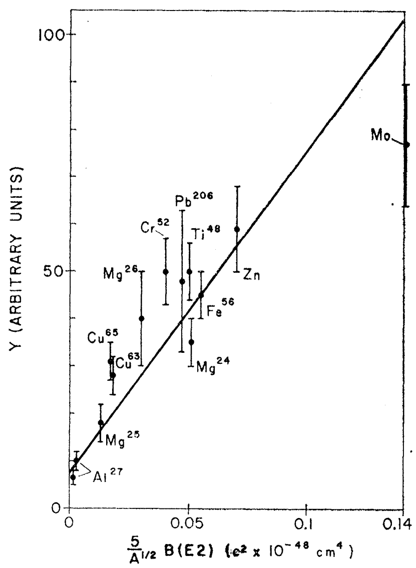

FIG. 9. The measured relative probability Y for exciting a nuclear state by neutron inelastic scattering vs the quantity $5A^{-1/2}\,B(E2)$ for that state. The straight line is a least-square fit of the type described in the text.

quantity $5A^{-1/2}B(E2)$, together with the line which gives the best fit to the data. As can be seen, the deviations of the data from this line are greater than those in Fig. 8. The chi-square value of the fit is 2.2 times the expected value. The best fits to the data when the parameter n in Eq. (4) is 0.0 and -1.0, have, respectively, chi-square values of 3.1 and 3.6 times the expected value. To summarize, if Eq. (4) is considered a valid expression for all of the data presented in Table I, then the parameter n has a most probable value of -0.5. However, no value of n could be found for which Eq. (4) gives a good description of the data.

The significance of the results illustrated in Figs. 8 and 9 is not clear, since the median energy of the neutrons used in these experiments, approximately 2–3 MeV, is such that contributions from direct interactions are not expected to be important. However, for the neutron energies used, and for the nuclear levels studied in these experiments, it seems clear that a correlation does exist between the probability of exciting a nuclear state by neutron inealstic scattering and the reduced quadrupole transition probability, $B(E2)$, of that state.

ACKNOWLEDGMENTS

The author thanks Dr. P. Signell, who prepared the computer program used in the statistical analyses of the data, and the members of the Pennsylvania State reactor operating staff, who provided the neutrons.

PHYSICAL REVIEW VOLUME 128, NUMBER 3 NOVEMBER 1, 1962

Optical Model in the Interior of the Nucleus*

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(Received June 25, 1962)

It is shown how phase relationships between partial waves in the optical model result in focusing effects which are largest for surface partial waves. In the case of very low incident energies where the wavelength of a partial wave is large compared to the nuclear surface thickness, it is shown how the phase relationships may enable surface- and volume-reaction mechanisms for a direct reaction to be distinguished by looking at gross features of the angular distribution. Thus, low-energy direct interactions may be used as a probe for the nuclear interior.

1. INTRODUCTION

It is well known that elastic scattering cross sections are sensitive only to the optical-model potential in the nuclear surface.[1] However, it is also true that optical-model wave functions calculated in potentials which fit elastic scattering data are not negligibly small in the nuclear interior[2] even for incident particles such as deuterons or α particles which may be expected on physical grounds to lose their identity when they get into the interior. One would, of course, expect the physical breakup of the particle to contribute to the imaginary part of the potential so that the model should give a good estimate of the probability of finding a particle inside.

This apparent contradiction has led some people to consider that the optical model must be regarded more as a parametrization of elastic scattering than as a physical description of a nuclear reaction. The success of the distorted-wave Born approximation for direct interactions, which consists of a sum of overlap integrals involving the optical-model wave function in the vicinity of the nucleus, shows that the wave function

* Supported by the Australian Atomic Energy Commission and the Australian Institute for Nuclear Science and Engineering.

[1] J. S. Nodvik, *Proceedings of the International Conference on the Nuclear Optical Model, Florida State University Studies, No. 32* (The Florida State University, Tallahassee, 1959), p. 16; G. Igo, Phys. Rev. **115**, 1665 (1959).

[2] I. E. McCarthy, Nuclear Phys. **10**, 583 (1959).

has physical meaning near the nucleus, not merely at infinity. This led Eisberg and McCarthy[3] to propose that direct reactions could give information about the nuclear interior. However, in most cases which have been studied it can be shown that the interior of the nucleus, even though the overlap integral is taken from $r=0$ to ∞, contributes very little to the cross section for reasons which are inherent in the optical model.

In a simple picture of nucleon inelastic scattering Elton and Gomes[4] showed that if the incident particle penetrates the nucleus and collides with an interior particle, the reaction product particle is almost certain to be totally internally reflected at the surface. Austern[5] showed in a detailed calculation of an optical model involving a large number of partial waves how phase averaging in an overlap integral involving partial waves of small l (i.e., ones which penetrate the interior) results in the integral averaging almost to zero.

The picture of Elton and Gomes is incomplete in that it considers contributions to the reaction only from regions of the nucleus where a particle traveling along a given trajectory (in a semiclassical picture) is likely to penetrate without first being involved in a collision and also from which it is likely to escape along a particular trajectory without a further collision. It was shown by McCarthy[6] that there is a region of the nucleus, called the "focus" where a particle is likely to be found, not because it is very likely to get there on a particular trajectory, but because many trajectories lead to the same point. Similarly, for outgoing particles, most of the particles originating at the focus are likely to come out in the same direction if they do come out at all.

At very low energies ($\cong 5$ MeV) the focus in the optical-model wave function is in the interior of the nucleus. Hence, effects which can be attributed to the focus can in these cases be attributed to the interior. Simple direct-interaction models[7] involving the focus at an energy where it is in the surface have confirmed the idea that the focus is responsible for the observed large departure of nucleon inelastic scattering cross sections from the spherical Bessel function shape which is predicted by the plane-wave Born approximation particularly at scattering angles $\Theta \cong 0$ and π.

