Inelastic Electron Scattering from Be^9 and C^{12}†

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High-energy inelastic electron scattering data for the first excited states in Be^9 and C^{12} are analyzed from the standpoint of three nuclear models, the alpha-particle model, the Nilsson model, and the spherical shell model. It is found that the spherical shell model with an enhancement of the quadrupole charge distribution by a factor of 2 gives a good fit to the inelastic data as a function of the momentum transfer. The alpha-particle model and Nilsson models, although correctly giving the magnitude of the low momentum transfer data, fail to predict the high momentum transfer data. These models can be used to determine closely the size of the low-energy properties of these nuclei such as the quadrupole moments and gamma-ray lifetimes.

I. INTRODUCTION

THE scattering of high-energy electrons from nuclei serves as a useful tool in determining the charge distribution of these nuclei. The short wavelength associated with these fast electrons allows one to see the finer details of the nuclear charge distribution. The elastic scattering data[1] of electrons from nuclei of the $1P$ shell which ranges from Li^6 to O^{16} have been used to fix rather accurately the spherically symmetric part of the nuclear charge distribution. The Coulomb interaction is well known, and the cross sections have been extensively investigated. The charge distributions correspond very well in many cases to that predicted by the harmonic-oscillator shell model. For nuclei of spin greater than 1/2, the nonspherically-symmetric portion of the charge distribution also contributes to the elastic scattering. The effect of the nonspherical part is small at low momentum transfers and is not readily determined except where the diffraction minima of the spherical contribution to the cross section occur. An analysis of just the elastic scattering does not allow one to separate out easily these different contributions to the nuclear charge density.

The cross section for inelastic excitation of certain collective nuclear levels depends entirely upon the nonspherically-symmetric portion of the nuclear transition charge density and in many cases the quadrupole distribution of charge gives the main contribution to the cross section. Hence by analyzing the inelastic scattering data one can find out further how well several of the nuclear models describe these light nuclei. In particular one can see how well the models, that have been involved to explain the enhanced electric quadrupole phenomena such as quadrupole moments and $E2$ gamma-ray transitions which are low momentum transfer reactions, behave for phenomena which involve larger momentum transfers. Also, to the extent that the uncertainty principle allows one to do so one can gain limited knowledge of the shape of the nonspherical component of the charge density assuming that one or more of the models give a reasonably good fit to the data. Of course, in general one does not obtain information concerning the charge distribution of the ground state or excited states of the nucleus, but the charge operator between the ground and excited state. However, in certain nuclear models such as the alpha-particle model, and collective model, the states that are predominantly excited have the same equilibrium shape and internal wave functions as the ground state and are characterized only by a different angular velocity of rotation. For these cases the operator between the ground and excited states is just that for the ground state except for factors depending upon the angular momenta of the two states. Thus one can infer some of the characteristics of the ground-state charge distribution although only to the extent of the validity of the model used. We will use three nuclear models to interpret the inelastic scattering data, the collective model of Bohr and Mottelson, the alpha-particle model, and a modification of the spherical shell model. The shell model has been considered extensively by others but has been included as a comparison to the other two models.

The collective model of Bohr and Mottelson, and Nilsson[2] is very successful in explaining the position and spin of low-lying energy levels, ground-state quadrupole moments, and the large matrix elements needed in electric quadrupole electromagnetic transitions. This model contains to adjustable parameters $\hbar\omega_0$, the size of the oscillator well in which the nucleons move, and ϵ, the distortion of the well. In this paper the procedure in fitting the data[1,3,4] is to use the elastic scattering to determine the well size $\hbar\omega_0$ and then to determine ϵ from the inelastic scattering data. We find, however, that this model predicts too low a cross section at the higher momentum transfers where the details of the charge density are important.

The alpha-particle model has been used with success

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among certain of the light nuclei,[5–7] particularly the energy levels of the nuclei. This model represents the correlations among the nuclei caused by the attractive nuclear forces. In many ways this model is similar to the collective model except that the description of the single-particle wave function is not given except in terms of the density of nucleons in the alpha particles. This model contains only one adjustable parameter, 2ξ, the equilibrium distance between the alpha particles. To bring the closer agreement between the predictions of the model and experimental data, the finite size of the alpha particles must be included in the calculations, although not included is the vibrational motion of the alpha particles. The one parameter, 2ξ, is adjusted to give the best fit to both the elastic and inelastic scattering data. Slightly different values of the parameter 2ξ are required to fit the two sets of data for Be^9. The alpha-particle model has been used by others[8] to compare the inelastic scattering data, but the model was included here because the data used here is more extensive and allows a more detailed comparison. This model predicts the low momentum transfer data well but predicts too much scattering at the high momentum transfers.

Analyses have been made on the inelastic data using the shell model[9] and a combination of the shell model with collective effects.[10] These analyses show that the pure shell model gives close to the correct energy dependence of the cross section but does not give the required magnitude. The failure of the shell model in predicting the magnitude of quantities involving matrix elements of the quadrupole operator is well known.[5,11] The addition of collective effects to the shell model can give the correct magnitude of the cross section. The shell model requires the definition of a transition charge density for its application to the inelastic scattering cross section. This transition density cannot be simply related to the ground state. However, one can define an intrinsic quadrupole moment Q_0 which is related to the actual transition quadrupole matrix element by factors which depend upon the angular momenta of the two states involved. This quantity, Q_0, is an adjustable parameter in the model as well as the size of the oscillator well. Similarly one can define an adjustable quadrupole moment for the ground-state charge density while keeping the shape of the charge density the same as for the usual shell model. These adjustable parameters can be interpreted as a different effective charge needed to give the proper magnitude to matrix elements involving this portion of the density. The low multipole moments of the two-body residual interactions in this model couple into single-particle wave function states of different angular momentum. This attractive interaction couples in states which enhance the single-particle quadrupole moments. In the $1P$ shell this enhancement is on the order of 100%. The question arises whether the radial dependence of the couples states will be different from the $1P$ shell-model wave functions. This will be answered by how good the fit to experimental data is. The data is fit by choosing the radius parameter, a, in the model to give a reasonable fit to the elastic data peak and then the quadrupole size parameter is adjusted to fit the inelastic data. It is found however a slight change in the size of the a parameter from the elastic valve gives a better fit to the inelastic data. This model gives the best fit to the inelastic data of the three models considered at all momentum transfers. This fit may be accidental at the highest momentum transfers where the approximations used become important.

