p}_1.$$

J^{PG}	(Pion charges) L	Matrix
0^{++}	$(+-)0$	a
	$(00)0$	a
	$(++--)0$	$2b$
	$(+-00)0$	b
	$(0000)0$	$3b$
0^{-+}	$(++--)5$ $(+-00)5$	$(p_1-p_2)\cdot(p_3-p_4)U$
0^{--}	$(+-0)6$	$\omega_1^2(\omega_2-\omega_3)+\omega_2^2(\omega_3-\omega_1)+\omega_3^2(\omega_1-\omega_2)$
1^{++}	$(++--)4$ $(+-00)4$	$[(\omega_1-\omega_2+\omega_3-\omega_4)(\mathbf{p}_1-\mathbf{p}_3)\times(\mathbf{p}_2-\mathbf{p}_4)+(\omega_1-\omega_2-\omega_3+\omega_4)(\mathbf{p}_1-\mathbf{p}_4)\times(\mathbf{p}_2-\mathbf{p}_3)]$
1^{+-}	$(+-0)3$	$\mathbf{p}_1(\omega_2-\omega_3)+\mathbf{p}_2(\omega_3-\omega_1)+\mathbf{p}_3(\omega_1-\omega_2)$
1^{-+}	$(++--)3$	$\{a(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)(\omega_1+\omega_2-\omega_3-\omega_4)+(2-a)[(\mathbf{p}_1-\mathbf{p}_2)(\omega_1-\omega_2)+(\mathbf{p}_3-\mathbf{p}_4)(\omega_3-\omega_4)]\}$
	$(+-00)3$	$\{(1-a)(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)(\omega_1+\omega_2-\omega_3-\omega_4)+a[(\mathbf{p}_1-\mathbf{p}_2)(\omega_1-\omega_2)+(\mathbf{p}_3-\mathbf{p}_4)(\omega_3-\omega_4)]\}$
	$(0000)3$	$4(\mathbf{p}_1\omega_1+\mathbf{p}_2\omega_2+\mathbf{p}_3\omega_3+\mathbf{p}_4\omega_4)$
1^{--}	$(+-0)2$	$\mathbf{A}$

[11] The factor Λ has been omitted whenever real photons occur, since in these cases there is less reason for distinguishing between L and Λ.

The last column lists those processes which can occur with one virtual gamma and which result in either two or three pion states, except that in the few cases for which neither of these is possible (0^+, $C=-1$), the $N=4$ decay is given. There further appear all decays with two real photons accompanied by zero or one pion[11]; only when neither of these is allowed will one find an entry for $(\pi\pi)\gamma\gamma$. Finally, every entry in the "e" column can appear in the "e^2" column with the gamma ray replaced by a Dalitz pair; occasionally the L value is lowered because a virtual photon which has even parity can carry away zero angular momentum. Only those Dalitz pairs not covered by this rule are written explicitly in the table (see $T=0$, 0^{--} and 1^{--}; $T=1$, 0^{-+} and 1^{-+}).

B. Discussion

As was mentioned in the Introduction, the main point to which we wish to draw attention in Tables III and IV is the appearance of the barrier factors L and Λ. Because of the symmetry requirement on states with a number of identical particles these factors are more often than not larger than the spin J of the initial state.

In order to see how to take these factors approximately into account, we consider an effective point interaction for the decay of an initial boson X into N pions. Such a Lagrangian density can be written in terms of a dimensionless coupling constant $g(R)$ and a range R ($\hbar=c=1$):

$$\mathcal{L}_{\text{int}}=g(R)R^{\nu+N-3}X(D^{\nu_1}\pi)\cdots(D^{\nu_N}\pi), \tag{1}$$

in which $\nu=\sum \nu_i$, D represents a derivative in four-space, and all the necessary labels relating to components in four-space and in isospin space have been omitted. The smallest possible value of ν for a given symmetry is equal to or greater than L, since there are time derivatives as well as space derivatives in Eq. (1). Further examination shows that the relativistic scalar matrix element takes the form (in the rest-frame of the X)

$$(g)(M_XR)^{\nu-L}[M_X(\omega_i-\omega_j)R^2]^{L-\Lambda}(|\mathbf{p}|R)^{2\Lambda-L}, \tag{2}$$

where each of the last factors schematically represents a term which is linear in the three-momenta, $|\mathbf{p}|$, in both the nonrelativistic and the extreme relativistic regions.[12] Terms that involve differences of pion energies, such as $[(\omega_i-\omega_j)R]$, reduce in these limits to $\frac{1}{2}(\mathbf{p}_i^2-\mathbf{p}_j^2)RR_\pi$ and $(|\mathbf{p}_i|-|\mathbf{p}_j|)R$, respectively, where R_π is the pion Compton wavelength.