Identification of such effects at low energies would confirm the idea that the interior plays a part in the reaction. The question of precisely what effects to look for can only be answered by a detailed investigation of distorted-wave angular distributions at low energies. The present work shows that the interior contributes a large amount to forward and backward cross sections at low energies and that large differences in angular distributions are to be expected between surface and volume-interaction theories.

To put the rather pictorial idea of the focal contribution to direct reactions into partial wave language, we may say that the partial waves with $l \cong kR$ (where R is the nuclear radius, k the exterior wave number), which are the ones which contribute principally to the distorted-wave overlap integral, are also the ones which are related to each other least like the partial waves of a plane wave. In the optical-model wave function this special phase relationship leads to constructive interference which produces the focus. This relationship can also lead to large differences between the distorted-wave overlap integrals for the surface- and volume-interaction cases when the wavelength of a partial wave is larger than the surface thickness, i.e., in the 5-MeV energy region. This is the case that was specifically excluded in Austern's paper[5] on optical-model wave functions. This paper will be referred to throughout the present work.

It may, of course, be true that the surface-reaction theory is true for reasons connected with the reaction mechanism rather than the optical model. For example, the surface-reaction theory is on a very sound basis for reactions which excite collective states where it predicts the right total cross section for the reaction.[8] It is also likely that for a two-body mechanism the two-body force is less in the nuclear interior than on the surface, because the Pauli principle inhibits collisions which lead to filled states.[9] Once the surface-reaction theory is shown to be wholly or partially true in a case where it is not a necessary consequence of the optical model, then the reaction can be used to get information, for example, about the relative strength of the two-body force in the surface and the interior.

Section 2 is a discussion of the phase relationships between partial waves in the optical model showing how they can be understood in general, and how they result in focusing effects.

In Sec. 3 a detailed calculation is made of the integrands of the overlap integrals for various combinations of partial waves using the optical-model wave function computed numerically for 5-MeV protons on C^{12}. The phase considerations of Sec. 2 are extended to the distorted-wave Born approximation case and it is shown how gross features of angular distributions may be understood without detailed calculations at low energy.

In Sec. 4 these considerations are shown to apply in an experimental case and the type of experiment required to distinguish surface from volume interaction is discussed.

The considerations are not regarded as applying specifically to any particular reaction, or to C^{12}, since

[3] I. E. McCarthy, in *Proceedings of the International Conference on the Nuclear Optical Model, Florida State University Studies, No. 32* (The Florida State University, Tallahassee, 1959), p. 24.
[4] L. R. B. Elton and L. C. Gomes, Phys. Rev. **105**, 1027 (1957).
[5] N. Austern, Ann. Phys. (New York) **15**, 299 (1961).
[6] I. E. McCarthy, Nuclear Phys. **11**, 574 (1959).
[7] I. E. McCarthy and D. L. Pursey, Phys. Rev. **122**, 578 (1961); A. J. Kromminga and I. E. McCarthy, Nuclear Phys. **31**, 678 (1962).
[8] E. Rost and N. Austern, Phys. Rev. **120**, 1375 (1960).
[9] N. K. Glendenning, Phys. Rev. **114**, 1297 (1959).

small changes in energy and nuclear radius do not appreciably change the phase relationships of the partial waves. Also, the Coulomb phase shifts are not very large for low values of l, although the Coulomb potential causes a difference in magnitude between proton and neutron partial waves.

2. GENERAL BEHAVIOR OF PARTIAL WAVES IN THE OPTICAL MODEL

If we neglect the spin of the incident particle, the optical-model wave function may be written as

$$\chi^{(+)}(\mathbf{r})=\sum_{l=0}^{\infty}\psi_l(\rho)P_l(\cos\theta), \tag{1}$$

where

$$\rho=kr, \tag{2}$$

$$\psi_l(\rho)=i^l(2l+1)f_l(\rho)e^{i\sigma_l}. \tag{3}$$

σ_l is the Coulomb phase shift. For large values of ρ, $f_l(\rho)$ is given by

$$f_l(\rho)\cong\rho^{-1}\{F_l(n,\rho)+C_l[G_l(n,\rho)+iF_l(n,\rho)]\}. \tag{4}$$

F_l and G_l are the Coulomb functions, regular and irregular at the origin, respectively; n is the Coulomb parameter

$$n=ZZ'e^2/\hbar v, \tag{5}$$

C_l is related to the nuclear phase shift δ_l by

$$\eta_l=\exp(2i\delta_l)=2iC_l+1. \tag{6}$$

For uncharged particles the Coulomb functions become spherical Bessel functions, and $f_l(\rho)$ may be written

$$f_l(\rho)\cong\tfrac{1}{2}[h_l^{(2)}(\rho)+\eta_l h_l^{(1)}(\rho)], \tag{7}$$

where $h_l^{(2)}$ and $h_l^{(1)}$ are, respectively, the incoming and outgoing spherical Hankel functions. In (7), η_l may be regarded as the reflection coefficient for the lth partial wave. For a complex potential, $|\eta_l|$ is small[5] for small values of l, and approximately unity for large values of l. This means that at some radius outside the nuclear potential we have approximately inward traveling waves for small l, and standing waves reflected from the centrifugal barrier for large l. For large l,

$$f_l(\rho)\cong j_l(\rho)$$

just as for plane waves.

