Energy-Density Relation for Nuclear Matter*

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In most previous calculations of nuclear matter the energy has been calculated only at the equilibrium density, which density has been determined by a minimum condition. In the present paper the author's theory of nuclear matter is applied to a study of the complete energy-density relation of nuclear matter, in the neighborhood of the equilibrium density. The emphasis here is not upon duplicating the accepted value for the equilibrium binding energy, but rather upon a study of the leading (diagonal) contribution of the quasi-particle interaction term $g_1(k_1k_2|k_3k_4)$, which is the matrix element of a reaction matrix G_1. It is shown that $g_1(k_1k_2|k_1k_2)$ must be evaluated partly by using observed nucleon-nucleon scattering phase shifts and partly by calculating the close-in behavior of the two-nucleon wave function, and that this second part receives a large contribution from the deuteron state. Curves are given for the dependence of $g_1(k_1k_2|k_1k_2)$ on the density and the center-of-mass momentum. It is also shown that $g_1(k_1k_2|k_1k_2)$ is sensitive to the size of the nucleon repulsive core, but not upon the character of the attraction, when agreement with scattering data has first been achieved. Finally, a comparison of $g_1(k_1k_2|k_1k_2)$ with the prediction of first-order perturbation theory is made.

I. INTRODUCTION

IN the Bethe-Weizsäcker, semiempirical mass formula for atomic nuclei there are four major terms which account for (1) the saturating effect of nuclear forces, (2) the repulsive effect due to unsaturated nuclear bonds at the nuclear surface, (3) the Coulomb repulsion between protons, and (4) the repulsive effect which occurs when the neutron number differs from the proton number. Green[1] has made the most accurate determination of the constants in the semiempirical mass formula and he has obtained a value of -15.8 MeV per nucleon for the average binding energy due to nuclear forces alone [effect (1)]. This is the number which one would like to derive for the theoretical idealization of nuclear matter.

The average density of nuclear matter is another important number. According to the most accurate determinations of nuclear radii by electron scattering experiments,[2] the central density of nuclei (with $A \gtrsim 30$) seems to be essentially constant. The measured value is $\rho \cong 0.2$ nucleon per fermi cubed. Corresponding to this number we define two other numbers d_N and k_N by

$$\rho = 2(3\pi^2)^{-1}k_N{}^3 = d_N{}^{-3}.$$

The average internucleon spacing d_N is then $d_N \sim (1.7)$ F and the Fermi momentum of nuclear matter is $k_N \sim (1.4-1.5)$ F^{-1}. In this paper we use for k_N the value

$$k_N = (1.48)\ \mathrm{F}^{-1}.$$

In a previous paper,[3] a theory of nuclear matter has been presented in which physical quantities are expanded in powers of the function $g_1(k_1k_2|k_3k_4)$, which is the matrix element of a reaction matrix G_1. It is the purpose of this paper to study the leading contribution of this function to the quasi-particle energy-momentum relation $\omega'(k)$, the thermodynamic potential g, and the average energy per particle $\langle E\rangle/\langle N\rangle$. The convergence of the expansions of these quantities is also examined.

We consider only the ground state of nuclear matter, which corresponds to the temperature $T=0$. In Sec. II, the general expressions for $\omega'(k)$, g, and $\langle E\rangle/\langle N\rangle$ to second order in $g_1(k_1k_2|k_3k_4)$ are summarized and written explicitly for the case of infinite nuclear matter.

In III it was shown that to first approximation the function $-g_1(k_1k_2|k_3k_4)$ is the quasi-particle interaction in momentum space. This function is therefore closely related to the scattering amplitude for "quasi-nucleons" in nuclear matter. In this sense it is appropriate to say that the leading term in the energy depends only upon the real part of the "forward scattering amplitude" $g_1(k_1k_2|k_1k_2)$. In fact, for high momentum values, $k \gg k_N$, $g_1(k_1k_2|k_1k_2)$ reduces to the real nucleon forward scattering amplitude $T_0(k_1,k_2)$, which is the correctly weighted sum of $\sin 2\delta_i(k_{12})$ for all the states (i) of the two-nucleon system. This is as one would expect, because in the high momentum limit the distinction between a quasi-nucleon and a real nucleon vanishes. The momentum dependence of the term $T_0(k_1,k_2)$ is determined in Sec. III using the measured phase shifts as deduced from nucleon-nucleon scattering experiments.

The difference between $g_1(k_1k_2|k_1k_2)$ and the $\sin 2\delta_i(k_{12})$ term is the quantity $T_{12}(k_1,k_2)$ which is studied in Secs. IV, V, and VI. For states below the Fermi sea, T_{12} is the same order of magnitude as T_0. This former term depends on the "close-in" behavior of the two-nucleon wave function, and therefore, a particular model for the nuclear interaction must be chosen before T_{12} can be calculated. In Sec. IV an exact expression, Eqs. (41) and (43), for the S-wave part of T_{12} is derived, using the model of a spin-dependent square well outside of an infinite repulsive core. Similarly, in Sec. V another exact expression is derived for

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[1] A. E. S. Green, Revs. Modern Phys. **30**, 569 (1958).

[2] D. G. Ravenhall, Revs. Modern Phys. **30**, 430 (1958).

[3] F. Mohling, Phys. Rev. **124**, 583 (1961), hereafter referred to as III.

T_{12} using the model of a spin-dependent, nonlocal, separable potential outside of an infinite repulsive core. In both cases the parameters of these interactions are fitted with the aid of low-energy S-wave scattering parameters.

A correct calculation of T_{12} must include the contribution of the wave function of the deuteron. This contribution is therefore included in the derivations of Secs. IV and V and its significance is also discussed. In the calculations of Sec. VI it is then shown that the effect of the deuteron state on the quasi-particle energy-momentum relation $\omega'(k)$ is very large.

In Sec. VI curves are given for the momentum dependence of T_{12}. These curves are computed after first making an effective mass approximation to $\omega'(k)$. The quantity T_{12} depends on both the center-of-mass momentum $2Q$ as well as on the density, and curves for these dependences are given. It is shown that T_{12} decreases with increasing density, an effect which helps to achieve saturation in nuclear matter. The dependence of T_{12} on the nuclear interaction is also studied. Thus, for parameters which fit the low-energy scattering experiments, it is shown that T_{12} decreases as the hard-core diameter is increased and that this effect is large. On the other hand, the difference between using a square-well potential and a nonlocal separable attraction is shown to have a very small effect on the momentum dependence of T_{12}. It is therefore suggested that (excepting for the core) the details of the nuclear interaction are not important for nuclear matter calculations, provided that agreement with nucleon-nucleon scattering data is reasonable.

The contribution of $[T_0+T_{12}]$ to $\omega'(k)$, g, and $\langle E\rangle/\langle N\rangle$ is also determined in Sec. V. Using the calculations of this paper, the binding energy is overestimated by a factor of 2 and the equilibrium density occurs at $k_F\cong(1.4)k_N$. This is not a disappointing result, however, because it is argued that the overestimate is primarily due to the omission of the D-wave contribution to T_{12}. Moreover, in Sec. VII the second-order terms in $g_1(k_1k_2|k_3k_4)$ are written down exactly for the S-wave square-well model and it is estimated that their contribution to the properties of nuclear matter is $\lesssim 10\%$ of the linear term.

The success of first-order perturbation theory, as applied to effective central forces, in producing a qualitative understanding of nuclear matter has been somewhat of a mystery, particularly in view of the tensor character of the nuclear force. Since first-order perturbation theory is the same as the first Born approximation to $\sin 2\delta_0$, the calculations of this paper readily lend themselves to a comparison with perturbation theory. Thus, in Sec. VIII, curves which show this comparison are presented for the case of no repulsive core. The remarkable approximate agreement of first-order perturbation theory with the S-wave part of $[T_0+T_{12}]$ is observed, and it is then further noted that the latter quantity has a simple generalization to the case of tensor forces.

II. GENERAL EXPRESSIONS

In a preceding paper,[3] we derived expressions for the thermodynamic quantities and distribution functions of the ground state ($T=0$) of nuclear matter using the methods of quantum statistics. In particular, we showed that the quantities of interest could be expressed as power series developments in the transformed pair function, Eq. (III.12), with a weighting function $\nu'(k)$ in momentum space integrals.

$$\nu'(k)=\exp\beta[g-\omega'(k)]/\{1+\exp\beta[g-\omega'(k)]\},\quad (1)$$
$$\omega'(k)=\omega(k)+\Delta(k),\quad \omega(k)=\hbar^2k^2/2M.$$

An important result of the investigations in III is that for nuclear matter the pair-function expansion is equivalent to an expansion in the function $g_1(k_1k_2|k_3k_4)$ of Eq. (III.13).[4] Moreover, it was shown that nuclear matter can be viewed as a composite of quasi-particles whose energy-momentum relation is $\omega'(k)$, and that to first approximation the interaction potential between these quasi-particles is $-g_1(k_1k_2|k_3k_4)$ [see Eq. (III.78)].

It was also shown that at $T=0$ the weighting function $\nu'(k)$ is simply the Fermi distribution function [but see discussion below Eq. (III.18)].

$$\lim_{T\to 0}\nu'(k)=1 \quad \text{if} \quad k<k_F,$$
$$=0 \quad \text{if} \quad k>k_F, \quad (2)$$

where k_F is related to the density $\rho\equiv\langle N\rangle/\Omega$ by

$$\rho\equiv 2(3\pi^2)^{-1}k_F^3. \quad (3)$$

Finally, the result of Eq. (III.19) is that the thermodynamic potential g is given by

$$g\equiv(\partial\langle E\rangle/\partial\langle N\rangle)|_{S,\Omega}=\omega'(k_F), \quad (4)$$

where $\langle E\rangle$ and $\langle N\rangle$ are, respectively, the average energy and number of particles in a given system. This expression shows that the mean separation energy of a nucleon from nuclear matter, which energy must correspond to the momentum $k=k_F$, is the quasi-particle energy $\omega'(k_F)$.

In the present paper we shall be concerned with the calculation, to leading order in g_1, of the average energy per particle and the quasi-particle energies, and we shall also be interested in the convergence of the energy expansion. In Sec. VI or III we have shown to second order in g_1 that $\langle E\rangle$ and $\omega'(k)$ are related by the Landau

[4] In III, Eqs. (73) and (74), it was shown that $-g_1(k_1k_2|k_3k_4)$ is the antisymmetrized matrix element of a reaction matrix G_1. This same reaction matrix also appears in the Puff-Martin theory of nuclear matter. See R. D. Puff, Ann. Phys. (New York) **13**, 317 (1961) and D. S. Falk and L. Wilets, Phys. Rev. **124**, 1887 (1961).

equations

$$\langle E\rangle=\sum_{k} \nu'(k)\omega(k)+\tfrac{1}{2}\sum_{k_1k_2} \nu'(k_1)\nu'(k_2)u(k_1,k_2), \qquad (5)$$

$$\begin{aligned}\omega'(k_1)&=\delta\langle E\rangle/\delta\nu'(k_1)\\ &=\omega(k_1)+\sum_{k_2} \nu'(k_2)u(k_1,k_2)\\ &\quad+\tfrac{1}{2}\sum_{k_3k_4} \nu'(k_3)\nu'(k_4)\frac{\delta u(k_3,k_4)}{\delta\nu'(k_1)}, \end{aligned}\qquad (6)$$

where[5]

$$\begin{aligned}u(k_1,k_2)&=-g_1(k_1k_2|k_1k_2)+\tfrac{1}{2}\sum_{k_3k_4}[g_1(k_1k_2|k_3k_4)]^2\\ &\times\left\{[1-\nu'(k_3)-\nu'(k_4)]\mathrm{P}\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\right.\\ &\left.-\mathrm{P}\left(\frac{1}{\omega_1'+\omega_2'-\omega_3-\omega_4}\right)\right\}+O(g_1^3). \end{aligned}\qquad (7)$$

Thus, the program of the paper will be to make a detailed investigation of the contribution of the linear term in g_1 to these equations and to make an estimate of the effect of the quadratic term in g_1. We shall only consider the idealization of infinite nuclear matter in which $\langle N\rangle$, $\Omega\to\infty$ while ρ remains finite.