The main point to be made is that at least $(L-\Lambda)$ powers of the range R are "used up" in this model in accompanying the mass M_X of the decaying boson, and in most cases one will not wish to assume M_XR to be small. The conclusions, to be used as a guide in semiquantitative estimates, can be summarized:

[12] Specific examples, without the constants in front, can be found in Tables V–VIII.

TABLE VI. Electromagnetic decay matrix elements for bosons with $T=0$ and $J\leqslant 1$ into $N\leqslant 3$ pions, except in the case 0^{+-} for which $N=4$ is the lowest possible. For explanation of symbols, see Table V.

J^{PG}	(Pion charges) L	Matrix
0^{++}	$(+-)0$	a
	$(00)0$	b
0^{+-}	$(++--)2$	$2(\omega_1+\omega_2-\omega_3-\omega_4)$
	$(+-00)2$	$(\omega_1-\omega_2)$
0^{-+}	$(+-0)0$	1
	$(000)0$	3
0^{--}	$(+-0)2$	$\omega_1-\omega_2$
1^{++}	$(+-0)1$	$\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3$
1^{+-}	$(+-0)1$	$\mathbf{p}_1-\mathbf{p}_2$
1^{-+}	$(+-0)4$	$(\omega_1-\omega_2)\mathbf{A}$
1^{--}	$(+-)1$	$a(\mathbf{p}_1-\mathbf{p}_2)$
	$(+-0)2$	$b\mathbf{A}$

(i) The matrix element for relativistic pions will carry a factor $\approx(|\mathbf{p}|R)^\Lambda$.

(ii) The corresponding matrix element for nonrelativistic pions will carry instead a factor

$$\approx(|\mathbf{p}|R)^\Lambda(|\mathbf{p}|R_\pi/2)^{L-\Lambda}.$$

In most applications, the distinction between the ranges R and $R_\pi/2$ will make little difference and can safely be ignored.

Many hypothetical cases in which information about the L values would be useful can be picked out of the tables. For example, the neutral component of a $T=1$, 0^{-+} boson with $M\lesssim 8m_\pi$ would have its strong four-pion decay rate inhibited not only by the available phase space but also by the appearance of a factor $(pR)^{14}$ in the case of $(++--)$ and $(pR)^{10}$ for the mode $(+-00)$. For this example, the decay to two pions and a photon, to three pions or to a pion and Dalitz pair (all with lower barrier factors than the four pion mode) might indeed overtake the strong decay rate. While the quantitative importance of such factors will depend strongly on the energy release in the process, it should be clear that the appearance of a high L serves in many cases as a warning that a particular decay may be reduced in intensity, so that electromagnetic channels must be considered simultaneously.

In using Tables III and IV to estimate branching ratios for competitive modes, it must be remembered that we have not included decays of the initial bosons into lighter ones (such as η, ζ) plus pions or gamma rays. It hardly seems worthwhile at present to include such tables since the quantum numbers of some of the lighter bosons are still very much in doubt, and also because Tables I and II show how to construct pion states which can be combined with those of heavier bosons by straightforward methods.

We could illustrate the uses of the decay tables by further hypothetical examples. However, their usefulness has already been demonstrated in an examination

TABLE VII. Strong interaction decay matrix elements for bosons with $T=1$ and $J\leqslant 1$ into $N\leqslant 4$ pions. For explanation of symbols, see Table V.