The cross sections are calculated[12] by first Born approximation using relativistic electron wave functions. The effect of magnetic dipole excitations, and nuclear dispersion effects are neglected. These approximations are estimated in the discussion. The effect of the nuclear recoil in the Nilsson and shell models are taken into account. The recoil effect is quite important at the large momentum transfers encountered in the data. The resulting cross sections are factored into a Coulomb cross section for scattering from a point charge of magnitude Z of the nucleus involved times a form factor squared, $|F(q)|^2$ which is the square of the Fourier transform of the nuclear charge density. The variable q is the momentum transferred in the reaction. Included in the form factor are the form factors for the single particles used in the models, the proton for the Nilsson and shell models and the alpha particle for the alpha-particle model. The form factors for the proton and alpha particle are taken from analyses of high-energy electron scattering[12,13] from the proton and alpha particle. For simplicity the shape of the factors are taken to be Gaussian. It is assumed that these factors which represent the shape of the charge density of these particles do not change in the nucleus although in the case of the alpha particle some change in size is indicated.

The expression for the form factors for the alpha particle and shell models can be given in terms of well known functions. The expressions for the Nilsson model require the use of an electronic computer to evaluate the Fourier transform integrals.

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II. ALPHA-PARTICLE MODEL

This model assumes that certain light nuclei can be considered to be constructed of an integral number of alpha particles plus or minus an extra nucleon, although for the nuclei consisting of an integral number of alpha particles plus or minus two nucleons is ambiguous in that there are two possible alpha configurations that can describe these nuclei. The alpha particles are assumed to execute small vibration about an equilibrium distance from each other, and the system of alpha particles is also allowed to rotate slowly. The energy levels of the nucleus are characterized by the total angular momentum, I, and its projection, K, upon a symmetry axis fixed with respect to the alpha particles. The energy levels for a fixed value of K are the usual rotational levels with an energy spacing proportional to $I(I+1)$ and are characterized by large quadrupole matrix elements between the levels with the same projection K.

The nucleus Be^9 can be considered to be two alpha particles and an extra neutron.[7] The ground state has a spin of 3/2, with $K=3/2$ and the first excited state in the ground state rotational band is taken to be the 2.43-MeV state which has a spin of 5/2. If we assume the equilibrium distance between the two alphas is given by 2ξ, we obtain the following expressions for the elastic scattering cross section and the inelastic scattering to the $I=5/2$ state:

$$\begin{aligned}\sigma_{\text{elastic}}(\theta)&=\sigma_c(\theta)\,|F_\alpha(q)|^2[j_0^2(q\xi)+j_2^2(q\xi)],\\ \sigma_{\text{inelastic}}(\theta)&=(18/7)\sigma_c(\theta)\,|F_\alpha(q)|^2[j_2^2(q\xi)+\tfrac{1}{6}j_4^2(q\xi)],\end{aligned}\qquad(1)$$

where $j_l(q\xi)$ is a spherical Bessel function of order l and q is the momentum transfer in the reaction. $F_\alpha(q)$ is the elastic scattering form factor for an alpha particle and given by[12,13]

$$F_\alpha(q)=\exp(-q^2a^2/6),\qquad(2)$$

with $a=1.61$ F, and $\sigma_c(\theta)$ is the Coulomb scattering cross section for a point-charge Be^9 nucleus. The elastic cross section is plotted as $q(\sigma_{\text{el}}/\sigma_c)^{1/2}$ in Fig. 1 for two values of ξ and the inelastic cross section is plotted as $\sigma_{\text{inel}}/\sigma_c$ in Fig. 2 for two values of ξ.

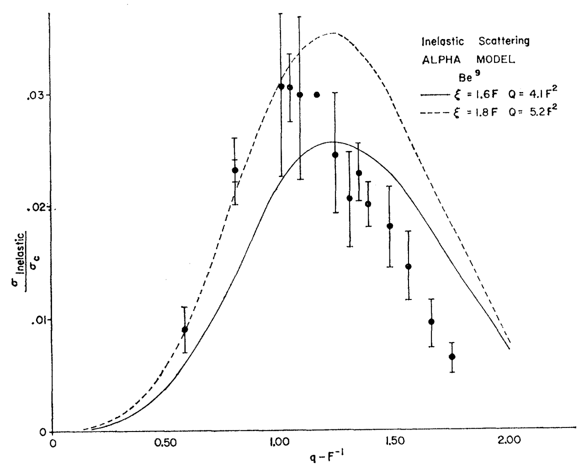

FIG. 2. Inelastic scattering of electrons from the Be^9 2.43-MeV state by the alpha-particle model.

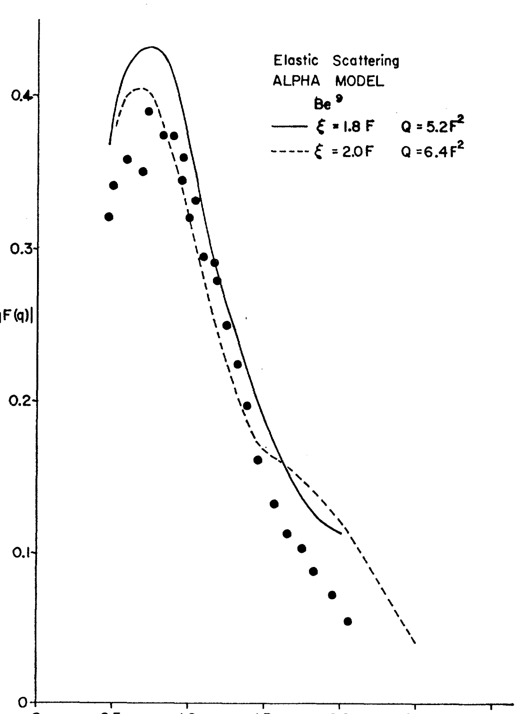

FIG. 1. Elastic scattering of electrons from Be^9 by the alpha-particle model.