The state labels k_i in Eqs. (5)–(7) refer to momentum, ordinary spin, and isotopic spin quantum numbers; i.e., $k=(\mathbf{k},m,q)$. We now become more explicit by using Eqs. (III.13) and (III.10) to rewrite Eq. (7) for $u(k_1,k_2)$ as follows:

$$\begin{aligned}\sum_{m_2q_2} u(k_1,k_2)&\\ =-(\pi\hbar^2/M)(\Omega k_{12})^{-1}&[T_0^{(1)}(k_1,\mathbf{k}_2)+T_{12}^{(1)}(k_1,\mathbf{k}_2)\\ &+T_{34}^{(2)}(k_1,\mathbf{k}_2)+T_5^{(2)}(k_1,\mathbf{k}_2)]+O(g_1^3), \end{aligned}\qquad (8)$$

where $\mathbf{k}_{12}=\frac{1}{2}(\mathbf{k}_2-\mathbf{k}_1)$, and where the superscript (n) refers to $O(g^n)$ terms. The $T_i^{(n)}(k_1,\mathbf{k}_2)$ are defined by the following set of equations:

$$T_0^{(1)}(k_1,\mathbf{k}_2)\equiv(8\pi^2M/\hbar^2)k_{12}\sum_{m_2,q_2}\tilde{f}_1(k_{12}|k_{12}),$$

$$\begin{aligned}T_{12}^{(1)}(k_1,\mathbf{k}_2)&\equiv(8\pi^2M/\hbar^2)k_{12}\sum_{m_2q_2k_0}\tilde{f}_2(k_{12}|k_0|k_{12})\\ &\times\left\{\mathrm{P}\left[\frac{1}{\omega_1+\omega_2-W_{12}-\omega(k_0)}\right]\right.\\ &\left.-\mathrm{P}\left[\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega(k_0)}\right]\right\},\end{aligned}$$

$$\begin{aligned}T_{34}^{(2)}(k_1,\mathbf{k}_2)&\equiv(2\pi\hbar^2/M)^{-1}\Omega k_{12}\sum_{k_3k_4,m_2q_2}[g_1(k_1k_2|k_3k_4)]^2\\ &\times\left[\mathrm{P}\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)\right.\\ &\left.-\mathrm{P}\left(\frac{1}{\omega_3+\omega_4-\omega_1'-\omega_2'}\right)\right],\end{aligned}$$

$$\begin{aligned}T_5^{(2)}(k_1,\mathbf{k}_2)&\equiv(2\pi\hbar^2/M)^{-1}\Omega k_{12}\sum_{k_3k_4,m_2q_2}[\nu'(k_3)+\nu'(k_4)]\\ &\times[g_1(k_1k_2|k_3k_4)]^2\\ &\times\mathrm{P}\left[\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right]. \end{aligned}\qquad (9)$$

In the expression for $T_{12}^{(1)}(k_1,\mathbf{k}_2)$

$$W_{12}=\tfrac{1}{4}(\hbar^2/M)(\mathbf{k}_1+\mathbf{k}_2)^2,$$

and the summation $\sum_{k_0}$ is over all states of the relative two-nucleon Hamiltonian $H^{(2)}$. The second-order terms $T_{34}^{(2)}$ and $T_5^{(2)}$ will be investigated in further detail in Sec. VII. The third term in Eq. (6) is called the rearrangement energy. It has a leading contribution which is second order in g_1 and we rewrite it as

$$\begin{aligned}\tfrac{1}{2}\sum_{k_3k_4}&\nu'(k_3)\nu'(k_4)\frac{\delta u(k_3,k_4)}{\delta\nu'(k_1)}\\ &=-\tfrac{1}{2}\sum_{k_2k_3k_4}\nu'(k_3)\nu'(k_4)[g_1(k_3k_4|k_1k_2)]^2\\ &\quad\times\mathrm{P}\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)+O(g_1^3)\\ &=-(\pi\hbar^2/M)(\Omega k_{12})^{-1}\sum_{\mathbf{k}_2}T_6^{(2)}(k_1,\mathbf{k}_2)+O(g_1^3), \end{aligned}\qquad (10)$$

where

$$\begin{aligned}T_6^{(2)}(k_1,\mathbf{k}_2)&\equiv(2\pi\hbar^2/M)^{-1}\Omega k_{12}\sum_{k_3k_4,m_2q_2}\nu'(k_3)\nu'(k_4)\\ &\times[g_1(k_3k_4|k_1k_2)]^2\\ &\times\mathrm{P}\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right). \end{aligned}\qquad (11)$$

The quantity $T_6^{(2)}$ will also be investigated in greater detail in Sec. VII.

We have assumed with Eq. (2) that the Fermi surface for nuclear matter is the same for all four spin projections, or in other words that the quasi-particle potential energy $\Delta(k)$, of Eq. (1), is spin independent. This will be true if the $T_i^{(n)}(k_1,\mathbf{k}_2)$ of Eqs. (9) and (11) are also spin independent (of m_1 and q_1), and we shall henceforth assume that they are. It is convenient for the subsequent analysis to introduce a coordinate change for the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$.

$$\begin{aligned}&\mathbf{l}_{12}=(2k_N)^{-1}(\mathbf{k}_2-\mathbf{k}_1)=k_N^{-1}\mathbf{k}_{12},\quad \mathbf{l}_1=k_N^{-1}\mathbf{k}_1,\\ &\mathbf{Q}_{12}=(2k_N)^{-1}(\mathbf{k}_2+\mathbf{k}_1),\\ &k_N=(1.48)\ \mathrm{F}^{-1},\end{aligned}\qquad (12)$$

[5] For large k_1 and k_2 values, such that $[\omega'(k_1)+\omega'(k_2)]>$ (binding energy of deuteron), Eq. (7) is incorrect and Eq. (III.39) must be used for $u(k_1,k_2)$. In this paper we shall not be interested in this high region of momentum values, and therefore we have only given the simpler form (7) for $u(k_1,k_2)$.

where we have arbitrarily chosen a momentum scale in units of k_N since we know that for nuclear matter $k_F \cong k_N$. We also introduce the parameter

$$x \equiv k_F/k_N. \tag{13}$$

We next redefine the $T_i^{(n)}$ in terms of the coordinates (12):

$$T_i^{(n)}(k_1,\mathbf{k}_2)=T_i^{(n)}(\mathbf{k}_1,\mathbf{k}_2)=T_i(\mathbf{l}_{12},\mathbf{Q}_{12}), \tag{14a}$$

and we finally make the approximation

$$T_i(\mathbf{l},\mathbf{Q})\cong T_i(l,Q). \tag{14b}$$

This approximation, i.e., that the $T_i(\mathbf{l},\mathbf{Q})$ depend only upon the magnitudes of $\mathbf{l}$ and $\mathbf{Q}$, will be shown in Sec. VI to be quite accurate for the purposes of this paper. In the case of $T_0(l,Q)$, Eq. (14b) is not an approximation but an identity.

With the aid of Eqs. (8), (10), and (12)–(14), it is possible to rewrite the terms of Eq. (6) in the following manner:

$$\begin{aligned}\omega'(l_1,x)&=[l_1^2+A^{(P)}(l_1,x)+A^{(R)}(l_1,x)]E_N,\\ E_N&=\hbar^2k_N^2/2M=45.3 \text{ MeV},\end{aligned} \tag{15}$$

where $A^{(P)}(l_1,x)$ is the second term in (6), (due to pair correlations) and $A^{(R)}(l_1,x)$ is the third or rearrangement term (due to higher order correlations). Equation (15) for $\omega'(l_1,x)$ is a rather complex integral equation since the $A(l_1,x)$ themselves have a functional dependence on ω'. In the limit of infinite volume the two quantities $A^{(P)}(l_1,x)$ and $A^{(R)}(l_1,x)$ are

$$\begin{aligned}A^{(P)}(l_1,x)&\equiv\lim_{\Omega\to\infty}\sum_{k_2}\nu'(k_2)u(k_1,k_2)\\ &=[A_0^{(P)}(l_1,x)+A_{12}^{(P)}(l_1,x)\\ &\quad+A_{34}^{(P)}(l_1,x)+A_5^{(P)}(l_1,x)]+O(g_1^3),\\ A^{(R)}(l_1,x)&\equiv\tfrac{1}{2}\lim_{\Omega\to\infty}\sum_{k_3k_4}\nu'(k_3)\nu'(k_4)\frac{\delta u(k_3,k_4)}{\delta\nu'(k_1)}\\ &=A_6^{(R)}(l_1,x)+O(g_1^3).\end{aligned} \tag{16}$$

After replacing the integration variables d^3k_2 by the coordinates (12), one obtains for each of the "pair" terms the general expression

$$\begin{aligned}A_i^{(P)}(l_1,x)&=-4(\pi l_1)^{-1}\Bigg[\int_0^{\frac{1}{2}(x-l_1)}dl\int_{|l_1-l|}^{l_1+l}QdQ\;T_i(l,Q)\\ &\quad+\int_{\frac{1}{2}(x-l_1)}^{\frac{1}{2}(x+l_1)}dl\int_{|l_1-l|}^{[\frac{1}{2}(l_1^2+x^2)-l^2]^{1/2}}QdQ\;T_i(l,Q\Bigg] \quad\text{if } l_1<x,\\ &=-4(\pi l_1)^{-1}\int_{\frac{1}{2}(l_1-x)}^{\frac{1}{2}(l_1+x)}dl\\ &\quad\times\int_{(l_1-x)}^{[\frac{1}{2}(l_1^2+x^2)-l^2]^{1/2}}QdQ\;T_i(l,Q) \text{ if } l_1>x.\end{aligned} \tag{17}$$

Similarly, for the "rearrangement" term one obtains

$$\begin{aligned}A_6^{(R)}(l_1,x)&=-4(\pi l_1)^{-1}\Bigg[\int_0^{x-l_1}dl\int_{|l_1-l|}^{l_1+l}QdQ\;T_6(l,Q)\\ &\quad+\int_{x-l_1}^{x+l_1}dl\int_{|l_1-l|}^{x}QdQ\;T_6(l,Q)\Bigg] \quad\text{if } l_1<x,\\ &=-4(\pi l_1)^{-1}\int_{l_1-x}^{l_1+x}dl\int_{|l_1-l|}^{x}QdQ\;T_6(l,Q) \quad\text{if } l_1>x,\end{aligned} \tag{18}$$

where according to Eq. (11), $T_6(l,Q)\equiv 0$ if $Q>x$.

According to Eqs. (5), (3), and (16) the expression for the ground-state energy per particle can be written

$$\frac{\langle E\rangle}{\langle N\rangle}\underset{\Omega\to\infty}{\longrightarrow}\left\{\tfrac{3}{5}x^2+\tfrac{3}{2}\int_0^x l^2dl\,A^{(P)}(l,x)\right\}E_N, \tag{19}$$

where $A^{(P)}(l,x)$ depends only upon the magnitude of $\mathbf{l}$. Thus, both the ground-state energy $\langle E\rangle$ and the quasi-particle energies $\omega'(k)$ have been expressed in terms of the $A_i(l,x)$ of Eqs. (16)–(18), which are, in turn, related to the $T_i^{(n)}(\mathbf{k}_1,\mathbf{k}_2)$ of Eqs. (9) and (11). The remainder of this paper is devoted to the details of the calculations of these latter functions.

III. EXPERIMENTAL DETERMINATION OF T_0

The first of the $T_i^{(n)}$ which we shall investigate is the function $T_0^{(1)}(\mathbf{k}_1,\mathbf{k}_2)=T_0(l_{12})$ of Eq. (9). According to Eq. (III.11), the most general expression for $T_0(l_{12})$ for the case of a spin-independent interaction is

$$T_0(l_{12})=2\sum_{L=0}^{\infty}(2L+1)[4-(-1)^L]\sin 2\delta_L(l_{12}) \quad\text{(for spin-independent forces)}, \tag{20a}$$

where $\delta_L(l_{12})$ is the phase shift for the Lth partial wave as a function of the dimensionless relative momentum l_{12}, defined by (12). Of course, the forces between nucleons are extremely spin dependent. In this case, the expression for $T_0(l_{12})$ can be shown to have the form

$$T_0(l_{12})=\sum_{\text{all states},i}(2J_i+1)(2T_i+1)\sin 2\delta_i(l_{12}) \quad\text{(for spin-dependent forces)}, \tag{20b}$$

where J_i and T_i are, respectively, the total angular momentum and isotopic spin of the ith state of the two-nucleon system. We see that the contribution of $T_0(l_{12})$ to the single-particle potential is spin independent in *both* of the above cases, as we have assumed it to be in Sec. II.