The smallness of $|\eta_l|$ for small l is due mainly to a phase averaging effect in the nuclear interior rather than to absorption of the outgoing wave by the potential. This has been shown in the WKB approximation by Austern.[5] This effect is more pronounced for large nuclei and high energies where the wavelength of the partial wave is small compared to the nuclear surface thickness and several wavelengths contribute to the integrand of the WKB expression for η_l, so that the oscillations have a chance to average to zero. At very high energies, of course, the potential becomes negligible and $\eta_l\rightarrow 1$.

Intermediate partial waves have intermediate values of η_l. These are the ones which sample the nuclear surface and give rise to the dependence of the scattering amplitude on the details of the potential.

In this section, we will be concerned with the phase of each partial wave.

$$\begin{aligned}\arg\psi_l(\rho)&=l\pi/2+\arg f_l(\rho)\\&=l\pi/2+\varphi_l(\rho).\end{aligned} \tag{8}$$

We will develop a general understanding of the behavior of $\varphi_l(\rho)$ for small ($\ll kR$), large ($\gg kR$), and intermediate values of l and ρ. In order to understand the interference of different partial waves we must remember that the magnitude of the lth partial wave is largest for $\rho\cong l$, and for a particular ρ the partial waves which contribute most to the wave function $\chi^{(+)}(\mathbf{r})$ are those for $l\cong\rho+1$, ρ and $\rho-1$. This will give us sufficient accuracy for general understanding. For the purposes of this section we will be concerned, except where it is stated otherwise, with $\varphi_l(\rho)$ only for values of ρ less than the first zero of the corresponding standing wave. In the standing wave there is a sudden phase change of π at the zeros. We will contrast the interference of the partial waves $\psi_l(\rho)$ in the optical-model case with that in the plane-wave case.

In the plane-wave case,

$$\varphi_l(\rho)=0 \tag{9}$$

for all l, and successive partial waves are 90° apart. This is also true for large l in the uncharged case. For large l in the charged case,

$$\varphi_l(\rho)=\sigma_l, \tag{10}$$

whose value is given for $n\lesssim 1$ and large l by

$$\sigma_l\cong n\ln l. \tag{11}$$

For small l and large ρ, $f_l(\rho)$ is approximately an incoming traveling wave whose phase is given if $|\eta_l|\cong 0$ by

$$\varphi_l(\rho)\cong-(\rho-n\ln 2\rho-l\pi/2+\sigma_l). \tag{12}$$

If $|\eta_l|$ is not very small, $\varphi_l(\rho)$ may still be expected to decrease for large ρ. For small l and a value of ρ well inside the nucleus but outside the centrifugal barrier, the phase of $f_l(\rho)$ is shifted from the plane-wave value by the nuclear potential.[5] Since $|\eta_l|$ is small in this case, this phase shift is given quite accurately by the lowest WKB approximation. In this approximation

$$\varphi_l(\rho)=\int_{\infty}^{\rho}[k'(\rho)-k]d\rho, \tag{13}$$

where k' is the internal wave number. In this case, the outside traveling wave joins smoothly through the nuclear surface region to the inside wave function, which for $\rho \cong 0$ must be proportional to the regular spherical Bessel function $j_l(\rho')$ apart from a phase factor. ρ' is a complex number

$$\rho' = k'(\rho) r. \tag{14}$$

In the case of nucleon scattering where the real and imaginary parts of the optical-model potential are something like

$$V = -40 \text{ MeV}, \quad W = -8 \text{ MeV},$$

$j_l(\rho')$ is approximately real for small ρ, so we expect $\varphi_l(\rho)$ to be close to the value given by the phase shift in the potential as ρ increases from zero to a value outside the centrifugal barrier. Here, $f_l(\rho)$ begins to assume the form of an inward traveling wave, the outward component being reduced by phase averaging, so $\varphi_l(\rho)$ will begin to decrease as ρ increases with a very rapid decrease at the zeros of $j_l(\rho')$.

These properties of φ_l are illustrated in Figs. 1 and 2 for the scattering of 30-MeV neutrons (case 1) and 5-MeV protons (case 2) from C^{12}. The optical-model parameters are in both cases:

$$V = -40 \text{ MeV}, \quad W = -8 \text{ MeV}, \quad r_0 = 1.2 \text{ F}, \quad a = 0.5 \text{ F}$$

Comparison of Figs. 1(a) and 2(a) with 1(b) and 2(b), respectively, will show that the magnitude of the lth partial wave becomes insignificant in comparison with that of higher waves by the time $\varphi_l(\rho)$ begins to decrease rapidly with ρ for small l. We may say that the effective value of φ_l for small l is the value given by the phase shift in the potential.

In Fig. 1 the inside partial waves are $l = 0, 1, 2$, the outside ones are $l = 4, 5$, etc. The intermediate partial wave, $l = 3$, has values of $\varphi_l(\rho)$ intermediate between the values for the inside and outside partial waves. In Fig. 2 the inside partial waves are $l = 0, 1$, $l = 2$ is intermediate and $l = 3, 4, 5$, etc. are outside partial waves. Notice that φ_l for the outside ones increases slowly with l because of the Coulomb phase shift in this case. Note also that in neither case is the nucleus