In the case of C^{12} one considers the nucleus to be formed by three alpha particles, the equilibrium positions of which are at the corners of an equilateral triangle. The energy levels of this nucleus are given by

$$E=(\hbar^2/2\mathscr{I}_1)I(I+1)+(\hbar^2/2)(1/\mathscr{I}_2-1/\mathscr{I}_1)K^2,\qquad(3)$$

where K is the projection of angular momentum on an axis perpendicular to the plane of the triangle containing the three alpha particles and takes on the values $K=0,\ 3,\ 6,\ \cdots$, and $\mathscr{I}_1$ and $\mathscr{I}_2$ are the moments of inertia perpendicular to and about this axis, respectively. The parity of the levels is given by $\Pi=(-)^K$.

The expressions obtained for the elastic cross section and inelastic cross section to the 2+, 4.43-MeV state are

$$\begin{aligned}\sigma_{\text{elastic}}(\theta)&=\sigma_c(\theta)\,|F_\alpha(q)|^2 j_0^2(\tfrac{2}{3}\sqrt{3}q\xi),\\ \sigma_{\text{inelastic}}(\theta)&=(5/4)\sigma_c(\theta)\,|F_\alpha(q)|^2 j_2^2(\tfrac{2}{3}\sqrt{3}q\xi),\end{aligned}\qquad(4)$$

where $\sigma_c(\theta)$ is the Coulomb scattering cross section from a point nucleus of charge +6, and 2ξ is again the inter-alpha-particle distance. The cross sections are plotted in Figs. 3 and 4 for several values of ξ.

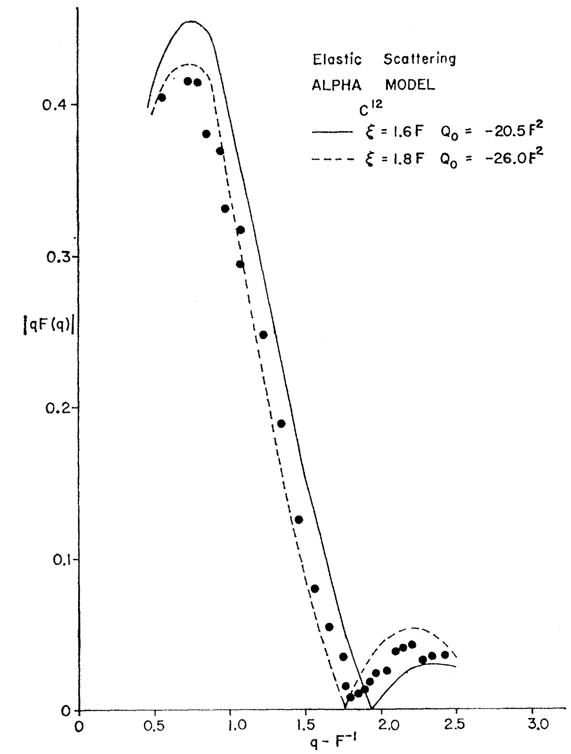

FIG. 3. Elastic scattering of electrons from C^{12} by the alpha-particle model.

III. NILSSON MODEL

In the Nilsson model[2] the nucleons are assumed to move in a deformed potential. This deformed potential in turn executes a rotational motion. The Hamiltonian describing the motion of the particles in the field is

$$H=-(\hbar^2/2m)\nabla'^2+\tfrac{1}{2}m(\omega_x^2X'^2+\omega_y^2Y'^2+\omega_z^2Z'^2) +C\mathbf{l}\cdot\mathbf{s}+D\mathbf{l}^2. \quad (5)$$

A parameter change is made by defining

$$\omega_x=\omega_y=(1+\tfrac{1}{3}\epsilon)\omega_0, \qquad \omega_z=(1-\tfrac{2}{3}\epsilon)\omega_0. \quad (6)$$

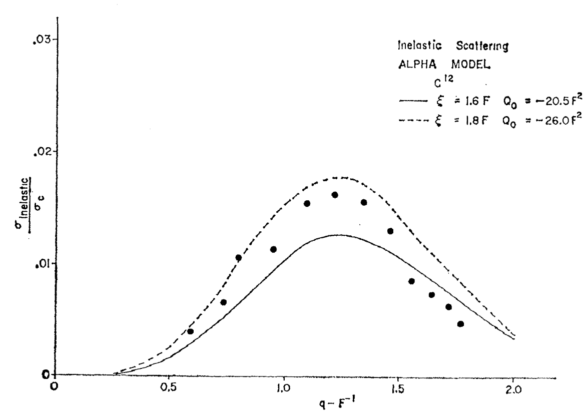

FIG. 4. Inelastic scattering of electrons from C^{12} 4.43-MeV state by the alpha-particle model.

Next we perform a further coordinate transformation

$$\xi=X'(m\omega_x/\hbar)^{1/2}, \quad \eta=Y'(m\omega_y/\hbar)^{1/2}, \quad \zeta=Z'(m\omega_z/\hbar)^{1/2}. \quad (7)$$

This gives for the Hamiltonian

$$H=\hbar\omega_z\left(-\frac{\partial^2}{\partial\zeta^2}+\zeta^2\right)+\hbar\omega_x\left(-\frac{\partial^2}{\partial\xi^2}+\xi^2\right) +\hbar\omega_y\left(-\frac{\partial^2}{\partial\eta^2}+\eta^2\right)+C\mathbf{l}\cdot\mathbf{s}+D\mathbf{l}^2. \quad (8)$$

For large values of ϵ one can approximately diagonalize The Hamiltonian with harmonic oscillator functions based upon the new coordinates (ξ,η,ζ). The new representation includes a major portion of the coupling between the oscillator shells of different total quantum number N caused by the distorted potential. A small coupling between states of different total quantum number still exists of course from the terms $\mathbf{l}^2$ and $\mathbf{l}\cdot\mathbf{s}$. The wave functions for the $1S$ and $1P$ shells are described in the Appendix.