The phase shifts in Eq. (20b) are those corresponding to the diagonal representation of the two-nucleon

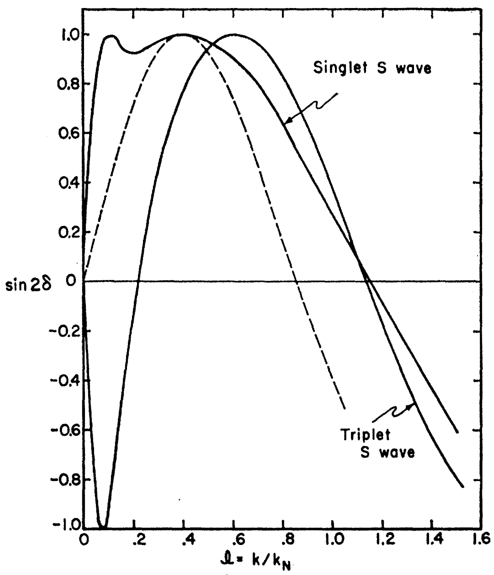

FIG. 1. Curves of $\sin 2\delta(l)$ as a function of $l=k/k_N\cong(0.074)\times(E_{lab})^{1/2}$. Shown are the experimentally determined triplet and singlet S-wave curves. The low-energy effective-range parameters corresponding to these curves are not the commonly accepted values given at the beginning of Sec. VI. The dashed curve is a plot of $\sin 2\delta_0(l)$ corresponding to the square-well resonance model of Eq. (46) ($k_N=1.48$ F^{-1}).

Hamiltonian $H^{(2)}$. This has been called the Blatt-Biedenharn representation[6] in the literature. Thus, in order to determine $T_0(l)$ we have only to use the Blatt-Biedenharn nuclear phase shifts which have been deduced from nucleon-nucleon scattering experiments.[7] In Fig. 1 are shown the curves of $\sin 2\delta_i(l)$ for the singlet and triplet S-wave states. For the case of an incident nucleon of energy E_{lab} scattered by a target nucleon at rest, the parameter l is given by

$$l=(\hbar k_N)^{-1}(\tfrac{1}{2}ME_{lab})^{1/2}\cong 0.074(E_{lab})^{1/2}, \qquad (21)$$

with E_{lab} given in MeV. Thus, phase shifts corresponding to laboratory scattering energies up to $E_{lab}\sim 300$ MeV, are represented by the curves of Fig. 1.

An important feature of the S-wave curves of Fig. 1 is that their sum exhibits a strong maximum at $l\sim\frac{1}{2}$. This means that their contribution to the energy of nuclear matter via Eqs. (19), (17), and (20b) exhibits a minimum at a density roughly equal to the observed nuclear density. Moreover, a numerical calculation demonstrates that the S-wave contribution of $T_0(l)$ to the nuclear binding energy is $\sim$25 MeV at $k_F\cong k_N$. We conclude that $T_0(l)$ is a very important term and that it must be precisely calculated if one is to achieve a quantitative understanding of the energy-density relation for nuclear matter.

In Fig. 2 are plotted the experimentally determined S-wave contribution to $T_0(l)$ (curve A) and the sum of the S-, P-, D-, and F-wave contributions to $T_0(l)$ (curve B). We see that only the S-wave part of $T_0(l)$ contributes for $l\lesssim 0.4$, which corresponds to a laboratory energy $E_{lab}\lesssim 30$ MeV. In the region $0.4\lesssim l\lesssim 1.0$, P, D, and F waves also make important contributions to $T_0(l)$ and they must all be included. For the region $l>1.0$, even higher angular momentum states become important, and curve B cannot be considered to be very accurate for $l>1.0$. One finds from an examination of these higher states (whose contribution can be estimated using the one-pion-exchange potential)[8], that their effect is mainly to lower the high-momentum values ($l\gtrsim 1.0$) of $T_0(l)$.

The effect on the energy per particle of nuclear matter when one uses curve B, instead of curve A, for the contribution of $T_0(l)$ is to shift the calculated minimum energy value to a higher density ($k_F\cong 1.5k_N$) than the observed density ($k_F\cong k_N$). (This is primarily a D-wave effect, since the P and F wave contributions in the region $0.4<l<1.0$ tend to cancel each other.) It is therefore clear that at least one other term besides $T_0(l)$ must have an important effect in the quantitative

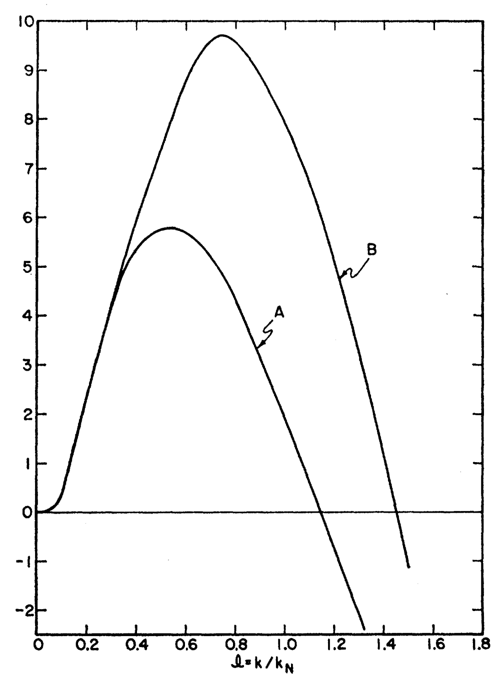

FIG. 2. Curve A: the S-wave part of $T_0(l)$ as calculated from experimental phase shifts. Curve B: the total of the S-, P-, D-, and F-wave contributions to $T_0(l)$ as calculated from experimental phase shifts ($k_N=1.48$ F^{-1}).

[6] J. M. Blatt and L. C. Biedenharn, Phys. Rev. **89**, 399 (1952).

[7] The most recent and accurate determination of the nuclear phase shifts has been made by Breit and co-workers. Their phase-parameter representations are reported in G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960), and in M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **122**, 1620 (1961). References to previous phase-shift analyses are given in the first of these two papers. The results of the second paper were unfortunately received too late to be included in the calculation of $T_0(l)$.

[8] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

calculation of the energy-density relation of nuclear matter. In fact, the only other important term is T_{12} of Eq. (9), as we shall show in Secs. VI and VII.

For calculational purposes it is useful to have an analytic expression for $T_0(l)$. For this reason we have made a least-squares fit of a fourth-order polynomial to curve B of Fig. 2. The result,

$$T_0(l)=-(1.267)+(13.88)l+(26.44)l^2 -(45.14)l^3+(14.26)l^4, \quad (22)$$

is a close approximation for the entire region $0.1<l<1.5$, and for $l<0.1$ the contribution of $T_0(l)$ to physical quantities is negligible. Equation (22) has been used in the calculations discussed in the second half of Sec. V.

IV. SQUARE WELL PLUS REPULSIVE CORE MODEL

In III, it was emphasized that the function $g_1(k_1k_2|k_3k_4)$ which appears in Eq. (7) can be written explicitly in terms of the eigenfunctions of the two-nucleon Hamiltonian $H^{(2)}$. Then, in the previous section we have shown that the function $T_0^{(1)}$ of Eq. (9) can be calculated by knowing only the asymptotic behavior of the two-nucleon wave functions. This fortunate circumstance does not hold for the calculation of any of the other $T_i^{(n)}$, for which the details of the two-nucleon wave function must be known. Now, according to Eq. (III.62) the two-nucleon wave functions for continuum states can be expressed in terms of the reaction matrices $\langle k|A^{(i)}|k_0\rangle$, each of which is equal to $\tan\delta_i(k_0)$ when $k=k_0$ (on the energy shell). It is therefore convenient to express the function $g_1(k_1k_2|k_3k_4)$ directly in terms of reaction matrices, and this has been done in Eqs. (III.10), (III.11), and (III.13) for the case of a spin-independent interaction.

In this section we shall neglect the complicated details of nuclear forces and use the very simple approximation of an S-wave square-well attraction outside of an infinite repulsive core. Thus, we consider the potential $U(r)=(M/\hbar^2)V(r)$ defined by

$$\begin{aligned} U(r)&=+\infty \quad &&\text{for} \quad r<a,\\ &=-U_0 &&\text{for} \quad a<r<b,\\ &=0 &&\text{for} \quad r>b, \end{aligned} \quad (23)$$

and which vanishes for angular momentum states with $L\neq 0$. In this case one finds[9] for the reaction matrix $\langle k|A^{(0)}|k_0\rangle$ the result

$$\begin{aligned} \langle k|A^{(0)}|k_0\rangle &= -(k_0/k)(y_0^2-k^2)^{-1}[y_0\cos k_0b\cos y_0(b-a) + k_0\sin k_0b\sin y_0(b-a)]^{-1}\\ &\quad\times\{y_0(k_0^2-k^2)\sin k_0a+U_0[y_0\sin kb\cos y_0(b-a) - k\cos kb\sin y_0(b-a)]\}, \quad (24)\\ y_0&=(U_0+k_0^2)^{1/2}. \end{aligned}$$

[9] The author is indebted to Sigurd Larsen of Columbia University for providing his calculation of the reaction matrix (24). The calculation is most easily performed with the aid of Eq. (73) in F. Mohling, Phys. Rev. 122, 1043 (1961).

In the limit $U_0\to 0$, this result approaches the well-known hard-core expression

$$\langle k|A^{(0)}|k_0\rangle=-k_0\sin ka(k\cos k_0a)^{-1}.$$

We next introduce the dimensionless parameters

$$A=k_Na,\quad B=k_Nb,\quad \mu^2=k_N^{-2}U_0, \quad (25)$$

and

$$\begin{aligned} l_0&=k_N^{-1}k_0, \quad Y_0=k_N^{-1}y_0=(l_0^2+\mu^2)^{1/2},\\ l&=k_N^{-1}k, \quad Y=(l^2+\mu^2)^{1/2}. \end{aligned} \quad (26)$$

In terms of these parameters, the expressions for the reaction matrix times $\cos\delta_0$ and the reaction matrix times $\cos^2\delta_0$ can be written as

$$ll_0^{-1}\langle l|A^{(0)}|l_0\rangle\cos\delta_0(l_0) = -[l_0^2+\mu^2\cos^2Y_0(B-A)]^{-1/2}N_1(l,l_0), \quad (27)$$

$$ll_0^{-1}\langle l|A^{(0)}|l_0\rangle\cos^2\delta_0(l_0) = -[l_0^2+\mu^2\cos^2Y_0(B-A)]^{-1}N_1(l,l_0)N_2(l_0), \quad (28)$$

where

$$\begin{aligned} N_1(l,l_0)&\equiv(Y_0^2-l^2)^{-1}\{Y_0(l_0^2-l^2)\sin lA +\mu^2[Y_0\sin lB\cos Y_0(B-A) - l\cos lB\sin Y_0(B-A)]\}\\ N_1(l,l)&=Y\sin lB\cos Y(B-A) -l\cos lB\sin Y(B-A), \end{aligned} \quad (29)$$

and

$$N_2(l)\equiv Y\cos lB\cos Y(B-A)+l\sin lB\sin Y(B-A). \quad (30)$$

The expressions (27) and (28) are needed for the general calculation of $g_1(k_1k_2|k_3k_4)$ using the square well plus repulsive core model. Thus, if Eq. (III.11) for the continuum contribution to $\bar{f}_2(k_{12}|k_0|k_{12})$ is substituted into the expression for $T_{12}^{(1)}(\mathbf{k}_1,\mathbf{k}_2)$, Eq. (9), then one obtains for the case of a spin-independent S-wave force the result

$$\begin{aligned} T_C(\mathbf{k}_1,\mathbf{k}_2) &= (24/\pi)(\hbar^2/M)k_{12}\int_0^\infty dk_0[\langle k_{12}|A^{(0)}|k_0\rangle\cos\delta_0(k_0)]^2\\ &\quad\times\left\{\mathrm{P}\left[\frac{1}{\omega_1+\omega_2-W_{12}-\omega(k_0)}\right] -\mathrm{P}\left[\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega(k_0)}\right]\right\}. \end{aligned} \quad (31)$$

