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Lifetime of a Quasi-Particle in a Superconductor at Finite Temperatures and Application to the Problem of Thermal Conductivity*

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The theory of the lifetime of a quasi-particle at finite temperatures is developed by using a generalization of the Green's function formalism of Nambu. The explicit results for the lifetime, as a function of the excitation energy and the temperature, are applied to the problem of electronic thermal conductivity when interaction with the phonons is the limiting mechanism for the heat flux. The lifetime appears in a general expression for the thermal conductivity which is derived from the corresponding Boltzmann equation by making a relaxation-time approximation. The resulting curve for the ratio of the thermal conductivity in the superconducting state to that in the normal state versus the reduced temperature is found to decrease monotonically and to have a slope of about 2 at the transition temperature.

I. INTRODUCTION

In a foregoing article[1] we determined the lifetime of a quasi-particle in a superconductor at zero temperature by using the Green's function formalism of Nambu.[2] In the present work we extend this theory from zero to finite temperatures. The generalization of Nambu's (2×2)-matrix propagator for a quasi-particle is obtained by replacing the expectation value in the ground state of the T product of the two-component electron-hole operators by an average over the grand canonical ensemble. According to the procedure developed by Abrikosov, Gor'kov, and Dzyaloshinskii[3] this propagator can be constructed by means of a diagrammatic analysis from the propagators of the noninteracting system in a way which is quite analogous to that used for zero temperature. Thus, the generalized self-consistency condition of Nambu is obtained, as for zero temperature, by equating an Ansatz for the self-energy part to the self-energy amplitude arising from the lowest order diagram in the dressed particle and interaction picture. The imaginary part of the pole of the particle propagator in its complex energy variable, which yields the inverse lifetime, can be determined with the help of this self-consistency condition.

However, the mathematical procedure has to be modified since the previous integration in the self-consistency condition over the energy variable of the intermediate state is now replaced by a sum over discrete imaginary values in the energy variable. The main problem is to carry out this sum and then to continue analytically the result for the sum, which is obtained as a function of imaginary values in the energy variable, from the imaginary to the real axis. The difference in the procedure of the calculation is a consequence of the boundary conditions which apply to the thermal Green's functions when they are considered for imaginary values of their time variables. These boundary conditions follow directly from the cyclic invariance of the trace appearing in the definition of the thermal Green's functions.

Since we have found that at zero temperature the decay of a particle is caused overwhelmingly by its interaction with single phonons, we shall consider here only this contribution to the decay rate. One expects intuitively that the decay rate consists of the sum of transition probabilities for all the particle-phonon processes allowed by energy and momentum conservation. At zero temperature there can only occur processes where the particle is scattered with the emission of a phonon. However, at finite temperatures processes where the particle is scattered with the absorption of a phonon and processes where the particle, together with a second particle of opposite spin, is annihilated with the creation of a phonon are also possible. One expects also that the transition probability contains the coherence factor which applies for the process in question, and that it is multiplied by the appropriate Fermi and Planck factors which take into account the occupation of states by particles and phonons.

The explicit results for the decay rate of a quasi-particle, obtained as a function of its excitation energy and the temperature, will enable us to develop a refined theory of the thermal conductivity when the interaction with phonons is the limiting mechanism for the heat flux. So far the corresponding experimental data have not been well understood. Bardeen, Rickayzen, and the author[4] treated this problem by considering the full Boltzmann equation for the deviation in the distribution function of quasi-particles from the equilibrium value. From an approximate solution of this transport equation they obtained a negative value for the slope of the ratio of the thermal conductivity in the superconducting state, κ_s, to that in the normal state, κ_n, vs reduced temperature, T/T_c, at the transition temperature T_c. This result disagrees with measurements made on pure tin[5] and mercury[6] which show a positive limiting slope

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[1] L. Tewordt, Phys. Rev. **127**, 371 (1962).

[2] Y. Nambu, Phys. Rev. **117**, 648 (1960).

[3] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, Soviet. Phys.—JETP **9**, 636 (1959).

[4] J. Bardeen, G. Rickayzen, and L. Tewordt, Phys. Rev. **113**, 982 (1959), hereafter referred to as BRT.

[5] A. M. Guenault, International Conference on Superconductivity, Cambridge, 1959 (unpublished).