The Be^9 nucleus is described in the Nilsson picture as a prolate spheroid, $\epsilon>0$; this is the shape of lowest energy. The first two protons are placed in the lowest state ψ_1, and the second pair of protons are placed in the next lowest state of ψ_3 (see the Appendix). The total wave function for Be^9 in the state of total angular momentum J, projection on the fixed space axis of M, and projection K on the symmetry axis of the deformed potential of J is given by

$$\Psi(J,M,K)=\left(\frac{2J+1}{16\pi^2}\right)^{1/2}[\phi_K D_{K,M}{}^J(\theta_i) +(-)^{J+1/2}\phi_{-K}D_{-K,M}{}^J(\theta_i)]. \quad (9)$$

The function ϕ_K is the product of single particle wave functions for the nine nucleons, and $D_{MK}{}^J(\theta_i)$ is the symmetric top function which describes the rotational motion of the nucleus. The $3/2^-$ ground state of Be^9 is given by

$$J=K=3/2,$$

and the $5/2^-$ first rotational state at 2.43 MeV by

$$J=5/2, \quad K=3/2.$$

The elastic cross section in first Born approximation is given by

$$\sigma_{\text{elastic}}(\theta)=\sigma_c(\theta)|F_p(q)|^2 \times\left[\left|\frac{1}{Z}\sum_i F_0{}^i(q)\right|^2+\left|\frac{1}{Z}\sum_i F_2{}^i(q)\right|^2\right], \quad (10)$$

where Z is the nuclear atomic number, the functions $F_0{}^i(q)$ and $F_2{}^i(q)$ are the monopole and quadrupole contributions to the cross section and are defined in the Appendix. In the sum the index i is taken over all filled proton states. The proton form factor $F_p(q)$ is given by

$$F_p(q)=\exp[-q^2b^2/6]. \quad (11)$$

For the proton, b is taken to be 0.72 F. The elastic cross section is plotted as $q(\sigma_{el}/\sigma_c)^{1/2}$ in Fig. 5 for several values of the parameters $\hbar\omega_0$ and ϵ.

The inelastic cross section for the transition from the ground state to the 2.43-MeV excited state is

$$\sigma_{\text{inelastic}}(\theta)=\frac{18}{7}\sigma_c(\theta)|F_p(q)|^2 \times\left[\left|\frac{1}{Z}\sum_i F_2{}^i(q)\right|^2+\frac{1}{6}\left|\frac{1}{Z}\sum_i F_4{}^i(q)\right|^2\right]. \quad (12)$$

The ratio $(\sigma_{\text{inel}}/\sigma_c)$ is plotted in Fig. 6 for several values of the parameter $\hbar\omega_0$ and ϵ.

In the Nilsson model the ground state of C^{12} is described by an oblate spheroid, $\epsilon<0$, with two protons in each of the states described by the wave functions ψ_1, ψ_2, and ψ_3. The total wave function for the 0^+ ground state is similar to that of Be^9 with

$$J=K=0.$$

The 2^+ state at 4.4 MeV on this model then is given by

$$J=2, \quad K=0.$$

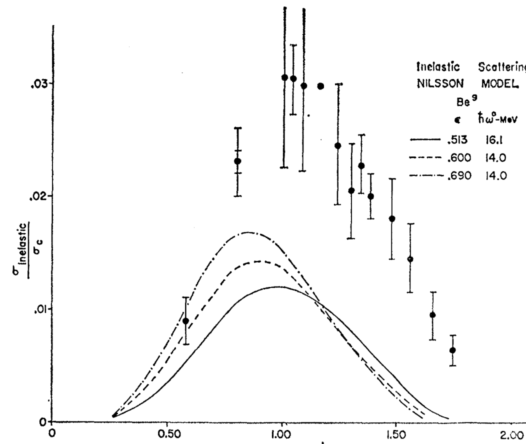

FIG. 6. Inelastic scattering of electrons from the 2.43-MeV state of Be^9 by the Nilsson model.

The elastic cross section is

$$\sigma_{\text{elastic}}(\theta)=\sigma_c(\theta)|F_p(q)|^2\left|\sum_i \frac{1}{Z}F_0{}^i(q)\right|^2. \quad (13)$$

The inelastic cross section is

$$\sigma_{\text{inelastic}}(\theta)=5\sigma_c(\theta)|F_p(q)|^2\left|\sum_i \frac{1}{Z}F_2{}^i(q)\right|^2. \quad (14)$$

These two results are plotted in Figs. 7 and 8 for several values of the two parameters $\hbar\omega_0$ and ϵ.

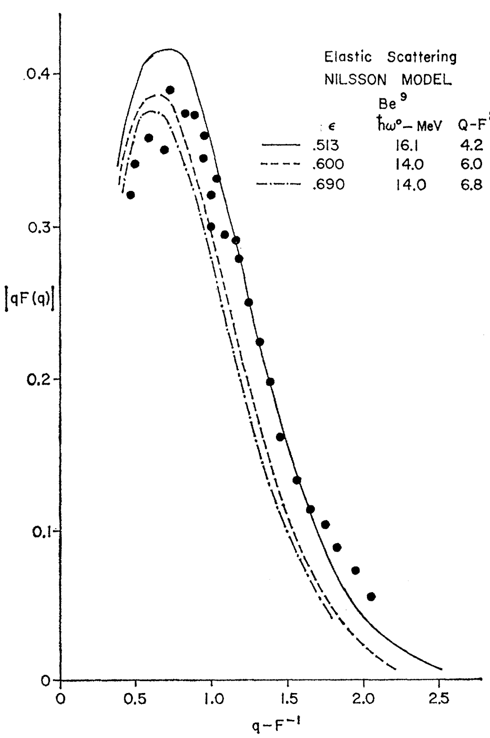

FIG. 5. Elastic scattering of electrons from Be^9 by the Nilsson model.