We introduce the dimensionless parameters of (12) and use Eq. (14a) to rewrite this expression as

$$T_C(l,\mathbf{Q})=(24/\pi)l\int_0^\infty dl_0[\langle l|A^{(0)}|l_0\rangle\cos\delta_0(l_0)]^2 \times\left[\mathrm{P}\left(\frac{1}{l^2-l_0^2}\right)+\mathrm{P}\left(\frac{1}{l_0^2+\epsilon}\right)\right], \quad (32)$$

where

$$\epsilon(\mathbf{l},\mathbf{Q}) \equiv Q^2 - (M/\hbar^2)[\omega'(\mathbf{Q}-\mathbf{l})+\omega'(\mathbf{Q}+\mathbf{l})] \quad (33)$$
$$\to -l^2 \quad \text{for} \quad \omega'=\omega.$$

Nuclear forces give rise not only to continuum states, but also to a bound triplet S state, i.e., the deuteron. This bound state, in turn, makes a contribution to the function $g_1(k_1k_2|k_3k_4)$ which must be included in a calculation of the properties of nuclear matter. According to Eq. (III.11) the contribution of the deuteron state to $T_{12}^{(1)}$, Eq. (9) is

$$T_B(\mathbf{k}_1,\mathbf{k}_2) = 6(\hbar^2/M)k_{12}^{-1}(k_{12}^2+\gamma^2)^2 k_N[\phi_0^2(k_{12})+\phi_2^2(k_{12})] \times\left\{ \mathrm{P}\left[\frac{1}{\omega_1+\omega_2-W_{12}-\omega(\gamma)}\right] - \mathrm{P}\left[\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega(\gamma)}\right]\right\}, \quad (34)$$

where $\omega(\gamma)=-(\hbar^2/M)\gamma^2=2.226$ MeV is the binding energy of the deuteron. The functions $\phi_0(k)$ and $\phi_2(k)$ are the Fourier transforms of the S and D components, respectively, of the wave function of the deuteron, normalized according to the convention

$$k_N^{-1}\int_0^\infty dk[\phi_0^2(k)+\phi_2^2(k)]=\pi/2. \quad (35)$$

One may inquire as to the meaning of such a term as (34), and we propose the following story in answer to such an inquiry. The quasi-particles of nuclear matter represent the way in which nature arranges a collection of nucleons so as to exhibit collective or normal modes in nuclear matter. That is to say, the effective single-particle eigenstates or quasi-particle states are these normal modes, and in momentum space nature arranges these states so that their distribution resembles a free-Fermion momentum distribution. On the other hand, the energy-momentum relation for the quasi-particles, $\omega'(k)$ of Eq. (1), differs radically from that of free Fermions. The question of the role of the deuteron state is then related to the determination of $\omega'(k)$. The point is that the strong pair correlations in nuclear matter, which determine the quasi-particle properties, are related much more closely to the *wave function* of a pair of isolated nucleons than they are to the potential which exists between a pair of isolated nucleons (which is, of course, quite singular). It is the Fourier transform of two-nucleon wave functions, then, which is most important in understanding the properties of nuclear matter. With regard to the deuteron state, one may imagine that if two nucleons find a "relative" wave number available for occupation in the deuteron state, then it will be energetically desirable for them to get into this state. Hence, we should expect, and shall presently show, that the deuteron state is very important in the determination of the ground-state properties of nuclear matter.

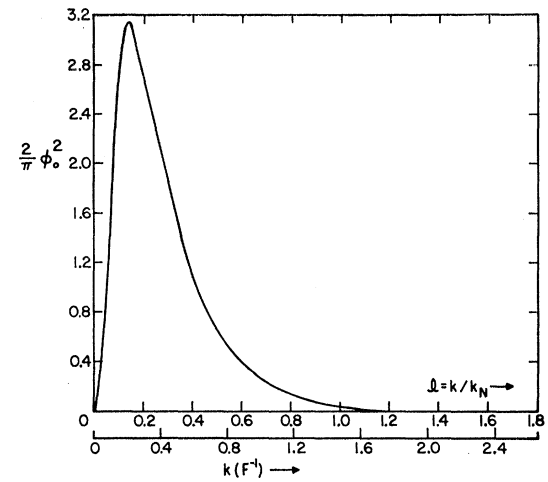

FIG. 3. The momentum distribution of the deuteron, as calculated for the square-well-repulsive-core model.

We wish to calculate T_B explicitly, and for this purpose we again use the square-well potential (23), thereby neglecting the D-state component ϕ_2 of the deuteron wave function. The expression for ϕ_0 in terms of the dimensionless parameter l_{12} of (12) is

$$\phi_0(l_{12})=\sqrt{2}\alpha\nu^{1/2}[1+\nu(B-A)]^{-1/2}(\alpha^2-l_{12}^2)^{-1} \times\{\sin l_{12}A+\mu(l_{12}^2+\nu^2)^{-1}(\nu\sin l_{12}B+l_{12}\cos l_{12}B)\}, \quad (36)$$

where

$$\nu=k_N^{-1}\gamma=0.157, \quad (37)$$
$$\alpha=k_N^{-1}(U_0-\gamma^2)^{1/2}.$$

In Fig. 3 we have plotted the momentum distribution $(2/\pi)\phi_0^2(l_{12})$ of the deuteron as a function of the relative momentum of the nucleons, using the triplet parameters for $a=0.4$ F from Table I. The distribution is characterized by a sharp maximum at $l_{12}=\nu=0.157$ which corresponds to the inverse of the "size of the deuteron." The curve agrees very closely with the momentum distribution given by the frequently used Hulthén wave function of the deuteron [which is Eq. (53) with $a=0.0$].

Use of the definition (33) together with Eq. (14a) yields for T_B the expression

$$T_B(\mathbf{l},\mathbf{Q})=6l^{-1}(l^2+\nu^2)^2\phi_0^2(l) \times[P(\epsilon-\nu^2)^{-1}+(l^2+\nu^2)^{-1}]. \quad (38)$$

Explicit computation, using the effective mass approximation (57) for ϵ, reveals that T_B is the same order of magnitude as the continuum term T_0 of Sec. III. The second term in the brackets gives the dominant (order of magnitude larger than the first) contribution to T_B. Thus, we see that the contribution of the deuteron state to $\omega'(l_1)$ and $\langle E\rangle/\langle N\rangle$ is very important.

TABLE I. Nuclear S-wave parameters, Eqs. (23) and (48). The values given in bold type represent the "best" fits in each column.

	Singlet		Triplet	
Hard-core diameter a (F)	Square well, range b (F)	Square well, depth U_0 (F^{-2})	Square well, range b (F)	Square well, depth U_0 (F^{-2})
0.0	2.494	0.3655	2.494[a]	0.6162[a]
0.2	**2.286**	**0.5295**	2.286[a]	0.8226[a]
0.4	1.69[b]	1.421[b]	**1.69**	**1.875**
	Separable attraction, cutoff momentum C' (F^{-1})	Separable attraction, strength σ	Separable attraction, cutoff momentum C' (F^{-1})	Separable attraction, strength σ
0.2	**1.434**	**0.9448**		
0.4[c]	1.79	0.9557	2.295	1.2121
0.45			**2.479**	**1.1957**

[a] Values correspond to $a_3=5.61$ F.
[b] Values correspond to $r_{01}=2.1$ F.
[c] Values in this row do not give good agreement with the curves of Fig. 1 for $l>0.8(E_{\rm lab}>100$ MeV.)

In the Appendix of III, we have shown that the use of the idealization of an infinite repulsive core for the nuclear interaction leads to an additional term in $g_1(k_1k_2|k_3k_4)$,[10] thereby resulting in the contribution

$$T_{HC}^{(1)}(\mathbf{k}_1,\mathbf{k}_2)=(8\pi M/\hbar^2)k_{12}\sum_{m_2q_2}\tilde{f}_3(k_{12}|k_{12})\times(\omega_1+\omega_2-\omega_1'-\omega_2'),\quad(39)$$

which we shall henceforth include with $T_{12}^{(1)}$. From Eqs. (A3) and (A11) of III, we obtain for the S-wave part of T_{HC} the expression

$$T_{HC}(\mathbf{l},\mathbf{Q})=-6l^{-2}(l^2+\epsilon)(lA-\tfrac{1}{2}\sin 2lA),\quad(40)$$

where $\epsilon(\mathbf{l},\mathbf{Q})$ is given by Eq. (33) and we have again used Eq. (14a). The term T_{HC} gives a small, but appreciable, positive contribution to $\omega'(l_1)$ and $\langle E\rangle/\langle N\rangle$.

As has already been observed in connection with Eq. (20b) and Fig. 1, the nuclear force is spin dependent. It also has a tensor character that mixes S and D wave functions, but we shall not consider the complication of tensor forces until the end of Sec. V. For a spin-dependent nuclear force, Eq. (9) for T_{12} can be written as follows:

$$T_{12}(\mathbf{l},\mathbf{Q})=[T_s(\mathbf{l},\mathbf{Q})+T_t(\mathbf{l},\mathbf{Q})]+[\text{higher angular momentum terms}],\quad(41)$$

where

$$T_s(\mathbf{l},\mathbf{Q})=\tfrac{1}{2}[T_C(\mathbf{l},\mathbf{Q})+T_{HC}(\mathbf{l},\mathbf{Q})]_s,$$
$$T_t(\mathbf{l},\mathbf{Q})=[\tfrac{1}{2}T_C(\mathbf{l},\mathbf{Q})+T_B(\mathbf{l},\mathbf{Q})+\tfrac{1}{2}T_{HC}(\mathbf{l},\mathbf{Q})]_t,\quad(42)$$

and where the explicit expressions for T_C, T_B, and T_{HC} have all been written down in this section using the square well plus repulsive core model.

Using the methods of contour integration the integral (32) for T_C can be performed exactly after substituting the explicit expression (27). The final expression for both T_s and T_t is the same, namely,

$$T_{s,t}(\mathbf{l},\mathbf{Q}=-3\sin 2\delta_0(l)+3l^{-2}(l^2+\epsilon)(l^2-Y_\epsilon^2)^{-1}\{(l^2+\epsilon)(\sin 2lA[\tfrac{1}{2}-l^2(l^2-Y_\epsilon^2)^{-1}]-lA)+\mu^2(lB-\tfrac{1}{2}\sin 2lB)\}$$
$$+6\mu^2(l^2-Y_\epsilon^2)^{-1}[\mu^2\cos^2 Y_\epsilon(B-A)-\epsilon]^{-1}\{N_1(l,i\epsilon^{1/2})]Y_\epsilon\cos Y_\epsilon(B-A)\cos lB-\epsilon l^{-1}\sin lB\sin Y_\epsilon(B-A)]$$
$$+l^{-1}Y_\epsilon(l^2-Y_\epsilon^2)^{-1}(l^2+\epsilon)\sin lA[(l^2+\epsilon)\sin lA\cos Y_\epsilon(B-A)\sin Y_\epsilon(B-A)+lY_\epsilon\cos lB\cos Y_\epsilon(B-A)$$
$$-\epsilon\sin lB\sin Y_\epsilon(B-A)]\}-6l^{-1}\epsilon^{1/2}[\mu^2\cos^2 Y_\epsilon(B-A)-\epsilon]^{-1}[N_1(l,i\epsilon^{1/2})]^2\theta(\epsilon),\quad(43)$$

where

$$Y_\epsilon=(\mu^2-\epsilon)^{1/2},\quad(44)$$

$$\theta(\epsilon)=0\quad\text{if}\quad\epsilon<0$$
$$=1\quad\text{if}\quad\epsilon>0,\quad(45)$$

and $N_1(l,i\epsilon^{1/2})$ is given by Eq. (29). Although the expressions for T_s and T_t are formally similar, we shall see that they are quite different when evaluated for realistic nuclear parameters. In particular T_t has a singularity[5] at $\epsilon=\nu^2$, which can be traced directly to the contribution of T_B, Eq. (38). There are no other singularities in either T_s or T_t.