[6] J. K. Hulm, Proc. Roy. Soc. (London) **A204**, 98 (1950).

of about unity and five, respectively. Kadanoff and Martin[7] treated the problem of thermal conductivity anew by the method of thermal Green's functions. Their first basic assumption is that a replacement of the transport cross section by the scattering cross section does not lead to a significant error in the case of the thermal conductivity. In this work we shall assume this approximation to be valid, and we shall make the corresponding simplification in the Boltzmann equation of BRT. We shall see that this approximation leads to exactly the same general expression for the thermal conductivity as that obtained by KM. This expression consists essentially of an integral over the excitation energy of the particle where the integrand contains the lifetime of the particle as a factor. In evaluating this expression KM made two further approximations. They neglected the dependence of the lifetime on the excitation energy, and they assumed that the lifetimes of the excitations in the superconducting and the normal state are the same. Under these approximations the lifetimes, which are unknown parameters in the work of KM, drop out of the expression for the ratio κ_s/κ_n, and the temperature dependence of this ratio can easily be calculated.

However, from our results for zero temperature it seems likely that the lifetimes do strongly depend on the excitation energy, and further, that the occurrence of the gap and the coherence factors does markedly alter the lifetime of a quasi-particle in comparison to that of a normal state excitation. Therefore, it seems desirable to calculate explicitly the lifetimes in both states and, if necessary, to modify Kadanoff and Martin's calculation of the temperature dependence of κ_s/κ_n by inserting these quantities into the expressions for κ_s and κ_n.

II. GENERAL THEORY OF LIFETIMES AT FINITE TEMPERATURES

The generalization of Nambu's one-particle matrix Green's function from zero to finite temperatures is obtained by replacing the ground-state expectation value of the T product by an average over the grand canonical ensemble, i.e.,

$$G_{ij}(\mathbf{p},t)=-i\,\mathrm{Tr}e^{\beta(\Omega-K)}T(\psi_i(\mathbf{p},t)\psi_j^\dagger(\mathbf{p},0)).\quad(2.1)$$

The operator K is equal to $H-\mu N$, where H is the Hamiltonian of the system, N is the electron-number operator, and μ is the chemical potential. The quantity $\exp(-\beta\Omega)$, with $\beta=(k_BT)^{-1}$, is the grand partition sum. If we measure the one-particle energies from the energy μ, then the old Hamiltonian H is replaced by the new one $K=H-\mu N$, and the Heisenberg operators ψ and $\psi^\dagger$ develop in time according to

$$\psi(\mathbf{p},t)=e^{iKt}\psi(\mathbf{p})e^{-iKt},\text{ etc.},\quad(2.2)$$

where

$$\psi(\mathbf{p})=\begin{pmatrix}c_{p\uparrow}\\ ac_{-p\downarrow}^\dagger\end{pmatrix},\quad \psi^\dagger(\mathbf{p})=(c_{p\uparrow}^\dagger,a^\dagger c_{-p\downarrow}).\quad(2.3)$$

The $c_{p\sigma}$ and $c_{p\sigma}^\dagger$ are annihilation and creation operators for electrons with momentum $\mathbf{p}$ and spin component σ along the axis of quantization. The operator $a=C\sum c_{p\uparrow}c_{-p\downarrow}$ subtracts two electrons from the system without changing its total momentum and spin.[7]

The analytic properties of the thermal Green's function $G_{ij}(\mathbf{p},t)$ can be derived from a spectral representation. Denoting by $|n\rangle$ the eigenstates of the operator K and by K_n the eigenvalues, we obtain

$$\begin{aligned}G_{ij}(\mathbf{p},t)&=-i\int_{-\infty}^{+\infty}d\epsilon\, e^{-i\epsilon t}\zeta_{ij}(\mathbf{p},\epsilon),\quad\text{for}\quad t>0,\\ &=i\int_{-\infty}^{+\infty}d\epsilon\, e^{-i\epsilon t}\zeta_{ij}'(\mathbf{p},\epsilon),\quad\text{for}\quad t<0,\end{aligned}\quad(2.4)$$

where the two spectral weight functions for $t>0$ and $t<0$ are connected through the relationship

$$\zeta_{ij}'(\mathbf{p},\epsilon)=e^{-\beta\epsilon}\zeta_{ij}(\mathbf{p},\epsilon).\quad(2.5)$$