J^{PG}	(Pion charges) L	Matrix
0^{++}	$(++--)2$	$2(\omega_1+\omega_2-\omega_3-\omega_4)$
	$(+-00)2$	$(\omega_1-\omega_2)$
	$(++-0)2$	$(\omega_1+\omega_2-2\omega_4)$
	$(+000)2$	$(3\omega_1-\omega_2-\omega_3-\omega_4)$
0^{-+}	$(+-00)5$ $(-0++)5$	$a(\omega_3-\omega_4)U$
	$(++--)7$	$b(\omega_1+\omega_2-\omega_3-\omega_4)(p_1-p_2)\cdot(p_3-p_4)U$
	$(+000)7$	$b[\omega_2(p_1-p_2)\cdot(p_3-p_4)+\omega_3(p_1-p_3)\cdot(p_4-p_2)+\omega_4(p_1-p_4)\cdot(p_2-p_3)]U$
0^{--}	$(+-0)0$	1
	$(000)0$	3
	$(++-)0$	2
	$(+00)0$	1
1^{++}	$(+-00)2$ $(-0++)2$	$(\mathbf{p}_1-\mathbf{p}_2)\times(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$
	$(++--)4$	$a\{(\mathbf{p}_1-\mathbf{p}_3)\times(\mathbf{p}_2-\mathbf{p}_4)[\omega_1-\omega_2-\omega_3+\omega_4]-(\mathbf{p}_2-\mathbf{p}_3)\times(\mathbf{p}_1-\mathbf{p}_4)[\omega_1-\omega_2+\omega_3-\omega_4]\}$ $+b(\mathbf{p}_1-\mathbf{p}_2)\times(\mathbf{p}_3-\mathbf{p}_4)(p_1-p_2)\cdot(p_3-p_4)$
	$(+000)4$	$a\{(\mathbf{p}_1-\mathbf{p}_2)\times(\mathbf{p}_3-\mathbf{p}_4)(\omega_3-\omega_4)+(\mathbf{p}_1-\mathbf{p}_3)\times(\mathbf{p}_2-\mathbf{p}_4)(\omega_2-\omega_4)+(\mathbf{p}_1-\mathbf{p}_4)\times(\mathbf{p}_2-\mathbf{p}_3)(\omega_2-\omega_3)\}$ $-2b\{\mathbf{p}_2\times\mathbf{p}_3(p_1-p_4)\cdot(p_2-p_3)+\mathbf{p}_2\times\mathbf{p}_4(p_1-p_3)\cdot(p_2-p_4)+\mathbf{p}_3\times\mathbf{p}_4(p_1-p_2)\cdot(p_3-p_4)\}$
1^{+-}	$(+-0)1$ $(++-)1$ $(00+)1$	$\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3$
1^{-+}	$(+-)1$ $(+0)1$	$a(\mathbf{p}_1-\mathbf{p}_2)$
	$(++--)1$	$2b(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$
	$(+-00)1$	$b(\mathbf{p}_1-\mathbf{p}_2)$
	$(++-0)1$	$b(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_4)$
	$(+000)1$	$b(3\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$
1^{--}	$(+-0)4$ $(++-)4$ $(00+)4$	$(\omega_1-\omega_2)\Lambda$

of the possible quantum-number assignments for the η and ζ mesons,[13] and we refer the reader to this article.

IV. MATRIX ELEMENTS

For a detailed analysis of the boson decay modes, the explicit forms of the matrix elements are required. For three pions such forms have repeatedly been written down in the literature[14] in those cases where they were needed. In the expectation that a general table will prove useful, we present a more complete collection of matrix elements in this section.

The tables presented here give the matrix elements for strong decay into $N\leqslant 4$ pions, and for electromagnetic (e^2) decays (i.e., with one virtual quantum) for $N\leqslant 3$. In the two cases for which there can be no decay with $N\leqslant 3$, the $N=4$ electromagnetic elements have been listed.

For each J^{PG}, each N, and each charge specification we have included all amplitudes with the lowest possible L value; the only exception to this rule is that configurations $(\pi^0)^3$ and $(\pi^0)^4$ have been omitted unless they compete with some mode containing charged pions.

Tables V and VI give the strong and e^2 matrix elements for an initial $T=0$ boson, and Tables VII and VIII do the same for $T=1$. The electromagnetic tables differ in the two cases for two reasons: firstly, $G=C$ for $T=0$ and $G=-C$ for $T=1$, $T_z=0$; and secondly, the selection rule $\Delta T\leqslant 2$ (valid for one virtual quantum) imposes more restrictions on the $T=0$ than on the $T=1$ initial state.