In the literature there has frequently appeared a spin-independent version of the square well plus repulsive core model which we are using. This is the square-well resonance model of Gomes, Walecka, and Weisskopf,[11] in which the attraction gives rise to a "just-bound" state at zero energy. The parameters which these authors have chosen are

$$a=0.4\text{ F},$$
$$b=2.3\text{ F},\quad(46)$$
$$U_0^{1/2}(b-a)=\pi/2,\quad[(\hbar^2/M)U_0\cong 28.3\text{ MeV}].$$

[10] One can show that the function $B(k)$, which appears in Eq. (III.A4), is given by the expansion $B(k)=(3\pi)^{-1}(k_Fa)^3\times[1+O(k_Fa)^2f(k_1/k_F)]$. For $a=0.4$ F we find $B\sim 0.02$, and therefore the contribution of $B(k)$ to the pair functions in the expansion of physical quantities can be neglected to first approximation.

[11] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (New York) 3, 241 (1958).

When one sets $l_0=l$ in Eq. (28), then this expression becomes equal to $\frac{1}{2}\sin 2\delta_0(l)$. Upon using Eqs. (25) and the numbers of (46) in this expression for $\sin 2\delta_0(l)$, one then obtains the dashed line curve shown in Fig. 1. This curve (multiplied by 6) is only a fair approximation to curve A of Fig. 2, which represents the experimentally averaged values of $\sin 2\delta$ for the singlet and triplet S states. One might use the model for the estimates of second order (in g_1) terms presented in Sec. VII, but we do not feel that it is good enough for the calculation of the leading $O(g_1)$ term made in Sec. VI. In particular, it cannot reproduce the important bound-state effect discussed below Eq. (35).

V. SEPARABLE POTENTIAL PLUS REPULSIVE CORE MODEL

It is of interest to investigate how the character of the two-nucleon interaction affects the calculation of the properties of nuclear matter. At first thought, it would seem that the character of the nuclear interaction, i.e., the shape of the potential and its velocity dependence, would have very important effects. At second thought, however, one remembers that the parameters of a phenomenological nuclear interaction must always be fitted to the nucleon-nucleon scattering phase shifts and these phase shifts are closely related to the off-diagonal reaction matrix elements which determine T_C, Eq. (32). This leads one to the hypothesis that the properties of nuclear matter are *not* very sensitive to the character of any two-nucleon interaction which is fitted to experimental phase shifts. The purpose of this section is to repeat the derivations of Sec. IV for a very different S-wave interaction in order that this hypothesis can be quantitatively pursued.

An S-wave interaction which is very different from a square well is a nonlocal, or velocity-dependent attraction. With such an interaction the $L=0$ radial wave equation becomes (for $r>a$)

$$\left(\frac{d^2}{dr^2}+k_0^2\right)\langle r|k_0\rangle=r\int_0^\infty (r')^2dr'\,U_0(r,r')(r')^{-1}\langle r'|k_0\rangle, \quad (47)$$

where the boundary condition due to the repulsive core (which we still assume to be present) is that $\langle r|k_0\rangle=0$ for $r<a$. We shall choose the simplest case of a nonlocal potential, namely, the separable potential of Yamaguchi[12]:

$$rr'U_0(r,r')=-2\sigma C'u(r)u(r'), \quad (48)$$
$$u(r)=C'\exp[-C'(r-a)].$$

The strength σ of the separable attraction has been defined so that a "just bound" state occurs when $\sigma=1$. For the separable interaction of (48), the equations corresponding to (27) and (28) are

$$ll_0^{-1}\langle l|A^{(0)}|l_0\rangle\cos\delta_0(l_0) =(l_0^2+C^2)[l_0^4+2(1+\sigma)C^2l_0^2+(1-\sigma)^2C^4]^{-1/2}\times N_1(l,l_0), \quad (49)$$

$$ll_0^{-1}\langle l|A^{(0)}|l_0\rangle\cos^2\delta_0(l_0) =[l_0^4+2(1+\sigma)C^2l_0^2+(1-\sigma)^2C^4]^{-1}\times N_1(l,l_0)N_2(l_0), \quad (50)$$

where

$$N_1(l,l_0)=2\sigma g(l)g(l_0)[\sin lA+lC^{-1}\cos lA]-[1+\sigma g(l_0)]\sin lA, \quad (51)$$
$$N_2(l)=[l^4+(2+\sigma)l^2C^2+(1-\sigma)C^4]\cos lA+2\sigma lC^3\sin lA,$$

and

$$g(l)=C^2(C^2+l^2)^{-1}, \quad C=k_N^{-1}C'. \quad (52)$$

Similarly, the bound-state wave function $\phi_0(l)$ is given by

$$\phi_0(l)=[2\nu C^{-1}(C+\nu)]^{1/2}(l^2+\nu^2)^{-1}\times\{(C+\nu)g(l)[\sin lA+lC^{-1}\cos lA]-C\sin lA\}, \quad (53)$$

where the experimental value for $\nu=(\sigma^{1/2}-1)C$ is given by Eq. (37).

Upon substituting Eq. (49) into Eq. (32) for T_C, and again using the methods of contour integration, one obtains an explicit expression for T_C. The final expression for T_s or T_t of Eq. (42) is found to be

$$\begin{aligned}T_{s,t}(\mathbf{l},\mathbf{Q}) =&-3l^{-2}(l^2+\epsilon)(lA-\tfrac{1}{2}\sin 2lA)\\ &+12\sigma l^{-1}C[l^4+2(1+\sigma)C^2l^2+(1-\sigma)^2C^4]^{-1}\\ &\times\{[l^2+(\sigma-1)C^2]N_3(l)+[l^2+(1-\sigma)C^2]N_4(l)\}\\ &+12\sigma l^{-1}C[\epsilon^2-2(1+\sigma)C^2\epsilon+(1-\sigma)^2C^4]^{-1}\\ &\times\{[\epsilon+(1-\sigma)C^2]N_3(l)+[\epsilon+(\sigma-1)C^2]N_4(l)\}\\ &+6l\epsilon^{-1/2}[\langle l|A^{(0)}|i\epsilon^{1/2}\rangle\cos\delta_0(i\epsilon^{1/2})]^2\theta(\epsilon), \end{aligned}\quad (54)$$

where

$$N_3(l)=\sigma g^2(l)(C\sin lA+l\cos lA)^2+C^2\sin^2 lA,$$
$$N_4(l)=2Cg(l)(C\sin lA+l\cos lA)\sin lA, \quad (55)$$

and $\theta(\epsilon)$ is given by (45). As with Eq. (43), the only singularity in either T_s or T_t occurs at $\epsilon=\nu^2$ for T_t.

VI. CALCULATIONS OF T_{12} AND THE GROUND-STATE ENERGY

In this section we shall present curves of T_{12}, for a wide range of acceptable parameters corresponding to the S-wave nuclear interactions of Secs. IV and V. In Table I we have made a list of the parameters a, b, and U_0, Eq. (23), for the square-well potential and of a, σ, and C', Eq. (48), for the separable attraction. In each case the parameters have been determined from (1) the binding energy of the deuteron $=2.226$ MeV, (2) the triplet scattering length, $a_3=5.38$ F, (3) the

[12] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

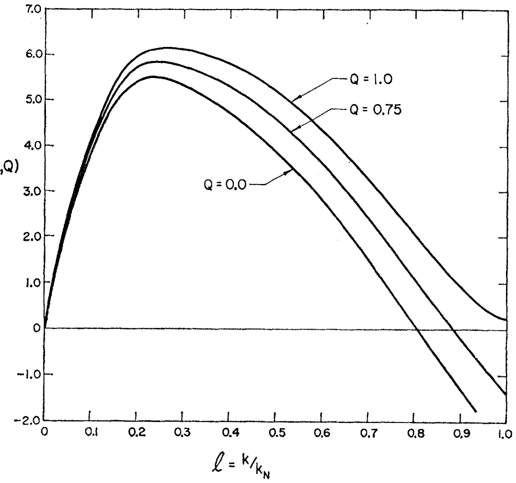

FIG. 4. Q dependence of the S-wave part of the function $T_{12}(l,Q)$ for $x=1.0$. The effective-mass parameters used were those of Table II, and the interaction used was the square-well model of Sec. IV with the $a=0.4$ F parameters of Table I. The singularity at $\epsilon=\nu^2$ occurs for $Q=1.0$ at $l=1.072$.

singlet scattering length, $a_1=-23.68$ F, and (4) the singlet effective range, $r_{01}=2.6$ F.

Agreement with the experimental values of $\sin 2\delta$, Fig. 1, up to $l=1.0$ ($E_{\text{lab}}\sim 200$ MeV) has also been required. For some of the parameter sets presented in Table I, this last requirement has resulted in a poor fit either to the singlet effective range, or to the triplet scattering length as has been indicated. The parameters given in bold type represent the "best" fits in each column, and it is remarkable that these best fits reproduce the entire range of experimental S-wave phase shifts from $E_{\text{lab}}=0$ to 300 MeV. For the square-well model we have, for reasons of simplicity, always used the same range b with both singlet and triplet states for a given value of the hard-core diameter. That the ranges are essentially the same follows from the experimental curves of Fig. 1.

In order to integrate or evaluate numerically the function T_{12} of Eqs. (9) and (41) it is necessary to know the explicit dependence of ϵ, Eq. (33) on the momenta $\mathbf{l}$ and $\mathbf{Q}$. This is precisely the self-consistency problem of the theory, because the $\omega'(l_1)$ are determined solely by the integral equation (15) together with Eqs. (16)–(18). In order to simplify the solution to this integral equation we approximate $\omega'(l_1)$ by a quadratic expression (later justified by Fig. 9)

$$\omega'(l_1,x)\cong\{-U(x)+[M/M^*(x)]l_1^2\}E_N. \qquad (56)$$

This is the effective-mass approximation, in which both U and the effective mass M^* are density-dependent parameters. Upon substituting the approximation (56) into the expression (33) for ϵ, one obtains

$$\epsilon(\mathbf{l},\mathbf{Q})\cong\epsilon(l,Q)=U-(M/M^*)l^2-[(M/M^*)-1]Q^2, \quad (57)$$

where the ratio M/M^* is greater than 1 for nuclear matter. From (57) we see that the effective mass

TABLE II. Effective mass parameters used in the calculations of Sec. VI.

$x=k_F/k_N$	U	M^*/M
0.8	1.930	1.624
1.0	3.039	1.855
1.2	4.226	2.086

approximation is equivalent to the approximation (14b) of Sec. II.

Use of (57) in either of the expressions (43) or (54) makes it possible to perform the integration (17) (numerically) for any given choice of U and M^*. Trial values of these parameters can be chosen and used to evaluate $A_{12}^{(P)}$. Then $A_0^{(P)}$ and $A_{12}^{(P)}$ can be substituted into Eq. (15) for $\omega'(l_1,x)$ and new values of U and M^* can be determined for a recalculation of $A_{12}^{(P)}$. This iteration procedure is, in fact, rapidly convergent and it results in the one-pair approximation to $\omega'(l_1,x)$. That the two-pair terms give only small corrections to the one-pair term is shown in Sec. VII. The effective mass parameters used for the studies of the function T_{12} discussed below are given in Table II. They were obtained by using the above described iteration procedure (see Fig. 9).

Figures 4 through 8 represent numerical computations of the S-wave part of the function T_{12} as given by Eq. (41) and either Eq. (43) or (54). In each case the approximation (57) was used in $T_{12}(l,Q)$, which was then plotted as a function of $l=k/k_N$. The figures show how T_{12} depends on Q, x, a, character of nuclear attraction, and strength of nuclear attraction. We discuss below each of these cases separately, noting first that positive values of T_{12} correspond to an

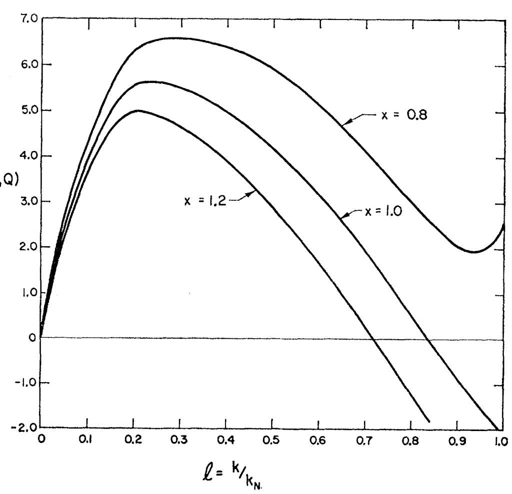

FIG. 5. x dependence of the S-wave part of the function $T_{12}(l,Q)$ for $Q=0.5$. The effective-mass parameters used were those of Table II, and the interaction used was the square-well model of Sec. IV with the $a=0.4$ F parameters of Table I. The singularity at $\epsilon=\nu^2$ occurs for $x=0.8$ at $l=1.038$.

attractive effect, whereas negative values correspond to a repulsive effect.