If terms of order $1/N$ are neglected, the matrix $\zeta_{ij}(\mathbf{p},\epsilon)$, for instance, is given by

$$\zeta_{ij}(\mathbf{p},\epsilon)d\epsilon=e^{\beta\Omega}\sum_{n,m}e^{-\beta K_n}\begin{pmatrix}|\langle m|c_{p\uparrow}^\dagger|n\rangle|^2 & \langle n|c_{p\uparrow}|m\rangle\langle m|a^\dagger c_{-p\downarrow}|n\rangle\\ \langle n|ac_{-p\downarrow}^\dagger|m\rangle\langle m|c_{p\uparrow}^\dagger|n\rangle & |\langle m|a^\dagger c_{-p\downarrow}|n\rangle|^2\end{pmatrix},\quad(2.6)$$

with the restriction

$$\epsilon<(K_m-K_n)<\epsilon+d\epsilon.$$

Following the procedure introduced by Abrikosov *et al.*,[3] and presented in a convenient form by Schultz,[8] we define still another one-particle matrix Green's function by

$$g_{ij}(\mathbf{p},\tau)=-\mathrm{Tr}e^{\beta(\Omega-K)}T(\psi_i(\mathbf{p},\tau)\psi_j^\dagger(\mathbf{p},0)),\quad(2.7)$$

where

$$\psi(\mathbf{p},\tau)=e^{K\tau}\psi(\mathbf{p})e^{-K\tau},\text{ etc.}\quad(2.8)$$

Now the T product is understood to order the operators according to the relative sizes of the τ's. One can show that the new Green's function satisfies the following boundary conditions

$$g_{ij}(\mathbf{p},\tau)=-g_{ij}(\mathbf{p},\tau-\beta).\quad(2.9)$$

Thus, g_{ij} is periodic with period 2β. If we expand $g_{ij}(\mathbf{p},\tau)$ in a Fourier series,

$$g_{ij}(\mathbf{p},\tau)=\beta^{-1}\sum_{n=-\infty}^{+\infty}e^{-iE_n\tau}g_{ij}(\mathbf{p},iE_n),\quad(2.10)$$

[7] L. P. Kadanoff and P. Martin, Phys. Rev. **124**, 670 (1961), hereafter referred to as KM.

[8] T. D. Schultz, *Quantum Field Theory and the Many-Body Problem* (Space Technology Laboratories, Inc., Los Angeles, 1960).

we must require that

$$iE_n = i(2n+1)\pi\beta^{-1},\ n \text{ integer.} \tag{2.11}$$

The expansion coefficients in Eq. (2.10) are given by

$$g_{ij}(\mathbf{p},iE_n) = \frac{1}{2}\int_{-\beta}^{+\beta} d\tau \exp(iE_n\tau) g(\mathbf{p},\tau). \tag{2.12}$$

If we make a spectral representation of $g_{ij}(\mathbf{p},\tau)$ in terms of eigenstates $|n\rangle$ of K and calculate the Fourier transform according to Eq. (2.12), we obtain with the help of the Eqs. (2.11) and (2.5)

$$g_{ij}(\mathbf{p},iE_n) = \int_{-\infty}^{+\infty} d\epsilon \frac{\zeta_{ij}(\mathbf{p},\epsilon)(1+e^{-\beta\epsilon})}{iE_n - \epsilon}, \tag{2.13}$$

where the matrix $\zeta_{ij}(\mathbf{p},\epsilon)$ is given by Eq. (2.6). The function $\tilde{g}_{ij}(\mathbf{p},z)$, defined by the integral representation in Eq. (2.13) and iE_n set equal to z, is analytic in the upper and lower half of the complex z plane and has a branch cut running along the real axis. $\tilde{g}(\mathbf{p},z)$ goes over into its complex conjugate value if z crosses the branch cut. The spectral weight function, $\zeta_{ij}(\mathbf{p},\epsilon)$, can be obtained by taking the imaginary part of $\tilde{g}_{ij}(\mathbf{p};\epsilon+i\eta)$, where η is a positive infinitesimal:

$$\zeta_{ij}(\mathbf{p},\epsilon) = \frac{i}{2\pi}(1+e^{-\beta\epsilon})^{-1} \times[\tilde{g}_{ij}(\mathbf{p};\epsilon+i\eta) - \tilde{g}_{ij}^{*}(\mathbf{p};\epsilon+i\eta)]. \tag{2.14}$$