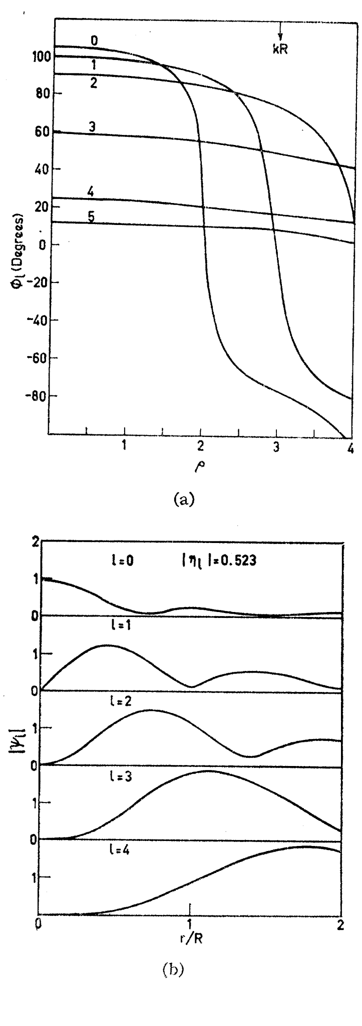

FIG. 1. (a) The phase angles $\varphi_l(\rho)$ and (b) the magnitudes $|\psi_l(\rho)|$ of the first few partial waves for the elastic scattering of 30-MeV neutrons on C^{12}. The optical-model potentials are given in the text.

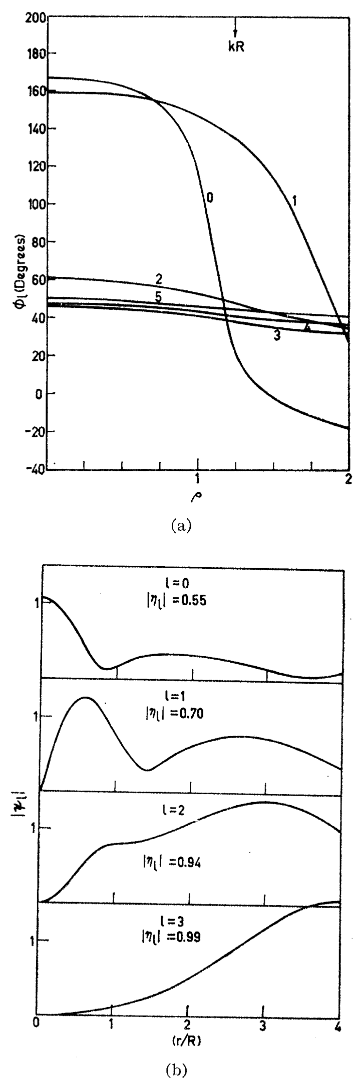

FIG. 2. (a) The phase angles $\varphi_l(\rho)$ and (b) the magnitudes $|\psi_l(\rho)|$ of the first few partial waves for the elastic scattering of 5-MeV protons on C^{12}. The optical-model parameters are given in the text.

very "black" to any of the partial waves. The inside partial waves have important characteristics of both standing and traveling waves. There is a strong vestige of the abrupt phase change at the zeros of the standing wave component, but the magnitude is not particularly small where the phase is changing rapidly. This provides a key to the understanding of the next section where we are concerned with products of pairs of partial waves and cannot confine ourselves to consideration only of the phases at values of ρ before the first zero of the standing wave component.

Now that we have seen the behavior of $\varphi_l(\rho)$, in general, we are in a position to discuss the interference of partial waves. For a given ρ, we are only interested in the partial waves $l-1$, l and $l+1$, where l is the nearest integer to ρ, because the others will not be large enough in magnitude to effect the gross considerations in which we are interested.

We are mainly interested in whether these partial waves interfere constructively or destructively, i.e., whether successive $\psi_l(\rho)$ have a phase difference less than or greater than 90°. This property, together with the question of whether a partial wave is an inside, outside, or intermediate one will not vary much with changes of radius or internal wave number of the order of 10%; so the computed cases are typical of quite a wide range of energies and nuclei.

In the plane wave case, successive partial waves are 90° apart. The wave function in the forward ($\theta<\pi/2$) hemisphere is the complex conjugate of that in the backward hemisphere, since

$$P_l(\cos\theta)=(-1)^l P_l[\cos(\pi-\theta)]. \qquad (15)$$

i.e., one obtains the forward wave function from the backward wave function by reversing the odd partial waves. The effect of the phase shift $\varphi_l(\rho)$ on each $\psi_l(\rho)$ is to rotate it in the positive direction. Different $\psi_l(\rho)$ are rotated by different amounts so that they are no longer at right angles to each other. This produces a forward-backward asymmetry in $|\chi^{(+)}(\mathbf{r})|$.

We will be mainly concerned with the forward-backward asymmetry of $|\chi^{(+)}(\mathbf{r})|$ on the axis where $P_l(\cos 0)=1$, $P_l(\cos\pi)=(-1)^l$. The magnitude of the wave function off the axis can easily be understood by an extension of our considerations.

If successive $\varphi_l(\rho)$ (regarded now as practically independent of ρ where $|\psi_l(\rho)|$ is large enough to be effective) are different, then the corresponding partial waves will contribute differently to $|\chi^{(+)}(\mathbf{r})|$ at $\theta=0$ and π. The rule for this is as follows.

If $\varphi_{l+1}<\varphi_l$ then ψ_l and ψ_{l+1} interfere constructively at 0 and destructively at π. Since the major contribution to $|\chi^{(+)}(\mathbf{r})|$ at a particular value of ρ comes from only three or four successive partial waves, this rule enables us to tell from the phase diagram whether we have large or small values of $|\chi^{(+)}(\mathbf{r})|$.

Consider the forward direction, $\theta=0$. Then φ_l for inside partial waves are all very similar. This can be seen in the WKB approximation where the classical rays corresponding to them all go near the center. In the uncharged case we always have $\varphi_{l+1}\lesssim\varphi_l$, so for small ρ we will always have slight constructive interference. The value of $|\chi^{(+)}(\mathbf{r})|$ will be somewhere near 1 but decreased by the attenuation due to the absorption. In the charged case the Coulomb phase shift may reverse the situation giving $\varphi_{l+1}\gtrsim\varphi_l$ and slight destructive interference.