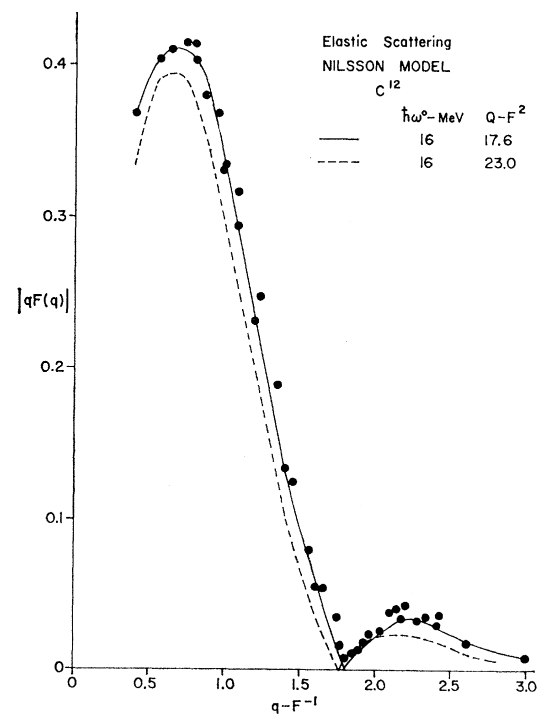

FIG. 7. Elastic scattering of electrons from C^{12} by the Nilsson model.

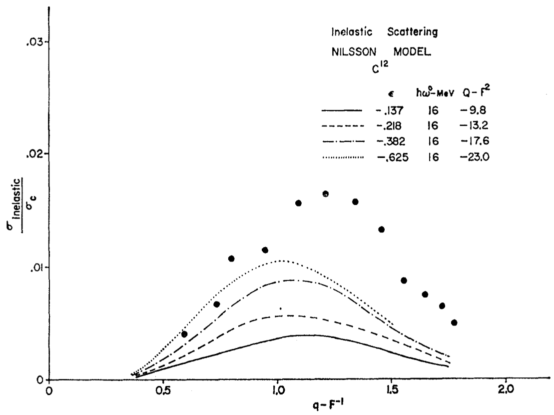

FIG. 8. Inelastic scattering of electrons from the 4.43-MeV state of C^{12} by the Nilsson model.

IV. SHELL MODEL

The third model to describe the scattering of electrons from the Be^9 and C^{12} nuclei is an undeformed shell-model charge distribution. The essential difference from the usual shell model will be an arbitrary scale parameter for the quadrupole term of the charge distribution. We shall assume that there are two protons in $1S$ harmonic oscillator states and the remaining protons in $1P$ harmonic oscillator states.

The ground-state charge density, $\psi^*\psi$, can be expanded in a series of spherical harmonics

$$\rho(r)=\rho_0(r)Y_{00}+\rho_2(r)Y_{20}+\cdots. \quad (15)$$

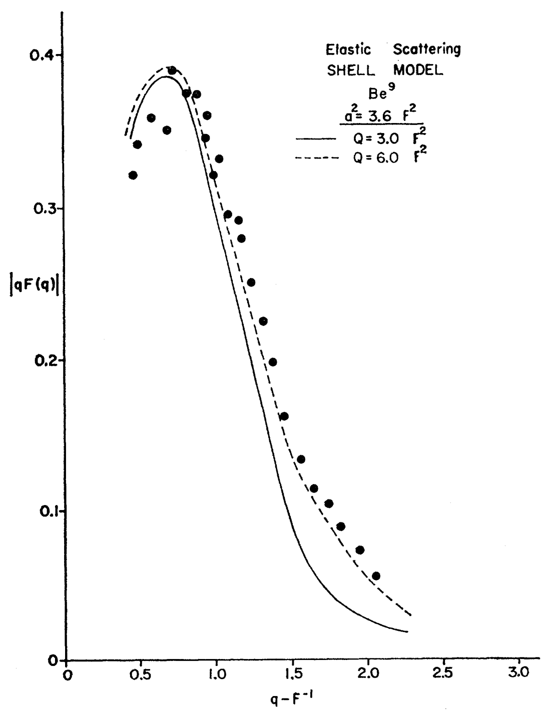

FIG. 9. Elastic scattering of electrons from Be^9 by the shell model.

The undeformed shell model gives

$$\begin{aligned}\rho_0(r)&=[2/(\pi a^3)][2+\tfrac{2}{3}(Z-2)r^2/a^2]\exp(-r^2/a^2),\\ \rho_2(r)&=[4/(3\sqrt{5}\pi a^2)]Qr^2\exp(-r^2/a^2),\end{aligned} \quad (16)$$

where Q is the quadrupole moment of the ground state of the nucleus. The elastic cross section is

$$\sigma_{\text{elastic}}(\theta)=\sigma_c(\theta)|F_p(q)|^2[|F_0(q)|^2+|F_2(q)|^2]. \quad (17)$$

Here

$$F_0(q)=[1-\tfrac{1}{6}(Z-2)a^2q^2/Z]\exp(-q^2a^2/4) \quad (18)$$

and

$$F_2(q)=(1/30)(5/P_I)^{1/2}(Q/Z)q^2\exp(-q^2a^2/4) \quad (19)$$

where $P_I=I(2I-1)/(I+1)(2I+3)$. Since Be^9 has a ground-state quadrupole moment, the term $F_2(q)$ is