A simple physical interpretation of the propagator $G_{ij}(\mathbf{p},t)$ is possible only if all four components of the spectral weight function ζ_{ij} are strongly peaked around the same excitation energy, say $\epsilon = E_{rp}$, with the same half-width, Γ_p. We shall assume in the following that this peak is due to a simple pole of $\tilde{g}_{ij}(\mathbf{p},z)$ at $z=(E_{rp}-i\Gamma_p)$, with real residue A_{ij}. Note that this pole belongs to the analytic continuation of $\tilde{g}_{ij}(\mathbf{p},z)$ from the upper right into the lower right plane. If one inserts $g_{ij} = A_{ij}/[\epsilon-(E_{rp}-i\Gamma_p)]$ into Eq. (2.14), one obtains from Eq. (2.4) in the case $t>0$ the following expression for $G_{ij}(\mathbf{p},t)$:

$$G_{ij}(\mathbf{p},t) \approx -iA_{ij}[1+\exp(-\beta E_{rp})]^{-1} \times \exp[-(iE_{rp}+\Gamma_p)t] - A_{ij}\sum_{n=0}^{\infty} \frac{2\Gamma_p \exp[-(2n+1)\pi\beta^{-1}t]}{\beta\{[E_{rp}+i(2n+1)\pi\beta^{-1}]^2+\Gamma_p^2\}}. \tag{2.15}$$

One recognizes that the second term on the right-hand side of Eq. (2.15) can be neglected in comparison to the first term if the conditions $\Gamma_p \ll E_{rp}$ and $\Gamma_p\beta\ \ \pi$ are fulfilled. From the definition of $G_{ij}(\mathbf{p},t)$ and from the time dependence of the first term in Eq. (2.15), we see then that $2\Gamma_p$ has to be identified with the inverse decay time of a quasi-particle $(\mathbf{p},\uparrow)$. An analogous consideration for the case $t<0$ reveals that the imaginary part of the pole of the analytic continuation of $\tilde{g}(\mathbf{p},z)$ from the lower left into the upper left z plane determines the inverse decay time of a quasi-particle $(-\mathbf{p},\downarrow)$.

The rules for obtaining $g(\mathbf{p},iE_n)$ by a diagrammatic analysis are completely analogous to those valid for the zero temperature case if the appropriate modifications are made. In the dressed particle and interaction picture these prescriptions turn out to be the following:

(1) Each directed particle line carries a matrix factor $-g(\mathbf{p},iE_n)$. This complete propagator can be obtained from the proper irreducible self-energy part, $\sum(\mathbf{p},iE_n)$, and the propagator for the noninteracting system, $g_0(\mathbf{p},iE_n)$, by means of Dyson's equation:

$$g(\mathbf{p},iE_n) = [g_0^{-1}(\mathbf{p},iE_n) - \textstyle\sum(\mathbf{p},iE_n)]^{-1}. \tag{2.16}$$

$g_0(\mathbf{p},iE_n)$ is easily calculated from Eqs. (2.7) and (2.12) and found to be

$$g_0(\mathbf{p},iE_n) = \frac{iE_n + \epsilon_p\tau_3}{(iE_n)^2 - \epsilon_p^2}. \tag{2.17}$$

The quantity ϵ_p denotes the one-electron energy measured from μ. τ_i are the usual Pauli spin matrices.[9]

(2) Each phonon plus Coulomb interaction line carries a factor $-d(\mathbf{q},i\nu_n)$. This complete interaction propagator is determined by

$$d(\mathbf{q},i\nu_n) = \left(\frac{v_q^{1/2}\omega_{\mathrm{pl}}}{K_q(i\nu_n)}\right)^2 \times\left[(i\nu_n)^2 - \frac{\omega_{\mathrm{pl}}^2}{K_q(i\nu_n)}\right]^{-1} + \frac{v_q}{K_q(i\nu_n)}. \tag{2.18}$$

Here ω_{pl} denotes the ion-plasma frequency, $K_q(i\nu_n)$ is the analog to the dielectric constant, and $v_q = 4\pi e^2/q^2$ is the Fourier transform of the Coulomb potential. From the boundary conditions valid for the boson propagators one finds that $i\nu_n$ can take on only the values

$$i\nu_n = i2n\pi\beta^{-1},\ n \text{ integer.} \tag{2.19}$$

(3) Each vertex carries the matrix factor

$$\tau_3\delta(\mathbf{p}-\mathbf{p}'\mp\mathbf{q})\delta(E_n - E_m \mp \nu_1), \tag{2.20}$$

where the first delta stands for the delta function and the second one for the Kronecker symbol.