For outside partial waves in the uncharged case at $\theta=0$, φ_l is always near zero. Any interference will be constructive. In the charged case the phase shift is σ_l which increases slowly with l, so we will have slight destructive interference corresponding to the bending of the rays away from the nucleus by the Coulomb potential.

The intermediate case is more interesting. Here, the values of φ_l are intermediate between the large values for small l and the small values for large l. We always have $\varphi_{l+1}<\varphi_l$ and the difference amounts to a large angle, sometimes as much as 90°. This results in extremely strong constructive interference at $\theta=0$ and strong destructive interference at $\theta=\pi$.

At $\theta=\pi$ the strong destructive interference for $l\cong kR$ allows other partial waves to contribute a higher proportion to $|\chi^{(+)}(\mathbf{r})|$.

For other angles the important angular factor near $\theta=0$ is $P_l(\cos\theta)$ where $l\cong kR$. This decreases rapidly as θ increases so the strong constructive interference exists for a small range of θ near 0 and a small range of ρ near the surface. It appears as an intense spot in $|\chi^{(+)}(\mathbf{r})|$. Near $\theta=\pi$ lower values of l contribute to the angular factor and the falloff of $|\chi^{(+)}(\mathbf{r})|$, with θ, is not so steep.

All these properties of $|\chi^{(+)}(\mathbf{r})|$ are illustrated in the calculations of the divergence of the probability flux [actually $W(r)|\chi^{(+)}(\mathbf{r})|^2$] performed by Eisberg, McCarthy, and Spurrier[10] and McCarthy.[2]

In classical language, constructive interference in the forward hemisphere between successive partial waves can be described as focusing, destructive interference (such as occurs outside the nucleus in the charged case) as defocusing. The focusing effect is most prominent for the intermediate values of l (i.e., the last ones whose classical rays do not miss the nucleus) as is shown in the detailed comparison of quantal and classical calculations in reference 6.

Austern[5] prefers to regard the intense spot on the forward surface in the case of 40-MeV α-particle scattering, when the nucleus is more "black," as due to diffraction of the wave function round the nucleus rather than refraction. Of course, it is rather arbitrary to distinguish diffraction from refraction in a complex potential, but the reason for the intense spot can be

[10] R. M. Eisberg, I. E. McCarthy, R. A. Spurrier, Nuclear Phys. **10**, 571 (1959).

seen here. It is due to different partial waves having phase differences less than 90° which is in turn due to the fact that the nucleus has a surface. The phase is determined mainly by the real part of the potential, so the term "focus" seems appropriate.

At very low energies, the wavelength of each partial wave is long compared to the nuclear surface and the last nonexternal partial wave has its maximum magnitude well inside the surface so that the focus is in the interior. In case 2 the relevant interfering partial waves are those for $l=1$ and 2 and the focus occurs where the sum of their magnitudes is largest in the region where $\varphi_1-\varphi_2\cong 90°$. This is at the first peak of $|\psi_1|$ [see Fig. 2(b)] which occurs well inside the nucleus. The focus occurs well inside as can be seen from the plot of $|\chi^{(+)}(\mathbf{r})|$ on the axis in Fig. 4. The focus in case 1 can be seen in Fig. 3 to be nearer the surface.

The reason why the focus is much smaller in magnitude for the α-particle case, in which many partial waves are in the surface, can be seen. There are many values of φ_l intermediate between the inside and outside values, so successive φ_l are not so far apart and the interference is not so strong. This is also the reason for the small focal intensity for high-energy nucleons. In this case the inside value of φ_l is not very different from the outside value, so again the interference is not so strong.

The smaller focus in the case of heavier particles is associated with their shorter mean free path. The small focus is due to the large number of partial waves in the surface which varies as the square root of the mass number. The mean free path varies inversely as the square root of the mass number.[10] This is a property to be expected if the focus corresponds to the classical refraction effect.

The focus for charged particles is smaller than that for uncharged particles of the same energy and mass because φ_l for the first outside partial wave is closer to the inside value in the charged case because of the Coulomb phase shift. Hence, the successive intermediate partial waves are not so far from 90° apart in phase in the charged case. This effect would again be expected from classical considerations.

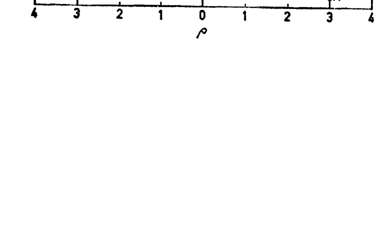

FIG. 3. The square of the magnitude of the optical-model wave function $|\chi^{(+)}(\mathbf{r})|^2$ on the axis of scattering in case 1.

3. THE DISTORTED WAVE BORN APPROXIMATION AT EXTREME ANGLES

It was seen in the last section that for very low energy the optical-model wave function focuses in the interior of the nucleus. It would be expected that the overlap integral involved in the distorted wave Born approximation would have a large contribution from the interior of the nucleus at scattering angles where the entrance and exit channel wave functions are so oriented that the focus in one of them overlaps a part of the other where the magnitude is large, either the rear surface or the focus. This occurs at scattering angles $\Theta=0$ and π, respectively.

In this section the overlap integral I of the wave function in case 2 in both entrance and exit channels with a $1p$ wave function for C^{12} in both channels is computed. The contributions $I_{ll'}$ for different combinations (l,l') of partial waves are illustrated in detail for $L=1$ and 2, where L is the angular momentum transfer.