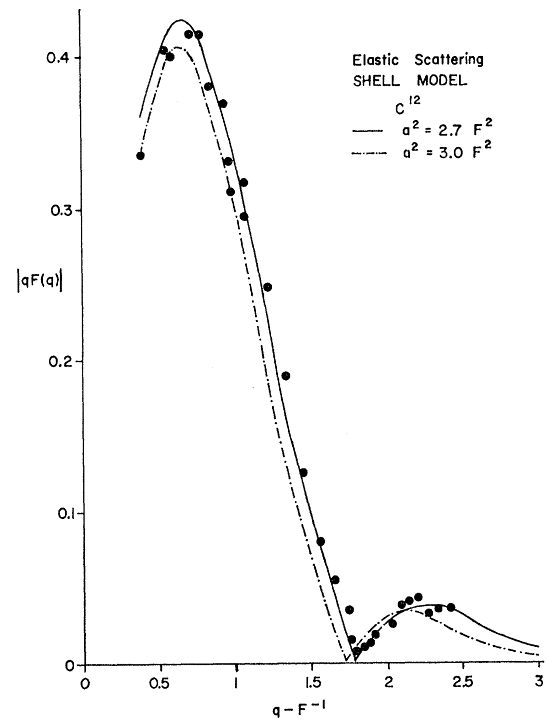

FIG. 10. Elastic scattering of electrons from C^{12} by the shell model.

nonzero. In this case the ground-state spin is $I=3/2$, and

$$F_2(q)=\tfrac{1}{6}(Q/Z)q^2\exp(-q^2a^2/4). \quad (20)$$

The nucleus C^{12} has no quadrupole moment and the quadrupole contribution to the elastic scattering is zero.

The theoretical elastic cross sections are compared with the experimental ones in Fig. 9 for Be^9 and Fig. 10 for C^{12} for various values of the parameters a^2 and Q.

In a similar fashion to the ground-state charge distribution we define a transition charge density between the ground and excited state of the nucleus. The quadrupole part of the inelastic-scattering cross section can be expressed by

$$\sigma_{\text{inelastic}}(\theta)=\sigma_c(\theta)|F_p(q)|^2|F_2'(q)|^2, \quad (21)$$

where

$$F_2'(q)=(5^{1/2}/30)(I2I0|I'I)(Q_0/Z)q^2 \times\exp(-q^2a^2/4). \quad (22)$$

Here Q_0 is related to the quadrupole matrix element between the states I and I', and $(I2I0/I'I)$ is a Clebsch-Gordan coefficient. The quantity Q_0 is similar to the intrinsic quadrupole moment Q_0 of the alpha-particle model and Nilsson model.

The expression $F_2'(q)$ for the Be^9 inelastic scattering from $3/2^-$ ground state to the $5/2^-$ 2.43-MeV level is

$$F_2'(q)=(18/7)^{1/2}(1/30)(Q_0/Z)q^2\exp(-q^2a^2/4). \quad (23)$$

For C^{12} the expression $F_2'(q)$ for the inelastic scattering

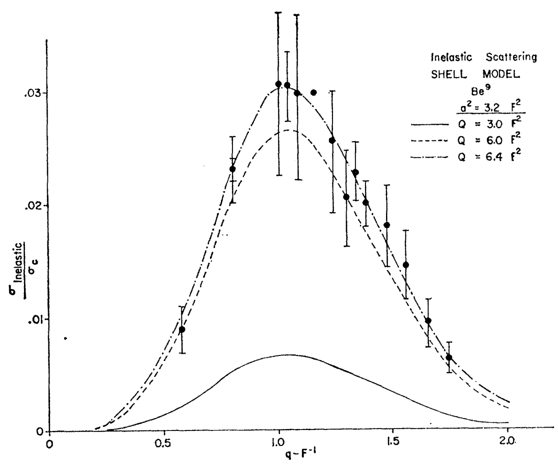

FIG. 11. Inelastic scattering of electrons from the Be^9 2.43-MeV state by the shell model.

from the 0^+ ground state to the $2+4.43$-MeV level is

$$F_2'(q)=(5^{1/2}/30)(Q_0/Z)q^2\exp(-q^2a^2/4). \quad (24)$$

The ratio of the inelastic scattering to the point nucleus Coulomb scattering is compared to experimental data in Fig. 11 for Be^9 and Fig. 12 for C^{12} for various values of Q_0.

V. DISCUSSION

The comparison of the three models to the inelastic-scattering data produces varying degrees of agreement. The analysis of the data should bear in mind two major approximations made in deriving the cross sections (a) the use of the Born approximation and (b) the neglect of magnetic dipole excitation. The use of the Born approximation for these nuclei gives markedly different results from the more accurate phase shift calculation wherever a diffraction minimum occurs. The main effect of the nuclear dispersion is to fill in this minimum and enhance the cross section over the Born approximation on either side of the minimum. The inelastic cross

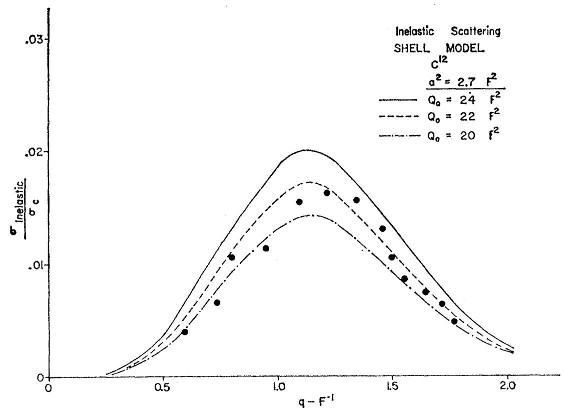

FIG. 12. Inelastic scattering of electrons from the C^{12} 4.43-MeV state by the shell model.

sections for both Be^9 and C^{12} seem to have a zero just beyond the range of the data at about $q=2.0$ F^{-1}. This enhancement would seem to be important for values of q greater than 1.5 F^{-1}. This would modify the fit of the shell model at the high momentum transfer end but would not be enough enhancement for the Nilsson model which begins to fail to give a good fit for q values of less than 1.0 F^{-1} where one expects the Born approximation to be good. The effect on the alpha-particle model would be to make an even worse fit at the large q values.

The effect of the magnetic dipole excitations were estimated using an extreme model in which the density distributions for this process is concentrated at the nuclear surface. A radius of 3.14 F for these distributions was used in estimating these effects. This particular radius gave the maximum contribution to the Be^9 inelastic cross section at $q=0.5$ F^{-1} and $q=1.5$ F^{-1}. The resulting cross sections at these points were increased by 10%. A more reasonable density distribution of the nuclear currents and magnetic moment proportional to the $1P$ shell distribution gives a 5% addition to the inelastic cross section. The effect of the magnetic excitations could lead to a sizeable contribution to the cross section at the diffraction minimum.