Q Dependence

In Fig. 4 the Q dependence of the S-wave part of $T_{12}(l,Q)$ is plotted using Eq. (43) together with the $a=0.4$ F parameters from Table I and the effective mass parameters for $x=1.0$ from Table II. It is seen that increasing values of Q correspond to a greater attractive effect. In the literature the center-of-mass effect is often eliminated, for convenience, by setting $Q=0.0$ in calculations. It is seen that this choice greatly underestimates the attraction, and that if any average Q value is to be selected, then it should be $Q\cong 0.75$. Actually, the Q integration limits in Eqs. (17) and (18) show that there is no "best" average Q value which can be selected, independent of l_1 and l, for nuclear matter calculations.

x Dependence

In Fig. 5, the x dependence of the S-wave part of $T_{12}(l,0.5)$ is plotted using Eq. (43) together with the $a=0.4$ F parameters from Table I. It is seen that T_{12} decreases rapidly with increasing density, i.e., with x, as it should if the saturation of nuclear matter is to be explained. The effect shown by these curves is due entirely to the variation of the quasi-particle energy-momentum relation, i.e., of the effective-mass parameters, with density. The effect is not great enough, however, to give saturation at $x\cong 1.0$, and the reason for this will be explained in connection with Fig. 10.

a Dependence

In Fig. 6, curves are plotted which show the variation of the S-wave part of $T_{12}(l,0.5)$ with the diameter of

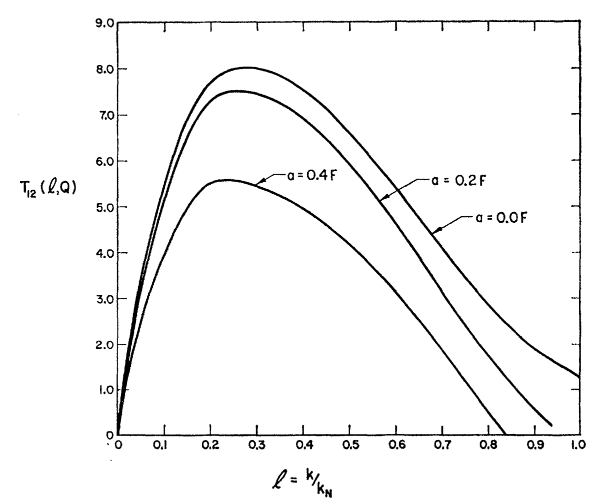

FIG. 6. a dependence of the S-wave part of the function $T_{12}(l,Q)$ for $x=1.0$ and $Q=0.5$. The effective-mass parameters used were those of Table II, and the interaction used was the square-well model of Sec. IV with the parameters of Table I.

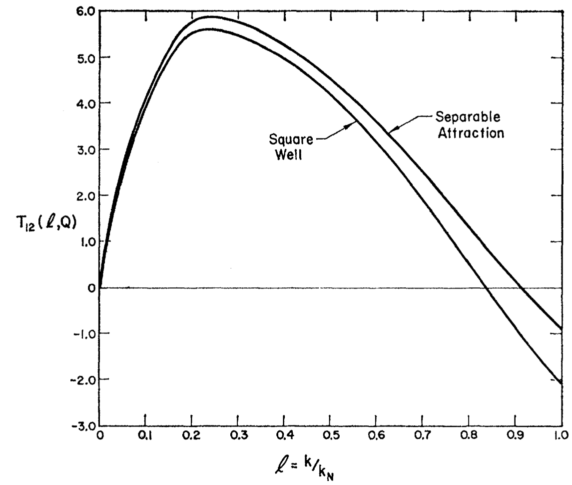

FIG. 7. Effect of the type of nuclear attraction on the S-wave part of the function $T_{12}(l,Q)$ for $x=1.0$ and $Q=0.5$. The effective-mass parameters used were those of Table II, and the interaction parameters used were those corresponding to $a=0.4$ F in Table I.

the nuclear repulsive core. It is to be emphasized that the square-well parameters used to obtain these curves all produce at least fair agreement with the experimental S-wave phase shifts, as has previously been discussed. Therefore, we may conclude from these curves that the properties of nuclear matter, and in particular, the saturation density, are very sensitive to the diameter of the nuclear repulsive core.

We note here that the reason why the $a=0.4$ F curve looks different from the other two curves is because the latter curves correspond to a triplet scattering length $a_3=5.61$ F. The triplet parameters corresponding to $a_3=5.38$ F are $b=2.05$ F and $U_0=0.8465$ F^{-2} for $a=0.0$ and $b=1.87$ F and $U_0=1.195$ F^{-2} for $a=0.2$ F. Using these parameters one obtains poorer agreement with the triplet S-wave phase shifts above $E_{\mathrm{lab}}=100$ MeV than with the parameters of Table I, but the curves of Fig. 6 become uniformly spaced.

Dependence upon the Character of the Nuclear Attraction

The curves of Fig. 7 show how $T_{12}(l,0.5)$ varies (for $a=0.4$ F) when calculated using a separable attraction instead of a square-well potential. The variation is seen to be very small, and, in fact, is no greater than the corresponding variation in the fits to the phase shift curves of Fig. 1. Thus, the hypothesis made at the beginning of Sec. V that the properties of nuclear matter are not very sensitive to the character of any two-nucleon interaction which is fitted to experimental phase shifts has been verified in a particular case. We shall cite this result at the end of the section as evidence for the fact that the tensor character of the nuclear force is not essential to an understanding of nuclear saturation.

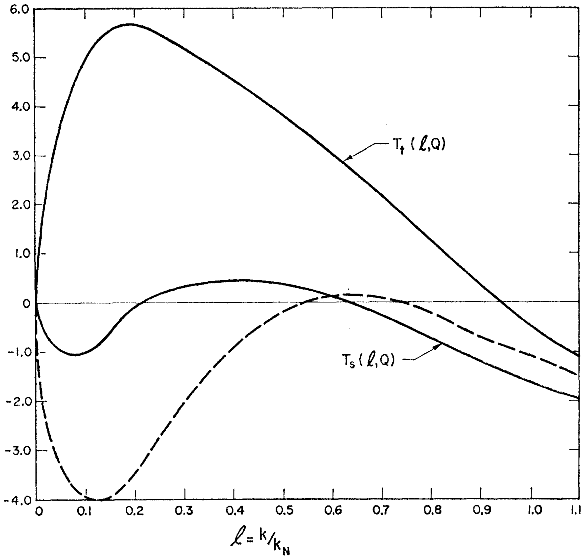

FIG. 8. Curves of $T_s(l,Q)$ and $T_t(l,Q)$ for $x=1.0$ and $Q=0.5$. The effective-mass parameters used were those of Table II, and the interaction used was the square well model of Sec. IV with the $a=0.4$ F parameters of Table I. The sum of these two curves appears in Figs. 5, 6, and 7. The dashed curve is the "continuum" part of T_t.

Dependence upon the Strength of the Nuclear Attraction

The singlet and triplet S-wave nuclear interactions are almost the same strength, and therefore one might expect to gain insight into the dependence of $T_{12}(l,Q)$ on the strength of attraction by plotting T_s and T_t, Eq. (43), separately. At first sight one would not expect the great variation shown by the curves in Fig. 8. Clearly, the triplet contribution to T_{12} greatly dominates over the singlet contribution for $l<0.8$. The reason for this great variation can be partly attributed to the bound-state term $T_B(l,Q)$, which as pointed out below Eq. (38) is the same order of magnitude as $T_0(l,Q)$, Fig. 2. In fact, when the bound-state contribution is not included (dashed line in Fig. 8), T_t has a behavior similar to that of T_s.

If one refers to Eq. (32) for $T_C(l,Q)$, one sees that $T_C(l,Q)$ can, in general, be expected to be a negative quantity for l values less than the maximum l_0 values of $[\langle l|A^{(0)}|l_0\rangle \cos\delta_0(l_0)]^2$. Unless these maximum l_0 values occur where $l_0\ll x$, then one way expect that $T_C(l,Q)$ will never become positive. Thus, we can understand the difference in Fig. 8 between T_s and the dashed line curve for the continuum part of T_t, because the singlet interaction is closer to a zero-energy resonance than the triplet interaction [see also comment following Eq. (40)].

If the maximum l_0 values of $[\langle l|A^{(L)}|l_0\rangle \cos\delta_L(l_0)]^2$ occur in the region $l_0\gg x$, then one can expect that the contribution of the Lth partial waves to $T_{12}(l,Q)$, Eq. (41), will be small and negative. We thus conclude that besides the $L=0$ terms there is probably only one other set of terms in $T_{12}(l,Q)$ which it is essential to include, namely, the D-wave contributions which we expect to be large and negative.

We now define a quantity $\Delta^{(1)}(l_1,x)$:

$$\Delta^{(1)}(l_1,x)\equiv[A_0^{(P)}(l_1,x)+A_{12}^{(P)}(l_1,x)]E_N \cong\{-U(x)+[(M/M^*(x))-1]l_1^2\}E_N. \quad (58)$$

According to Eq. (15), $\Delta^{(1)}(l_1,x)$ is the $O(g_1)$ part of the quasi-particle potential energy. It has been numerically computed using Eqs. (17), (22), (41), (43), and (57) together with the effective mass parameters of Table II, and the $a=0.4$ F square well parameters of Table I. The results are plotted in Fig. 9. Also included as dashed lines are the effective mass approximations to these curves, using the same parameters from Table II. It is clear from Fig. 9 that the quasi-particle energies are strongly density dependent, as they must be. The effective mass approximation is also seen to be fairly good for $k_1<k_F$. For $k_1>k_F$, the approximation is bad, and the curves depart markedly from a quadratic fit. The flattened portions of the curves for $k_1<\frac{1}{2}k_F$ reflect the influence of the bound-state contributions and show that quasi-particles have a great preference for low-momentum states in nuclear matter. The reason for this has been discussed below Eq. (35).

We next use the results represented by Fig. 9 to calculate the $O(g_1)$ contribution to the thermodynamic potential and the ground-state energy per particle. The thermodynamic potential is given by Eq. (4), which can be written in terms of the effective-mass parameters

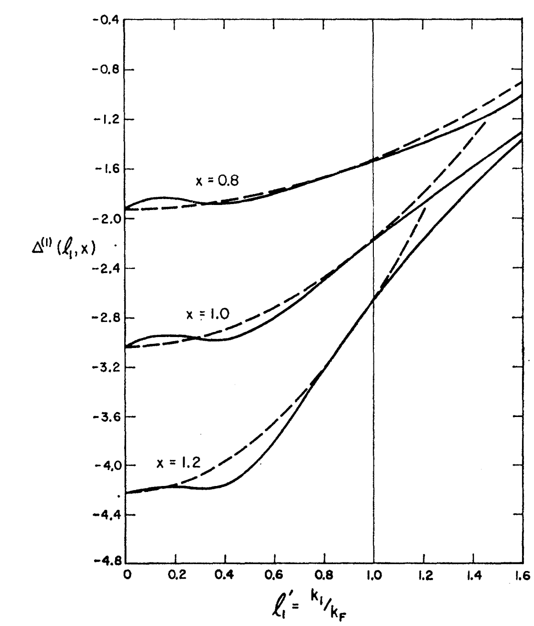

FIG. 9. The $O(g_1)$ part of the quasi-particle potential energy Δ as a function of $l_1'=k_1/k_F$ for various values of $x=k_F/k_N$ ($k_N=1.48$ F^{-1}). The dashed lines are the effective-mass approximations to the curves (fitted at $l_1'=0.0$ and $l_1'=1.0$). The interaction used was the square-well model of Sec. III with the $a=0.4$ F parameters of Table I.

of Eq. (56) as

$$g^{(1)}\cong[-U+(M/M^*)x^2]E_N. \tag{59}$$

Similarly, according to Eqs. (19), (15), and (56), the $O(g_1)$ part of the ground-state energy can be written in terms of the effective-mass parameters as

$$\langle E\rangle^{(1)}/\langle N\rangle\cong[-\tfrac{1}{2}U+\tfrac{3}{10}(1+M/M^*)x^2]E_N. \tag{60}$$

[Note that the leading contribution to the rearrangement energy of Eq. (15) is $O(g_1^2)$.] By plotting the effective mass parameters of Table II as a function of x and extrapolating the resulting curves, one can obtain the values of these parameters necessary to calculate $g^{(1)}$ and $\langle E\rangle^{(1)}/\langle N\rangle$ over a wide range of densities. The results are shown in Fig. 10 as a function of $x^{-1}=d/d_N$, where d is the average internucleon spacing in nuclear matter.

$$\rho\equiv d^{-3},\qquad 2(3\pi^2)^{-1}k_N{}^3=d_N{}^{-3},\quad d_N=1.66\text{ F}. \tag{61}$$

From the definition of the thermodynamic potential as $g=\langle E\rangle/\langle N\rangle+\rho^{-1}\mathcal{P}-T\langle S\rangle/\langle N\rangle$ and the fact that the pressure at $T=0$ is given by

$$\mathcal{P}=-\partial(\langle E\rangle/\langle N\rangle)/\partial(1/\rho), \tag{62}$$

it is clear that the curve for the thermodynamic potential should pass through the minimum of the energy curve. The curves of Fig. 10 come as close to satisfying this condition as one could expect from the accuracy of the calculation which has been made.