Although the form of all matrix elements refers explicitly to the Lorentz frame in which the decaying

[13] E. M. Henley and B. A. Jacobsohn, Phys. Rev. **127**, 1829 (1962). No distinction between L and Λ was made in that paper. *Note added in proof:* The accumulation of recent evidence points to 0^{-+} for the η, rather than to 0^{--}. See, for example, H. Foelsche, E. C. Fowler, H. L. Kraybill, J. R. Sanford, and D. Stonehill, Phys. Rev. Letters **9**, 223 (1962).

[14] See, for example, reference 2.

particle is at rest, each entry is relativistically correct. In some cases this may not be immediately apparent, since we have used relations of which the following is typical:

$$\mathbf{p}_1\times\mathbf{p}_2\omega_3+\mathbf{p}_2\times\mathbf{p}_3\omega_1+\mathbf{p}_3\times\mathbf{p}_1\omega_2=(\mathbf{p}_1\times\mathbf{p}_2)(\textstyle\sum_j\omega_j)$$
$$=\tfrac{1}{3}M(\mathbf{p}_1\times\mathbf{p}_2+\mathbf{p}_2\times\mathbf{p}_3+\mathbf{p}_3\times\mathbf{p}_1),$$

and the irrelevant constant $\frac{1}{3}M$ has been dropped.

In several instances, the decay amplitudes included here are not unique. One of the terms for $T=0$, 1^{-+} corresponds to the symmetry class (4) and the other to (2^2); as a consequence, the two components of the matrix interfere in the correlations but not in the total decay rate. On the other hand, the fact that both terms of the matrix for the $L=4$ entries of the $T=1$, 1^{++} boson belong to the same symmetry class (3,1) means that interference will occur in the total decay rate as well as in the correlations.

A brief explanation of the relative normalizations is needed. Where there is no relation between the entries for a single J^{PG}, this fact has been indicated by writing explicit arbitrary constants. The relative magnitudes in other cases are written in such a form that, in calculating decay rates, for each group of n identical particles one should multiply the integral over phase space by $1/n!$ if this integral is calculated without regard to the identity of the products.

TABLE VIII. Electromagnetic decay matrix elements for bosons with $T=1$ and $J\leqslant 1$ into $N\leqslant 3$ pions, except for the neutral 0^{++}, in which case $N=4$ is the lowest possible. For explanation of symbols, see Table V.

J^{PG}	(Pion charges) L	Matrix
0^{++}	$(++--)2$	$a(\omega_1+\omega_2-\omega_3-\omega_4)$
	$(+-00)2$	$b(\omega_1-\omega_2)$
	$(+0)0$	c
0^{+-}	$(+-)0$	a
	$(00)0$	b
	$(+0)0$	c
0^{-+}	$(+-0)2$	$a(\omega_1-\omega_2)$
	$(++-)0$	b
	$(+00)0$	c
0^{--}	$(+-0)0$	a
	$(000)0$	b
	$(++-)0$	c
	$(+00)0$	d
1^{++}	$(+-0)1$	$a(\mathbf{p}_1-\mathbf{p}_2)$
	$(++-)1$	$b(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)$
	$(00+)1$	$c(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)$
1^{+-}	$(+-0)1$	$a(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)$
	$(++-)1$	$b(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)$
	$(00+)1$	$c(\mathbf{p}_1+\mathbf{p}_2-2\mathbf{p}_3)$
1^{-+}	$(+-)1$	$a(\mathbf{p}_1-\mathbf{p}_2)$
	$(+0)1$	$b(\mathbf{p}_1-\mathbf{p}_2)$
	$(+-0)2$	$c\mathbf{A}$
	$(++-)2$	$d\mathbf{A}$
	$(00+)2$	$e\mathbf{A}$
1^{--}	$(+0)1$	$a(\mathbf{p}_1-\mathbf{p}_2)$
	$(+-0)4$	$b(\omega_1-\omega_2)\mathbf{A}$
	$(++-)4$	$c(\omega_1-\omega_2)\mathbf{A}$
	$(00+)4$	$d(\omega_1-\omega_2)\mathbf{A}$

V. SUMMARY

A pair of tables (III and IV) presented here should serve as a useful guide in estimating branching ratios for the decay of zero strangeness bosons with various quantum numbers via strong and electromagnetic processes, since they include information on the barrier factors in each case together with the other consequences of selection rules. In addition, we have systematically listed explicit forms of the matrix elements for the decay of these bosons into four or fewer pions.