(4) One has to integrate over all internal three-momenta and to sum over all internal energy values.

Using these rules we obtain the following analog to the Nambu self-consistency condition for the self-energy part of a quasi-particle:

$$\textstyle\sum(\mathbf{p},iE_n) = -\int \frac{d^3q}{(2\pi)^3}\beta^{-1}\sum_{m=-\infty}^{+\infty} \tau_3 g(\mathbf{p}-\mathbf{q},\ iE_m) \times\tau_3 d(\mathbf{q},\ iE_n - iE_m). \tag{2.21}$$

[9] The notation is:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Nambu's Ansatz for the self-energy part generalizes now to

$$\sum(\mathbf{p},iE_n)=iE_n\xi_p(iE_n)+\chi_p(iE_n)\tau_3+\phi_p(iE_n)\tau_1. \quad (2.22)$$

Inserting Eqs. (2.17) and (2.22) into Eq. (2.16) one obtains

$$g(\mathbf{p},iE_n)=\left.\frac{zZ_p(z)+\bar{\epsilon}_p(z)\tau_3+\phi_p(z)\tau_1}{z^2[Z_p(z)]^2-[E_p(z)]^2}\right|_{z=iE_n}, \quad (2.23)$$

with

$$\begin{aligned}E_p(z)&=\{[\bar{\epsilon}_p(z)]^2+[\phi_p(z)]^2\}^{1/2},\\ \bar{\epsilon}_p(z)&=\epsilon_p+\chi_p(z), \quad Z_p(z)=1-\xi_p(z).\end{aligned} \quad (2.24)$$

We have seen that, in general, the energy E_{rp} and the half-width Γ_p for a quasi-particle $(\mathbf{p},\uparrow)$ are given by the real and imaginary parts, respectively, of the pole of the analytic continuation of $\tilde{g}(\mathbf{p},z)$ from the upper into the lower right plane. From the particular expression for the particle propagator given by Eq. (2.23), we obtain in this way the following approximate expressions for E_{rp} and Γ_p:

$$E_{rp}=\mathrm{Re}[\tilde{E}_p(z)/\tilde{Z}_p(z)]_{z=E_{rp}+i\eta}, \quad (2.25)$$

$$\Gamma_p=-Z_p^{-1}\,\mathrm{Im}\left\{z\tilde{\xi}_p(z)+\frac{\bar{\epsilon}_p}{E_p}\tilde{\chi}_p(z)+\frac{\phi_p}{E_p}\tilde{\phi}_p(z)\right\}_{z=E_{rp}+i\eta}. \quad (2.26)$$

The quantity η is a positive infinitesimal. The functions $\tilde{\xi}_p(z)$, $\tilde{\chi}_p(z)$, and $\tilde{\phi}_p(z)$ are the analytic continuations of the corresponding quantities given originally for the argument $z=iE_n$. The other quantities appearing in Eqs. (2.25) and (2.26) are defined by

$$\begin{aligned}\bar{\epsilon}_p&=\epsilon_p+\mathrm{Re}\tilde{\chi}_p(E_{rp}+i\eta), & \phi_p&=\mathrm{Re}\tilde{\phi}_p(E_{rp}+i\eta),\\ E_p&=(\bar{\epsilon}_p^2+\phi_p^2)^{1/2}, & Z_p&=1-\mathrm{Re}\tilde{\xi}_p(E_{rp}+i\eta).\end{aligned} \quad (2.27)$$

It should be remarked that a consideration of the z dependence of $\tilde{E}_p(z)/\tilde{Z}_p(z)$ would yield a correction factor on the right-hand side of Eq. (2.26). (This factor has been denoted by A_p^{-1} in reference 1.) Since we do not evaluate this term explicitly, we neglect it here and in the sequel in order to simplify the notation.