For cases involving many partial waves it has been shown by Austern[5] that phase averaging due to the oscillations of the product of partial waves for small l

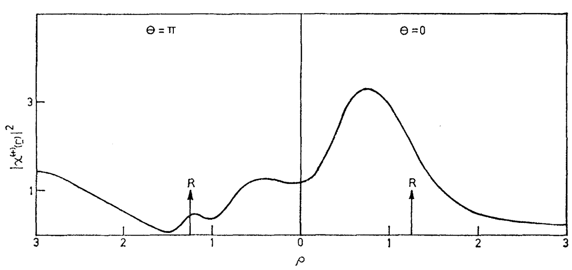

FIG. 4. The square of the magnitude of the optical-model wave function $|\chi^{(+)}(\mathbf{r})|^2$ on the axis of scattering in case 2.

and l' within the region of integration reduces the contribution from the inside partial waves almost to zero. The surface partial waves are the ones which contribute most to the integral. In case 2, as far as the focus is concerned, we have already seen that ψ_0 can be regarded as an inside partial wave but that ψ_1 gives the main contribution to the focus and must be regarded as a "surface" partial wave in this sense although it contributes mainly to the interior.

We will first compute the overlap integral for a particular (fictitious) reaction and use the computation as an example to show how considerations obtained from the phase diagrams may be used to understand low-energy reactions on light nuclei, in general.

For $\Theta=0$ and π, the distorted wave overlap integral is extremely simplified and quite easy to compute because it has axial symmetry. For many reactions on light nuclei only one angular momentum transfer L is allowed by the selection rules. This is true, in general, for a $0+$ initial state and we will use this case as an example. A zero-range interaction will be assumed for simplicity. In this case the overlap integral is

$$I=\int \chi^{(-)*}(\mathbf{k}',\mathbf{r})\chi^{(+)}(\mathbf{k},\mathbf{r})\varphi_i(r)\varphi_f{}^*(r)Y_L{}^M(\Omega_r)d^3r, \quad (16)$$

where $\chi^{(+)}(\mathbf{k},\mathbf{r})$ and $\chi^{(-)*}(\mathbf{k}',\mathbf{r})$ are, respectively, the entrance and exit (time-reversed) channel wave functions defined in Sec. 2. $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the incident and outgoing particles in the center of mass system. $\varphi_i(r)$ and $\varphi_f(r)$ are, respectively, the radial parts of the initial and final bound-state wave functions. M is the magnetic quantum number associated with L. Ω_r is measured from $\hat{k}$.

For $\Theta=\pi$ and 0, respectively, we may write

$$\begin{aligned}\chi^{(+)}(\mathbf{k},\mathbf{r})&=\sum_l\psi_l(\rho)P_l(\cos\theta),\\ \chi^{(-)*}(\mathbf{k},\mathbf{r})&=\sum_{l'}(\pm1)^{l'}\psi_{l'}(\rho)P_{l'}(\cos\theta),\end{aligned} \quad (17)$$

where θ is measured from $\hat{k}$. (Here, we are considering

TABLE I. The significant contributions $I_{ll'}$ to the overlap integral I at $\Theta=0$ and π for $L=2$.

l	l'	$I_{ll'}(\Theta=0)$ Re	Im	$I_{ll'}(\Theta=\pi)$ Re	Im
0	2	−0.11	−0.15	−0.11	−0.15
1	1	0.54	−1.39	−0.54	1.39
1	3	0.06	−0.00	−0.06	0.00
2	0	−0.11	−0.15	−0.11	−0.15
2	2	−0.00	0.13	−0.00	0.13
3	1	0.06	−0.00	−0.06	0.00
Total		0.44	−1.57	−0.90	1.24
$\lvert I\rvert^2$		2.66		2.35	

an artificial case where $k'=k$.) The integral becomes

$$I=\sum_{ll'}(\pm1)^{l'}\int r^2dr\,\psi_l(kr)\psi_{l'}(kr)\varphi_i(r)\varphi_f{}^*(r)$$
$$\times\int d\Omega_r\,P_l(\cos\theta)P_{l'}(\cos\theta)Y_L{}^M(\Omega_r). \quad (18)$$

The angular part of (18) reduces to

$$[4\pi/(2L+1)]^{\frac{1}{2}}(C_{ll'L}{}^{00})^2, \quad (19)$$

where $C_{ll'L}{}^{00}$ is a Clebsch-Gordan coefficient, giving

$$I=\sum_{ll'}I_{ll'}=\sum_{ll'}\int dr\,J_{ll'}(r)$$
$$=[4\pi/(2L+1)]^{\frac{1}{2}}\sum_{ll'}(\pm1)^{l'}(C_{ll'L}{}^{00})^2\int r^2dr\,\psi_l(kr)\psi_{l'}(kr)$$
$$\times\varphi_i(r)\varphi_f{}^*(r), \quad (20)$$

for $\Theta=\pi$ and $\Theta=0$, respectively.

For an initial state whose spin and parity are other than 0+ there is a term similar to (20) for each L allowed by the selection rules.

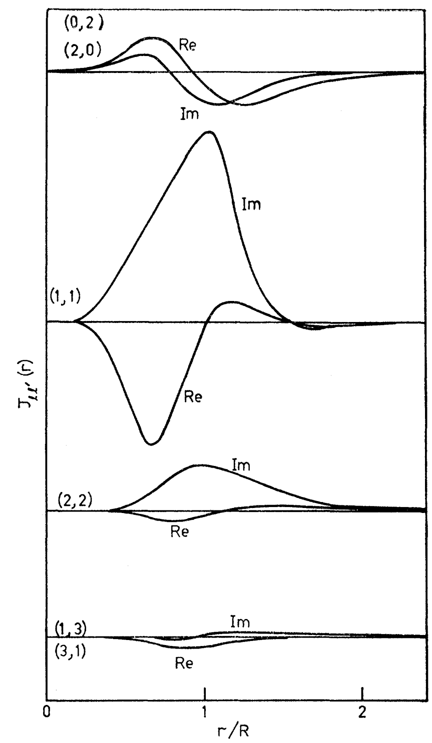

FIG. 5. The integrands $J_{ll'}(\rho)$ of the significant contributions $I_{ll'}$ to the overlap integral I for $L=2$.