A more serious question with respect to the curves of Fig. 10 is why the minimum of the energy curve occurs at $x^{-1}\cong0.75$ instead of at $x\cong1.0$, and why the computed binding energies are so much greater than 15.8 MeV, which is the value obtained for nuclear matter from the semiempirical mass formula. The discrepancy cannot be attributed to higher order terms, because, as the estimates of the next section show, the $O(g_1^2)$ contributions to $\langle E\rangle/\langle N\rangle$ are much smaller than the $O(g_1)$ term which we have calculated. Considerable insight into the reason why the curves of Fig. 10 are not in agreement with experiment can be obtained by subtracting away the P, D, and F contributions to $T_0(l)$, thereby using curve A of Fig. 2 instead of curve B. The result (which has not been done self-consistently) is represented by the dashed line in Fig. 10. This curve shows that it is the higher angular momentum contributions (mainly D wave) to T_0 which have led to the large energy values in the curves of Fig. 10. Moreover, it is the D-wave contribution to T_0 which is responsible for the shift of the energy minimum from $x^{-1}\cong1.0$ to $x^{-1}\cong0.75$.

From the preceding paragraph we may infer that the properties of nuclear matter are largely determined by only the S-wave part of the nuclear interaction, and that this is the reason why so many of the purely S-wave calculations which have been reported in the literature have resulted in approximate agreement of theoretical calculation with the experimental numbers for nuclear matter.[13] We must ask then how it occurs quantitatively that the D-wave contribution to the binding energy is small. The answer lies with the higher angular momentum contributions to T_{12}, Eq. (41), which we have not calculated in this paper. In particular, one can expect to calculate a large repulsive D-wave contribution to T_{12}, as we have already noted in connection with Fig. 8. The total D-wave contribution to (T_0+T_{12}) will then be small, as will be all other higher angular momentum contributions.

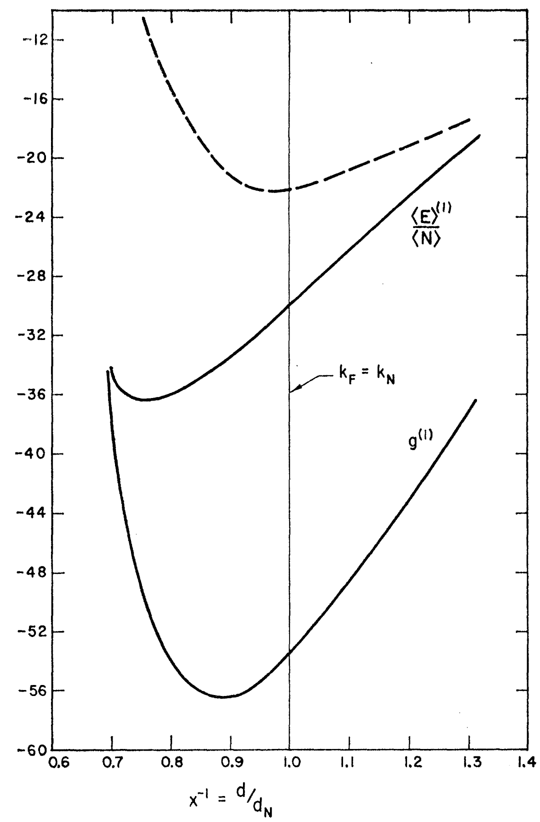

FIG. 10. The $O(g_1)$ contribution to the thermodynamic potential and the energy per particle as a function of $x^{-1}=d/d_N$ ($d_N=1.66$ F) when T_0 and the S-wave contribution to T_{12} are included. The dashed line is the $O(g_1)$ part of the energy per particle when only the S-wave parts of both T_0 and T_{12} are included.

We finally discuss the tensor character of the nuclear force. That the argument given in connection with Fig. 8, for the contribution of D-wave terms to T_{12}, continues to be valid for tensor forces can be demonstrated by showing that the most general form of the continuum part of $T_{12}(l,Q)$ is the same as $T_C(l,Q)$, Eq. (32). Therefore, the tensor *character* of the nuclear force, although important in a discussion of nuclear forces, is probably not important to an understanding of the properties of nuclear matter (see also Fig. 7).

[13] In this connection, see S. A. Moszkowski and B. L. Scott, Ann. Phys. (New York) 11, 65 (1960).

VII. ESTIMATES OF THE SECOND-ORDER TERMS

In the notation of Eqs. (8) and (10) the terms which are second order in $g_1(k_1k_2|k_3k_4)$, and which contribute to the quasi-particle potential energy $\Delta(k)$, are $T_{34}^{(2)}$, $T_5^{(2)}$, and $T_6^{(2)}$. These terms require a complete knowledge of the function g_1, whereas the calculations of $T_0^{(1)}$ and $T_{12}^{(1)}$ only require the diagonal elements $g_1(k_1k_2|k_1k_2)$. We write the off-diagonal matrix elements of g_1 as follows:

$$g_1(k_1k_2|k_3k_4)=\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)[(2\pi)^3/\Omega]^2 \times\tilde{g}_1(k_{12}\mathbf{K}_{12}|k_{34}), \quad (63)$$

where from Eqs. (III.10) and (III.13)

$$\tilde{g}_1(k_{12}\mathbf{K}_{12}|k_{34})=\tilde{f}_1(k_{12}|k_{34})+\sum_{k_0}\tilde{f}_2(k_{12}|k_0|k_{34})\left[\mathrm{P}\left(\frac{1}{\omega_{12}-\omega_0}\right)-\mathrm{P}\left(\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega_0}\right)\right], \quad (64)$$

with $\mathbf{K}_{12}=\mathbf{k}_1+\mathbf{k}_2$. For the case of a spin-dependent interaction, but in the absence of tensor forces, one can extract the spin factors from $\tilde{g}_1$ by generalizing Eqs. (III.11). One then obtains the expression

$$\begin{aligned}\tilde{g}_1(k_{12}\mathbf{K}_{12}|k_{34})= (2\pi)^{-2}(\hbar^2/M)\sum_{L=0}^{\infty}(2L+1)P_L(\hat{k}_{12}\cdot\hat{k}_{34})k_{12}^{-1}\{[\delta_{m_1m_3}\delta_{m_2m_4}-(-1)^L\delta_{m_1m_4}\delta_{m_2m_3}]\\ \times[\delta_{q_1q_3}\delta_{q_2q_4}+(-1)^L\delta_{q_1q_4}\delta_{q_2q_3}]\tilde{g}_S^{(L)}(\mathbf{k}_{12}\mathbf{K}_{12}|k_{34})+[\delta_{m_1m_3}\delta_{m_2m_4}+(-1)^L\delta_{m_1m_4}\delta_{m_2m_3}]\\ \times[\delta_{q_1q_3}\delta_{q_2q_4}-(-1)^L\delta_{q_1q_4}\delta_{q_2q_3}]\tilde{g}_S^{(L)}(\mathbf{k}_{12}\mathbf{K}_{12}|k_{34})\}, \quad (65)\end{aligned}$$

where

$$\begin{aligned}\tilde{g}_S^{(L)}(\mathbf{k}_{12}\mathbf{K}_{12}|k_{34})=\langle k_{34}|A_S^{(L)}|k_{12}\rangle\cos^2\delta_{LS}(k_{12})+2\pi^{-1}(\hbar^2/M)k_{12}\int_0^{\infty}dk_0\langle k_{12}|A_S^{(L)}|k_0\rangle\langle k_{34}|A_S^{(L)}|k_0\rangle\cos^2\delta_{LS}(k_0)\\ \times\left[\mathrm{P}\left(\frac{1}{\omega_{12}-\omega_0}\right)-\mathrm{P}\left(\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega_0}\right)\right]+(\hbar^2/M)k_{34}^{-1}\sum_{\gamma}(k_{12}^2+\gamma^2)(k_{34}^2+\gamma^2)\phi_{\gamma LS}(k_{12})\phi_{\gamma LS}(k_{34})\\ \times\left[\left(\frac{1}{\omega_{12}-\omega_\gamma}\right)-\mathrm{P}\left(\frac{1}{\omega_1'+\omega_2'-W_{12}-\omega_\gamma}\right)\right]. \quad (66)\end{aligned}$$

In analogy with the derivation of Eq. (43), one can derive an explicit expression for $\tilde{g}_S^{(0)}$ using the square-well-repulsive-core model. The final expression (again including a hard-core correction term) is, in dimensionless units,

$$\begin{aligned}\tilde{g}_S^{(0)}(\mathbf{l}\mathbf{Q}|k)=(k^2+\epsilon)(l^2-k^2)^{-1}\{(l^2+\epsilon)[lk^{-1}(l^2-Y_\epsilon^2)^{-1}\sin kA\cos lA-(k^2-Y_\epsilon^2)^{-1}\sin lA\cos kA]\\ +\mu^2(k^2-Y_\epsilon^2)^{-1}[\sin lB\cos kB-lk^{-1}\sin kB\cos lB]\}+\mu^2(k^2-Y_\epsilon^2)^{-1}[\mu^2\cos^2Y_\epsilon(B-A)-\epsilon]^{-1}\\ \times\{N_1(l,i\epsilon^{1/2})[Y_\epsilon\cos Y_\epsilon(B-A)\cos kB-\epsilon k^{-1}\sin kB\sin Y_\epsilon(B-A)]+k^{-1}Y_\epsilon(l^2-Y_\epsilon^2)^{-1}(k^2+\epsilon)\sin kA\\ \times[(l^2+\epsilon)\sin lA\cos Y_\epsilon(B-A)\sin Y_\epsilon(B-A)+lY_\epsilon\cos lB\cos Y_\epsilon(B-A)-\epsilon\sin lB\sin Y_\epsilon(B-A)]\}\\ -k^{-1}\epsilon^{1/2}[\mu^2\cos^2Y_\epsilon(B-A)-\epsilon]^{-1}N_1(k,i\epsilon^{1/2})N_1(l,i\epsilon^{1/2})\theta(\epsilon), \quad (67)\end{aligned}$$

where $\mathbf{l}=k_N^{-1}\mathbf{k}_{12}$, $k=k_N^{-1}k_{34}$, $\mathbf{Q}=\frac{1}{2}k_N^{-1}\mathbf{K}_{12}$, and ϵ is given by Eq. (33). We next substitute Eqs. (63) and (65) into the expressions for T_{34}, T_5, and T_6, Eqs. (9) and (11). Keeping only the S-wave contribution we obtain:

$$\begin{aligned}T_{34}(l,Q)&\cong 12(\pi l)^{-1}\int_0^{2.5}l_0^2dl_0\{[\tilde{g}_0^{(0)}(lQ|l_0)]^2+[\tilde{g}_1^{(0)}(lQ|l_0)]^2\}\{(M^*/M)\mathrm{P}(l_0^2-l^2)^{-1}-\mathrm{P}(l_0^2+\epsilon)^{-1},\\ T_5(l,Q)&\cong 24(\pi l)^{-1}(M^*/M)\int_0^{x-Q}l_0^2dl_0\{[g_0^{(0)}(lQ|l_0)]^2+[\tilde{g}_1^{(0)}(lQ|l_0)]^2\}\mathrm{P}(l^2-l_0^2)^{-1}\\ &\quad+6(\pi lQ)^{-1}(M^*/M)\int_{x-Q}^{x+Q}l_0dl_0[x^2-(l_0-Q)^2]\{[\tilde{g}_0^{(0)}(lQ|l_0)]^2+[\tilde{g}_1^{(0)}(lQ|l_0)]^2\}\mathrm{P}(l^2-l_0^2)^{-1}\quad \text{if}\quad Q<x\\ &\cong 6(\pi lQ)^{-1}(M^*/M)\int_{Q-x}^{Q+x}l_0dl_0[x^2-(l_0-Q)^2]\{[\tilde{g}_0^{(0)}(lQ|l_0)]^2+[\tilde{g}_1^{(0)}(lQ|l_0)]^2\}\mathrm{P}(l^2-l_0^2)^{-1}\quad \text{if}\quad Q>x,\\ T_6(l,Q)&\cong 12l\pi^{-1}(M^*/M)\int_0^{x-Q}dl_0\{[\tilde{g}_0^{(0)}(l_0Q|l)]^2+[\tilde{g}_1^{(0)}(l_0Q|l)]^2\}\mathrm{P}(l_0^2-l^2)^{-1}\\ &\quad+3l\pi^{-1}(M^*/M)Q^{-1}\int_{x-Q}^{(x^2-Q^2)^{\frac{1}{2}}}l_0^{-1}dl_0(x^2-Q^2-l_0^2)\{[\tilde{g}_0^{(0)}(l_0Q|l)]^2+[\tilde{g}_1^{(0)}(l_0Q|l)]^2\}\mathrm{P}(l_0^2-l^2)^{-1}\quad \text{if}\quad Q<x\\ &\equiv 0\quad \text{if}\quad Q>x. \quad (68)\end{aligned}$$

We have arbitrarily selected a cutoff at $l_0=2.5$ in the expression for $T_{34}(l,Q)$, because according to Fig. 9 the quasi-particle potential energy rapidly approaches zero at high momenta. Thus, at $l_0\sim 2.5$ we expect that $\omega'\cong\omega$, and the two energy denominators in the expression for $T_{34}(l,Q)$ cancel when $\omega'=\omega$. The computed expression for T_{34} should not be very sensitive to the choice of the l_0 integration cutoff. It is also expected that the use of the effective mass approximation for ω' in the region $x<l_0<2.5$ should not seriously impair the accuracy of the computed expression for T_{34}. In $T_6(l,Q)$ one must remember to replace l by l_0 in the expression for ϵ, Eqs. (33) or (57).

We have not carried out the full calculation of T_{34}, T_5, and T_6. Instead we have made estimates which are believed to be reliable to within a factor of 2. These estimates show that T_{34}, T_5, and T_6 are each an order of magnitude smaller that (T_0+T_{12}). Moreover, the total second-order contribution to the quasi-particle potential energy is estimated to be $\lesssim 10\%$ of the first-order terms. On the basis of these estimates we conclude that the convergence of expansions in powers of $g_1(k_1k_2|k_3k_4)$ for the physical quantities of nuclear matter is probably rapid.

VIII. PERTURBATION THEORY IN NUCLEAR MATTER

In this section we investigate the extent to which the results which we have calculated in Sec. VI can be understood by a perturbation theoretic treatment. Of course, one can not expect to understand the role of the repulsive core by using perturbation theory, unless one makes use of approximations such as in the pseudopotential method.[14] We therefore consider the case of no hard sphere (see Fig. 6). When $a=0.0$, the expression (43) for $T_s(\mathbf{l},\mathbf{Q})$ or $T_t(\mathbf{l},\mathbf{Q})$ in the case of a square-well attraction becomes

$$\begin{aligned}\tfrac{1}{3}T_{s,t}(\mathbf{l},\mathbf{Q}) &= -\sin 2\delta_0(l)+\mu^2 l^{-2}(l^2+\epsilon)(l^2-Y_\epsilon^2)^{-1}(lB-\tfrac{1}{2}\sin 2lB)\\ &\quad+2\mu^2(l^2-Y_\epsilon^2)^{-1}[\mu^2\cos^2 Y_\epsilon B-\epsilon]^{-1}N_1(l,i\epsilon^{1/2})\\ &\quad\times[Y_\epsilon\cos Y_\epsilon B\cos lB-\epsilon l^{-1}\sin lB\sin Y_\epsilon B]\\ &\quad-2l^{-1}\epsilon^{1/2}[\mu^2\cos^2 Y_\epsilon B-\epsilon]^{-1}[N_1(l,i\epsilon^{1/2})]^2\theta(\epsilon),\end{aligned}\qquad(69)$$

where from Eq. (28) we have

$$\sin 2\delta_0(l)=-2[\mu^2\cos^2 YB+l^2]^{-1}N_1(l,l)N_2(l),\qquad(70)$$

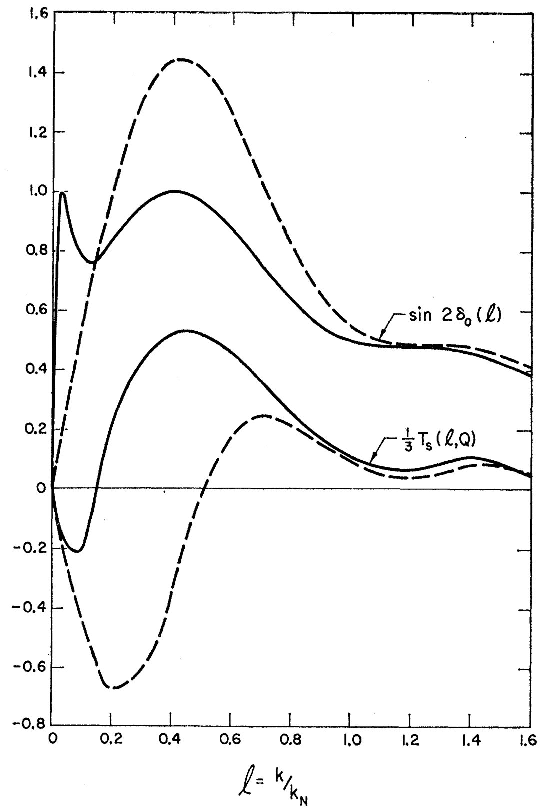

FIG. 11. Curves of $\sin 2\delta_0(l)$ and $\frac{1}{3}T_s(l,Q)$ for $Q=0.5$, the $x=1.0$ effective-mass parameters of Table II, and the square-well model of Sec. IV with the $a=0.0$ singlet parameters of Table I. The dashed lines give the Born approximation to these curves.

and

$$\begin{aligned}N_1(l,l_0)&=(Y_0^2-l^2)^{-1}\mu^2[Y_0\cos Y_0 B\sin lB\\ &\qquad -l\cos lB\sin Y_0 B],\\ N_2(l)&=Y\cos YB\cos lB+l\sin lB\sin YB.\end{aligned}\qquad(71)$$

We wish to compare the values of the expressions (69) and (70), using the $a=0.0$ parameters of Table I, with their leading terms in a perturbation expansion. We consider here only the case of the singlet nuclear interaction, because then we can be sure that the complication of the bound-state effect discussed in connection with Fig. 8 is not present. An expansion in powers of $\mu^2=k_N^{-2}U_0$ gives for T_s and $\sin 2\delta_0(l)$

$$\begin{aligned}\tfrac{1}{3}T_s(\mathbf{l},\mathbf{Q})&=\mu^4 l^{-2}(lB-\tfrac{1}{2}\sin 2lB)[(l^2+\epsilon)^{-1}+\tfrac{1}{2}l^{-2}(\tfrac{1}{2}+\cos^2 lB)]-\tfrac{1}{2}\mu^4 l^{-3}B\sin^2 lB+2\mu^4(l^2+\epsilon)^{-2}\\ &\quad\times\begin{cases}[\sin(-\epsilon)^{1/2}B\cos lB-(-\epsilon)^{1/2}l^{-1}\cos(-\epsilon)^{1/2}B\sin lB]\\ \qquad\times[\sin(-\epsilon)^{1/2}B\sin lB+l(-\epsilon)^{-1/2}\cos(-\epsilon)^{1/2}B\cos lB] & (\text{if } \epsilon<0)\\ [\sinh\epsilon^{1/2}B\cos lB-\epsilon^{1/2}l^{-1}\cosh\epsilon^{1/2}B\sin lB]\exp(-\epsilon^{1/2}B)[\sin lB+l\epsilon^{-1/2}\cos lB] & (\text{if } \epsilon>0)\end{cases}+O(\mu^6)\end{aligned}\qquad(72)$$

$$\begin{aligned}\sin 2\delta_0(l)&=\mu^2 l^{-2}[lB-\tfrac{1}{2}\sin 2lB]+O(\mu^4)\\ &=-(2\pi)^{-1}l\langle l|U_0|l\rangle+O(\mu^4).\end{aligned}\qquad(73)$$

The expression (72) can be calculated directly by making a perturbation expansion of the operator form of $T_s(\mathbf{k}_1,\mathbf{k}_2)$.

[14] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

From Eqs. (9), (III.11), and (III.76), the general expression for $T_s(k_1,\mathbf{k}_2)$ is

$$T_s(k_1,\mathbf{k}_2)=\frac{8\pi^2M}{\hbar^2}\frac{\Omega}{(2\pi)^3}k_{12}\sum_{m_2q_2}\Bigg\{\Bigg\langle k_1k_2\Bigg|V\Bigg[\mathrm{P}\Bigg(\frac{1}{\omega_1+\omega_2-H^{(2)}}\Bigg)-\mathrm{P}\Bigg(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\Bigg)\Bigg]V\Bigg|k_1k_2\Bigg\rangle$$
$$-\Bigg\langle k_2k_1\Bigg|V\Bigg[\mathrm{P}\Bigg(\frac{1}{\omega_1+\omega_2-H^{(2)}}\Bigg)-\mathrm{P}\Bigg(\frac{1}{\omega_1'+\omega_2'-H^{(2)}}\Bigg)\Bigg]V\Bigg|k_1k_2\Bigg\rangle\Bigg\}. \quad (74)$$

After replacing the two-nucleon Hamiltonian $H^{(2)}$ by the kinetic-energy operator $H_0^{(2)}$ in the energy denominators, and using the interaction (23) with $a=0.0$, one can derive (72) from (74). Similarly, Eq. (73) is nothing more than the first Born approximation to the reaction matrix, as has been indicated in the second line of (73).

In Fig. 11 we have drawn curves of $\frac{1}{3}T_s(l,Q)$ for $Q=0.5$ and $x=1.0$ using Eq. (69) and the leading perturbation term (72). We have also included the corresponding curves for $\sin 2\delta_0(l)$. It is seen that the Born approximation to these curves is excellent for $lB=(3.69)l\gg 1$, which is the criterion for its validity. An extremely interesting aspect of these curves is the Born approximation to $\sin 2\delta_0(l)$, which is often cited as the leading effect due to the nuclear attraction in calculations of nuclear matter.[15] It is seen that the first Born approximation (73) is fairly close to being equal to the sum $[\sin 2\delta_0(l)+\frac{1}{3}T_s(l,Q)]$ which appears in the present theory. Thus, we are able to understand this well-known result in the present connotation. A similar result holds for the triplet interaction.

We finally observe that the triplet-state nuclear interaction includes a strong tensor force, whose contribution to the properties of nuclear matter cannot be simply understood by using first Born approximation (which gives zero for a tensor force). However, the combination $[\sin 2\delta_0(l)+\frac{1}{3}T_t(l,Q)]$, which gives the $O(g_1)$ S-wave contribution to the properties of nuclear matter, continues to be physically significant when there are tensor forces. The detailed study of the role of tensor forces in nuclear matter is left for a subsequent investigation.

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[15] H. A. Bethe, Phys. Rev. **103**, 1353 (1956). See also reference 11.