The quantities $\xi_p(iE_n)$, $\chi_p(iE_n)$, and $\phi_p(iE_n)$ are obtainable from the self-consistency condition in Eq. (2.21). Comparing coefficients of the matrices 1, τ_3, and τ_1 in Eq. (2.21), one deduces three coupled integral equations for these quantities. From these integral equations and from Eq. (2.26), we derive the following expression for the half-width Γ_p:

$$\Gamma_p=\int\frac{d^3q}{(2\pi)^3}\frac{v_q\omega_{\mathrm{pl}}^2}{Z_p}\,\mathrm{Im}\tilde{\sigma}(E_{rp}+i\eta). \quad (2.28)$$

Here we have denoted by $\tilde{\sigma}(z)$ the analytic continuation

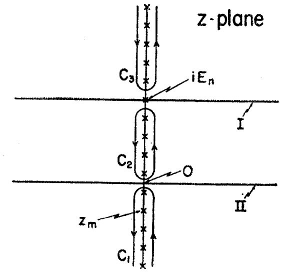

FIG. 1. Integration contours C_1, C_2, and C_3 and branch cuts I and II in the complex z plane. The contours encircle the poles $z_m=i(2m+1)\pi\beta^{-1}$ (m all integers) of the function $[1+\exp(-\beta z)]^{-1}$.

of the function $\sigma(iE_n)$ which is determined by the sum

$$\sigma(iE_n)=\beta^{-1}\sum_{m=-\infty}^{+\infty}H(iE_m)L(iE_n-iE_m). \quad (2.29)$$

The functions $H(z)$ and $L(z)$ are defined by

$$H(z)=\frac{E_pzZ_{p-q}(z)+\bar{\epsilon}_p\bar{\epsilon}_{p-q}(z)-\phi_p\phi_{p-q}(z)}{E_p\{z^2[Z_{p-q}(z)]^2-[E_{p-q}(z)]^2\}}, \quad (2.30)$$

and

$$L(z)=[K_q(z)]^{-2}\left[z^2-\frac{\omega_{\mathrm{pl}}^2}{K_q(z)}\right]^{-1}+\omega_{\mathrm{pl}}^{-2}[K_q(z)]^{-1}. \quad (2.31)$$

III. PARTICLE DAMPING DUE TO PHONON EMISSION AND ABSORPTION

In this section we will evaluate the damping of a particle caused by the interaction with phonons. This contribution to the damping, denoted by Γ_p^{ph}, can be obtained from the general expression which has been derived in the last section and is given by the Eqs. (2.28), (2.29), (2.30), and (2.31). Since we do not consider here the damping due to Coulomb interaction, we neglect the second term on the right-hand side of the Eq. (2.31) for $L(z)$.

First we carry out the summation over all integers m in the expression for $\sigma(iE_n)$ in Eq. (2.29). This is done by considering the expression $H(z)L(iE_n-z)$ as a function of the complex variable z, multiplying it by the function $[1+\exp(-\beta z)]^{-1}$, and integrating the resulting product along appropriate contours which enclose the poles $z_m=iE_m=i(2m+1)\pi\beta^{-1}$ of the latter function. These contours are then deformed in such a way that the integrals can be evaluated. In order to choose the right contours, we have to consider the analytic properties of the function $H(z)L(iE_n-z)$. From the spectral representation of the particle propagator we know that $\tilde{g}(\mathbf{p},z)$, and accordingly $H(z)$, has a branch cut, I, running along the real axis of the z plane and is analytic in the upper and lower half plane. One can derive the spectral representation for the phonon propagator in a way quite analogous to the procedure used in the case of the particle propagator, and one sees from this that $\tilde{d}(\mathbf{q},z)$ also has a branch cut running along the real axis and is otherwise analytic. Therefore, the function $L(iE_n-z)$ must possess a branch cut, II, running along the axis $\mathrm{Im}(z)=iE_n$. According to these general analytic properties we choose three different contours, C_1, C_2, and C_3, where C_1 encircles the poles of the function $[1+\exp(-\beta z)]^{-1}$ which lie below I, C_2 encircles the poles which lie between I and II, and C_3 encloses the poles lying above the branch cut II. The contours C_1, C_2, and C_3 and the branch cuts I and II are shown in Fig. 1. From the above consideration $\sigma(iE_n)$ is equal to

$$\sigma(iE_n)=(2\pi i)^{-1}\left(\oint_{C_1}+\oint_{C_2}+\oint_{C_3}\right)\times\frac{H(z)L(iE_n-z)}{(1+e^{-\beta z})}dz. \quad (3.1)$$