Neither the alpha-particle model nor the Nilsson model can reproduce the inelastic-scattering data even considering the corrections due to the approximations, although these models can give one magnitude of the cross sections at the lower momentum transfers where the details of the charge distribution are unimportant. The enhanced shell model can predict the proper shape for the cross sections, although the fit seems accidental at the highest moment transfers when one considers the errors in the approximations used. At the lower momentum transfers the quadrupole matrix element needs an enhancement by a factor of 2 over the usual shell model to predict the magnitude of the cross section. This result has been found by many others in analyzing the C^{12} data. These three models when the parameters are adjusted to give agreement at the lower

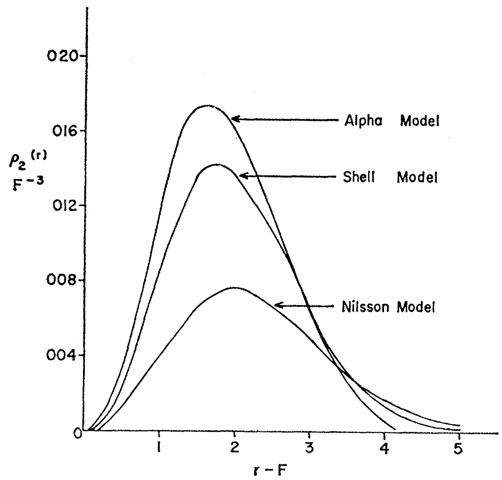

FIG. 13. The quadrupole charge distribution of the ground state of Be^9 as determined from the three models.

range of q differ considerably at the higher range of q where the details of the quadrupole charge distributions are the three models for the ground state of Be^9 are shown in Fig. 13.

The alpha model which consists of tightly bound alpha substructures tends to give a cross section too high at the high momentum transfers since the tight alpha structure gives a high central peak, and a fast rate of falloff for the charge density. The separation of the alpha particles gives the large quadrupole effect and also contributes to the decrease in cross section at high momentum transfers. The finite size of the alphas then adds to this decrease. In this model the distribution of charge in the alpha particles was taken to be that of the free alpha particle and possible distortion effects due to the alpha-alpha attraction, which may increase the size of the alpha particles, have been neglected.[14] A slight increase in the size of the alpha particles would lower the high momentum transfer prediction while keeping the low momentum transfer prediction unchanged. Also neglected is the effect of the zero-point variations of the alphas which also tends to cut down the cross section at large momentum transfers.

The Nilsson model which consists of protons and neutrons moving in a harmonic oscillator with the quadrupole part of the field proportional to $r^2P_2(\theta)$ predicts too low an inelastic cross section even at moderately low momentum transfers. This behavior is interpreted as a result of the quadrupole charge density being pushed out too far from the origin. For the large distortions present in the two nuclei, Be^9 and C^{12}, the major axis of the spheroid representing the nucleus is large compared to the size of an equivalent spherical nucleus. This large size of the major axis in the model gives a small peak and a slow rate of decrease of the charge density and this is then evident in the small predicted cross section at high momentum transfers. Also the peak in the curve $\sigma_{\text{inel}}/\sigma_{\text{el}}$ is shifted to a lower value of the momentum transfer from the large size of the major axis. A modification of this model to represent a large distortion of the wave functions of the particles to give the large quadrupole matrix elements yet to obtain a faster decrease in the charge density from the origin must be made in order to use this model for high-energy phenomena.

The third model, the modified shell model, is able to predict the shape dependence of the cross section on the momentum transfer over the range available from the experiments. The scaling of the quadrupole moment by a factor of approximately 2 in order to obtain the proper magnitude of the cross section is generally due to the lack of consideration in the model of mixing of different orbital angular momenta into nuclear wave functions by internuclear forces. For the $l=1$ orbital angular momentum states, a small amount of $l=3$ angular momentum is mixed into these states, and a large enhancement[11] of the quadrupole moment can be obtained. The shell model wave functions give the shape of the long-range behavior of the nuclear charge density to a good degree of approximation, although the model is simplified in that is does not contain correlation effects of the interparticle forces. This necessity for enhancement of the quadrupole excitation in C^{12} has been noted by others in their analysis of the inelastic C^{12} data.

The three models give good fits of the elastic-scattering data for the nucleus C^{12} while only the modified shell model gives a good fit for the nucleus Be^9. For these cases the same sets of parameters do not give good fits to both the elastic and inelastic scattering curves. The discrepancy between the theoretical curves and experiment for the elastic scattering of electrons from Be^9 for the alpha particle and Nilsson models is attributed mainly to the failure of these models to describe the behavior of the quadrupole component of the nuclear charge density. The nucleus Be^9, having a ground state spin of 3/2, contributes both a monopole and quadrupole to the elastic-scattering cross section. The nucleus C^{12}, having a ground-state spin of zero, makes no quadrupole contribution to the elastic scattering. The inelastic-scattering cross section is more sensitive than the elastic-scattering cross section to the quadrupole charge density.

The inelastic-scattering cross section data can be extrapolated to zero momentum transfer to give some information not otherwise available on the strength of electromagnetic transitions and quadrupole moments of these nuclei. From the alpha-particle model, the values of the interalpha distance, 2ξ, to reproduce the low-q data in the inelastic cross section is 3.6 F for Be^9 and 3.4 F for C^{12}. These values are somewhat small considering the values for this parameter from fitting the

[14] L. D. Pearlstein, Y. C. Tang, and K. Wildermuth, Nuclear Phys. **18**, 23 (1960). There is some evidence of increase in the alpha-particle size from the alpha-alpha interaction but a more probable contribution should come from the neutron-alpha interaction in Be^9.