In the present calculation $\varphi(r)$ was computed in a Woods-Saxon potential specified by the parameters:

$$V=-42.8\text{ MeV},\quad r_0=1.3\text{ F},\quad a=0.69\text{ F}.$$

Figure 5 shows the contributions to the integrand $J_{ll'}(r)$ for the combinations (l,l') of partial waves which are allowed by the selection rules and which give significant contributions in the case $L=2$. The values of $I_{ll'}$ at $\Theta=0$ and π are shown in Table I.

Since only the overlap integrals for forward and backward scattering have been computed, it is difficult to see whether they are large or small compared with the integrals for other scattering angles. A complete calculation of the angular distribution has not been carried out, but a good idea of the importance of the values at $\Theta=0$ and π can be obtained by comparing them with the value of the integral at the maximum in the angular distribution using plane waves instead of $\chi^{(+)}(\mathbf{r})$. This value is about 3.5. Since the total cross sections in distorted wave theory are generally less than those in plane-wave theory, it is clear that the values 2.66 and 2.35 for $\Theta=0$ and π, respectively, are very significant.

We can now check the idea that the interior of the nucleus plays an important part in the scattering at $\Theta=0$ and π. The term I_{11} is by far the most important in the present case. It contains the main contribution to the focus in each channel. About 60% of the total overlap integral I comes from radii less than R for the

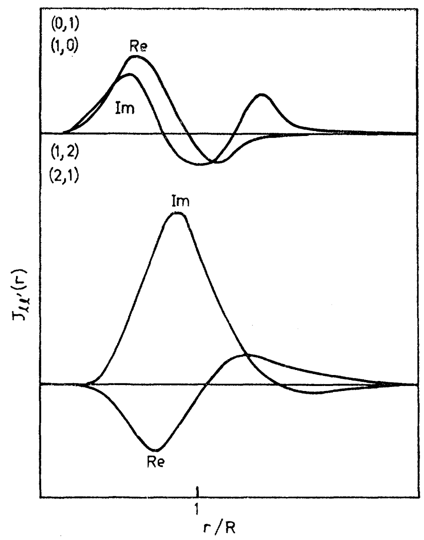

FIG. 6. The integrands $J_{ll'}(\rho)$ of the significant contributions $I_{ll'}$ to the overlap integral I for $L=1$.

entrance and exit channel optical models. If the radial integration is cut off at R, we have values 0.81 and 0.69 for $|I|^2$ at $\Theta=0$ and π, respectively. These are about one third of the values for the volume-interaction model. In this case, although the interior contributes a large amount to I, the forward and backward cross sections are cut down roughly proportionally by cutting off the integration. This is hardly likely to be true for all angles, but, since angular distributions due to direct interactions at low energies are difficult to measure owing to compound nucleus interference, we would like to be able to see a gross difference in the angular distributions.

The case $L=1$ is shown in Fig. 6. Here, the forward cross section is zero owing to the parity rule. The overlap integrals are much smaller in this case, because there are no contributions I_{ll} which are always larger than those for $l'\neq l$. The value of $|I|^2$ at $\Theta=\pi$ for integration over the whole volume is 0.087, which possibly corresponds to a backward peak, whereas for a cutoff at R it is only 0.006 which is much less than one-third of the volume-interaction value and may correspond to an angular distribution without a backward peak.

Some generalizations about extreme angle inelastic scattering can now be made using the ideas of Sec. 2. If the Q value for the reaction is not too large compared with the incident energy, we may get reliable information about the general behavior of $I_{ll'}$ from the magnitude and phase diagrams for one optical-model wave function which is assumed to represent both entrance and exit channels.

In general, we cannot confine ourselves to considering $\psi_l(\rho)$ only where its magnitude is near its maximum value because $\psi_{l'}(\rho)$ may have its maximum at another value of ρ so that their product is significant over a larger range of ρ, including places where the phase is changing rapidly.

From the phase diagram, Fig. 1, for higher energy reactions it may be seen that for low values of l or l' the phase of $\psi_l(\rho)\psi_{l'}(\rho)$ decreases rapidly over the range of ρ which contributes most to the integral $I_{ll'}$, thus giving rise to the phase-averaging effect noted by Austern[5] which makes $I_{ll'}$ small. The terms I_{ll} are an exception to this.

For low energies and surface partial waves, the phase of $J_{ll'}(\rho)$ can change by more than 90° over the overlap region, thus giving the possibility of a large difference between the values of $I_{ll'}$ for the volume-integration and cutoff models.

As an example, consider the terms $J_{ll'}(\rho)$ in Fig. 5. I_{02} and I_{20} are severely reduced by phase averaging. Figure 2 explains this. $\varphi_0+\varphi_2$ decreases by about 180° over the overlap region. This is not true, however, for I_{11}. The odd-parity case $L=1$ shown in Fig. 6 is different in that only combinations $I_{ll'}$ for $l'\neq l$ are allowed so there must be quite a large change in the phase of $J_{ll'}(\rho)$ over the overlap region. Note that in both cases $\psi_1(\rho)$, the partial wave which contributes most to the focus, is the dominant partial wave.