These three closed contours can be deformed in such a way that C_1 runs just below I, C_2 runs just above I and just below II, and C_3 runs just above II. The contributions from those parts of the contours which lie at infinity vanish. If we use the fact that the function H has a discontinuity equal to $2i\,\mathrm{Im}H$ on I, and that the function $L(iE_n-z)$ has a discontinuity equal to $2i\,\mathrm{Im}L(iE_n-z)$ on II, we obtain from Eq. (3.1)

$$\sigma(iE_n)=\pi^{-1}\int_{-\infty}^{+\infty}dx\frac{L(iE_n-x)\,\mathrm{Im}H(x+i\delta)}{[1+\exp(-\beta x)]}-\pi^{-1}\int_{-\infty}^{+\infty}dx\frac{H(iE_n-x)\,\mathrm{Im}L(x+i\delta)}{[1+\exp(\beta x)\exp(-i\beta E_n)]}. \quad (3.2)$$

The quantity δ is a positive infinitesimal. The function $H(z)$ is given by Eq. (2.30), and $L(z)$ is given by the first term of Eq. (2.31). We can assume, in a first-order approximation, that the quantities $-\mathrm{Im}[E_{p-q}(x)/Z_{p-q}(x)]=\alpha(x)$ and $-\mathrm{Im}[\omega_{\mathrm{pl}}^2/K_q(x)]=\beta(x)$ occuring in the denominators of the functions H and L are infinitesimals. From the general results for the analytic properties of $H(z)$ and $L(z)$, one sees that the signs of $\alpha(x)$ and $\beta(x)$ must be identical to that of x. With the help of the delta functions arising from the terms $\mathrm{Im}H(x)$ and $\mathrm{Im}L(x)$, the integrations in Eq. (3.2) can be carried out. We also consider that the term $\exp(-iE_n\beta)$ in the denominator of the second integrand is equal to -1, since iE_n takes on the discrete values given by Eq. (2.11). Then we find approximately from Eq. (3.2)

$$\sigma(iE_n)=-\frac{w(E_{r1})}{2E_{r1}}\frac{L(iE_n-E_{r1})}{[1+\exp(-\beta E_{r1})]}+\frac{w(-E_{r1})}{2E_{r1}}\frac{L(iE_n+E_{r1})}{[1+\exp(\beta E_{r1})]}+\frac{H(iE_n-\Omega)}{2\Omega|K_q(\Omega)|^2[1-\exp(\beta\Omega)]}-\frac{H(iE_n+\Omega)}{2\Omega|K_q(\Omega)|^2[1-\exp(-\beta\Omega)]}. \quad (3.3)$$

Here the zeros of the functions $\{x\mp\mathrm{Re}[E_{p-q}(x)/Z_{p-q}(x)]\}$ are denoted by $x=\pm E_{r1}=\pm E_{rp-q}$, and the zeros of the functions $\{x\mp[\omega_{\mathrm{pl}}(\mathrm{Re}K_q(x))^{1/2}/|K_q(x)|]\}$ are denoted by $x=\pm\Omega=\pm\Omega_q$. In deriving Eq. (3.3), we have used the abbreviation $w(z)$ defined by

$$w(z)=\{z^2-[E_{r1}(z)/Z_1(z)]\}H(z).$$

If we continue analytically the function σ in the z plane from $z=iE_n$ to $z=(E_{rp}+i\eta)$, and if we take thereafter the imaginary part of σ, we obtain once more delta functions from the terms L and H in Eq. (3.3). The result turns out to be