2.43-MeV level in Be^9 and 4.42-MeV level in C^{12} to a rigid body moment of inertia are 4.6 F for Be^9 and 3.8 F for C^{12}. However, the experimentally determined parameter, 2ξ, gives a ground-state quadrupole moment for Be^9 of 5.2 F^2 although the value of $2\xi=4.0$ F which gives a good fit to the peak of the elastic data gives a moment of 6.4 F^2. The intrinsic quadrupole moment for C^{12} (the quadrupole moment in the coordinate system in which the alpha particles are at rest) is $Q_0=-23$ F^2. Inopin, in his analysis, obtains a quadrupole moment of 5.8 F^2 for Be^9 using a value $2\xi=3.8$ and gives an intrinsic moment of -15 F^2 for C^{12} using a value of $2\xi=3.0$. The adjustment of the size of the harmonic-oscillator well parameter, $\hbar\omega_0$, in the Nilsson model to fit the elastic data and then the size of the distortion parameter, ϵ, to fit the low momentum transfer data in the inelastic data gives a quadrupole moment for Be^9 of 6.0 F^2 and a value of Q_0 for C^{12} of -23 F^2. In a similar fashion the modified shell model gives a quadrupole moment for Be^9 of 6.4 F^2 and an intrinsic moment for C^{12} of ± 22 F^2. These values are in contrast to the maximum values that the usual shell model can predict, of $Q=2.7$ F^2 for Be^9 and of $Q_0=-10.8$ F^2 for C^{12}. The experimental value[15] for the quadrupole moment of Be^9 is $Q=2.9$ F^2 which comes from the quadrupole splitting in magnetic resonance experiments on Be metal. The determinations of the quadrupole moments require the value of the gradient of the electron field at the Be^9 lattice site. This is a difficult quantity to obtain and the lack of agreement between the magnetic resonance method and electron scattering method of obtaining of the quadrupole moment is not serious. The intrinsic quadrupole moment of C^{12} can be calculated from the lifetime of the gamma ray emitted from the 4.43-MeV state of C^{12}. This lifetime[16] gives a value of

$$Q_0=\pm(19\pm 2)\ \mathrm{F}^2$$

for C^{12}.

The size of the oscillator well for these two nuclei given by the Nilsson model is $\hbar\omega_0=14$ MeV for Be^9 and $\hbar\omega_0=16$ MeV for C^{12}. These values are from fitting the peak in the elastic curve for the two nuclei. The shell model give the size of the oscillator well as $\hbar\omega=11.5$ MeV for Be^9 and $\hbar\omega=15.3$ MeV for C^{12}. The analysis of the high-energy electron inelastic scattering data can give useful information concerning the shape and size of the higher moments of the nuclear charge density. Also one can learn of the validity of predictions of these models in nuclear scattering where large momentum transfers occur at low energies.

ACKNOWLEDGMENTS

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[15] M. Pomerantz and T. P. Das, Bull. Am. Phys. Soc. 4, 251 (1959).

[16] V. K. Rasmussen, F. R. Metzger, and C. P. Swann, Phys. Rev. **110**, 154 (1958).

APPENDIX

The solutions of the Nilsson Hamiltonian which describe the four lowest states of the deformed nucleus are given in terms of the asymptotic solutions which are valid for the limit of very strong deformation. The spin-orbit term will mix some of these states but for the large deformations of Be^9 and C^{12} these functions will approximately describe the nucleons moving in the potential well.

The states are described by the state vector

$$|N_\zeta N_\perp \Lambda\Sigma\rangle, \tag{A1}$$

where N_ζ is the oscillator quantum number for the ζ direction, $N_\perp$ is the sum of the quantum numbers for the ξ and η directions, Λ is the quantum number for the ζ component of "angular momentum"

$$l_\zeta=-i\hbar\left(\xi\frac{\partial}{\partial\eta}-\eta\frac{\partial}{\partial\xi}\right), \tag{A2}$$

and Σ is the projection of the nucleon spin in the ζ axis.

The four wave functions for the representation of the $1P$ shell nuclei are

$$\begin{aligned}|000\tfrac{1}{2}\rangle&=(2/\sqrt{\pi})^{3/2}\exp[-(\xi^2+\eta^2+\zeta^2)]\chi_{1/2},\\ |100\tfrac{1}{2}\rangle&=\sqrt{2}(2/\sqrt{\pi})^{3/2}\exp[-(\xi^2+\eta^2+\zeta^2)]\chi_{1/2},\\ |011\tfrac{1}{2}\rangle&=(2/\sqrt{\pi})^{3/2}\exp[-(\xi^2+\eta^2+\zeta^2)]\chi_{1/2},\\ |011-\tfrac{1}{2}\rangle&=(2/\sqrt{\pi})^{3/2}\exp[-\xi^2+\eta^2+\zeta^2)]\chi_{-1/2},\end{aligned} \tag{A3}$$

where χ_m is the spin function. The approximate solutions to the Nilsson Hamiltonian are given by

$$\begin{aligned}\psi_1&=|000\tfrac{1}{2}\rangle,\\ \psi_2&=|011\tfrac{1}{2}\rangle,\\ \psi_3&=A|100\tfrac{1}{2}\rangle+B|001-\tfrac{1}{2}\rangle,\\ \psi_4&=-B|100\tfrac{1}{2}\rangle+A|011-\tfrac{1}{2}\rangle,\end{aligned} \tag{A4}$$

where

$$\begin{aligned}A&=2\sqrt{2}\{[((n+1)^2+8)^{1/2}+1-n]^2+8\}^{-1/2},\\ B&=(1-A^2)^{1/2},\\ n&=-2\epsilon\hbar\omega_0/C.\end{aligned} \tag{A5}$$

The function F_0^i, F_2^i, and F_4^i defined in the cross sections are as follows:

$$\begin{aligned}F_0^i&=\int j_0(qr)|\psi_i|^2d\mathbf{r},\\ F_2^i&=5^{1/2}\int j_2(qr)p_2(\theta)|\psi_i|^2d\mathbf{r},\\ F_4^i&=9^{1/2}\int j_4(qr)p_4(\theta)|\psi_i|^2d\mathbf{r}.\end{aligned} \tag{A6}$$