The case $L=0$ is particularly simple. Here, the only contributors to I are I_{ll}. The large ones, I_{00} and I_{11} are added at $\Theta=\pi$, subtracted at $\Theta=0$.

4. CONCLUSIONS

It has been shown that the partial waves which give the greatest contribution to the overlap integral in the distorted-wave Born approximation for a single-particle excitation mechanism are the same ones which contribute to the focus in each optical-model wave function. This is true whether the incident energy is high or low. In the vicinity of 5-MeV incident and outgoing energy this means that the overlap integral receives a large part of its value at scattering angles of 0 and π from the nuclear interior. Hence, it is clear that, if direct interactions at these energies sample only the nuclear surface, one must look beyond optical-model effects to establish the reason.

There can be such a large difference between the angular distributions including and excluding the interior, that there should be no difficulty in deciding from a well-chosen experiment whether the interior does in fact play an important part in the reaction.

The necessary experimental conditions are as follows. Firstly, k and k' must be sufficiently similar for roughly the same partial waves to be important in each at a particular value of r. Secondly, we should choose a reaction which we expect to be due to a single-particle excitation. Compound nucleus effects are, of course, important at low energies, but they can be roughly separated out by doing one of two things. A poor-resolution experiment can be used in a case where the compound nucleus has enough levels in the relevant energy region for the statistical model to apply, in which case the compound nucleus angular distribution is approximately isotropic and can be subtracted incoherently. Otherwise, one could perform a good resolution experiment at different energies and distinguish over-all trends in the angular distribution, ignoring fluctuations due to interference of compound and direct processes.

The second method may be applied to analyze the $C^{13}(p,n)N^{13}$ experiment of Dagley *et al.*[11] In this experiment the Q value is 3.237 MeV, and the conditions on k and k' are roughly satisfied for incident energies above about 6 MeV. Apart from fluctuations, angular distributions in the incident energy range 5 to 8 MeV are characterized by a small cross section at $\Theta=0$, and a large peak at $\Theta=\pi$.

Because of the half-integral spin of the initial and final states, the analysis of this case is a little more complicated than the one we have considered, but it may be seen that it reduces to an $L=0$ transition in which the relevant Clebsch-Gordan coefficient is $C_{ll'0}{}^{00}=\delta_{ll'}$.

Let us consider the case of proton energy $\cong$6 MeV, neutron energy $\cong$3 MeV. The approximation of Eq. (13) gives phases $\varphi_l(0)$ for $l=0$ and 1 which differ by only 10% for the entrance and exit channel. The Coulomb phase shift is also not large, so we may roughly consider $\psi_0(\rho)$ and $\psi_1(\rho)$ to be 90° apart for small ρ.

Although the proton and neutron energies are not the same, we may still use the phase diagram of Fig. 3 to give us a rough idea of how the contributions I_{00}, I_{11}, and I_{22} interfere. In this case I_{22} is not reduced by having a much smaller Clebsch-Gordan coefficient as it was for $L=2$.

Let us first consider the case where we integrate from $\rho=0$. I_{00} changes in phase by about 2π over the region where $\varphi(r)$ is large, so we may say that it averages to zero approximately. $\varphi_1-\varphi_2$ is about 90° over this region, so we have

$$\arg I_{11}-\arg I_{22}\cong\pi+2(\varphi_1-\varphi_2)\cong 2\pi.$$

This means that I_{11} and I_{22} interfere destructively at $\Theta=0$ and constructively at $\Theta=\pi$ in accordance with the experiment.

If we integrate only from $\rho=kR$, we find I_{11} and I_{22} having roughly the same relationship to each other. I_{00} is not now zero, but it is about 90° apart from the others and contributes roughly equal amounts to both forward and backward scattering. Backward peaking was found in the 5-MeV range in a calculation of the same reaction by Albert, Bloom, and Glendenning[12] which used a surface approximation.

Consideration of the magnitude and phase diagrams evidently is capable of giving a rough idea of one major feature of an angular distribution for energies about 5 MeV, namely, whether it is peaked forward or backward, or whether the cross sections at 0 and π are similar. The phase diagrams can be drawn to a good enough approximation from the general considerations of Sec. 2.

Although the nuclear interior contributes a very large proportion of the direct interaction cross section at 0 and π at these energies assuming a volume interaction, it is not clear that merely looking at extreme angles can distinguish surface from volume interaction in the cases $L=0$ and 2. The case $L=1$ is more hopeful. The backward cross section in this case is very much larger for volume interaction than surface interaction. Since the interior plays such a large part in the volume-interaction calculation, it is reasonable to expect that exact calculations over the whole angular distribution will reveal gross differences between surface and volume interaction in certain cases which would enable an experiment to be performed to distinguish one from the other.

ACKNOWLEDGMENTS

The author is grateful to K. A. Amos for computing the optical-model wave functions, to Dr. Melkanoff, Dr. Nodvik, Dr. Saxon, and Dr. Cantor for the use of their optical-model code SCAT 4, to Dr. I. N. Capon of the University of Adelaide Computing center and the staff of the Weapons Research Establishment, Salisbury, South Australia, for technical help with the computations, and to the Australian Institute for Nuclear Science and Engineering and the Australian Atomic Energy Commission for finances.

[11] P. Dagley, W. Haeberli, J. X. Saladin, and R. R. Borchers, in *Proceedings of the International Conference on Nuclear Structure, Kingston* (University of Toronto Press, Toronto, 1960), p. 359.

[12] R. D. Albert, S. D. Bloom, and N. K. Glendenning, Phys. Rev. **122**, 862 (1961).