$$\mathrm{Im}\bar{\sigma}(E_{rp}+i\eta)=\frac{\pi}{4E_{r1}\Omega|K_q(\Omega)|^2}\Bigg\{\frac{w(E_{r1})}{[1+\exp(-\beta E_{r1})]}[\delta(E_{rp}-E_{r1}-\Omega)-\delta(E_{rp}-E_{r1}+\Omega)]$$
$$+\frac{w(-E_{r1})}{[1+\exp(\beta E_{r1})]}[\delta(E_{rp}+E_{r1}+\Omega)-\delta(E_{rp}+E_{r1}-\Omega)]$$
$$+\frac{w(E_{rp}-\Omega)}{[1-\exp(\beta\Omega)]}[\delta(E_{rp}-\Omega+E_{r1})-\delta(E_{rp}-\Omega-E_{r1})]$$
$$+\frac{w(E_{rp}+\Omega)}{[1-\exp(-\beta\Omega)]}[\delta(E_{rp}+\Omega-E_{r1})-\delta(E_{rp}+\Omega+E_{r1})]\Bigg\}. \quad (3.4)$$

The expression in Eq. (3.4) can be simplified by rewriting it in terms of Fermi functions $f_p=f(\beta E_{rp})=[1+\exp(\beta E_{rp})]^{-1}$ and Planck functions $N_q=N(\beta\Omega_q)=[\exp(\beta\Omega_q)-1]^{-1}$ and by adding together the terms which belong to the same delta function. Introduction of the result into the Eq. (2.28) yields the following expression for the inverse decay time of a particle:

$$2\Gamma_p^{\text{ph}}=(1-f_p)^{-1}\int\frac{d^3q}{(2\pi)^3}\frac{\pi v_q\omega_{p1}^2}{Z_pZ_{p-q}\Omega_q|K_q(\Omega_q)|^2}\Bigg\{\frac{1}{2}\left(1+\frac{\bar{\epsilon}_p\bar{\epsilon}_{p-q}-\phi_p\phi_{p-q}}{E_pE_{p-q}}\right)$$
$$\times[(1+N_q)(1-f_{p-q})\delta(E_{rp-q}+\Omega_q-E_{rp})+N_q(1-f_{p-q})\delta(E_{rp-q}-\Omega_q-E_{rp})]$$
$$+\frac{1}{2}\left(1-\frac{\bar{\epsilon}_p\bar{\epsilon}_{p-q}-\phi_p\phi_{p-q}}{E_pE_{p-q}}\right)(1+N_q)f_{p-q}\delta(E_{rp-q}+E_{rp}-\Omega_q)\Bigg\}. \quad (3.5)$$

The physical meaning of the three different terms in the integrand of Eq. (3.5) becomes obvious if one examines the arguments of the delta functions, the statistical factors, and the coherence factors. The first and second terms correspond to particle scattering from the state **p** into the state **p**$-$**q** with the emission of a phonon **q** or, absorption of a phonon $-$**q**, respectively. The third term corresponds to the destruction of a pair of particles in states (**p**,↑) and $-$(**p**$-$**q**), ↓ with the simultaneous creation of a phonon **q**.

The physical meaning of the statistical factor $(1-f_p)^{-1}$ in front of the whole expression in Eq. (3.5) is not so obvious. A simple explanation for the origin of this statistical factor can be obtained, as has been pointed out by Kadanoff,[10] by considering the Boltzmann equation which governs the decay of a deviation from the equilibrium distribution. This Boltzmann equation has been set up by BRT[4] in order to develop the theory of the electronic thermal conductivity, when the heat current is limited solely by phonon scattering. The rate of change of the nonequilibrium distribution f_p for the particles due to collisions with phonons is found to be

$$\left(\frac{\partial f_p}{\partial t}\right)_{\text{coll}}=\int\frac{d^3q}{(2\pi)^2}|V_q|^2\Bigg\{\frac{1}{2}\left(1+\frac{\epsilon\epsilon'-\epsilon_0^2}{EE'}\right)[f'(1-f)(1+N)-f(1-f')N]\delta(E'-E-\Omega)$$
$$+\frac{1}{2}\left(1+\frac{\epsilon\epsilon'-\epsilon_0^2}{EE'}\right)[f'(1-f)N-f(1-f')(1+N)]\delta(E'-E+\Omega)$$
$$+\frac{1}{2}\left(1-\frac{\epsilon\epsilon'-\epsilon_0^2}{EE'}\right)[(1-\bar{f}')(1-f)N-\bar{f}'f(1+N)]\delta(E'+E-\Omega)\Bigg\}. \quad (3.6)$$

It has been assumed that one can neglect the departure from the equilibrium distribution $N=N_q=N_{-q}$ of the phonons. The matrix element for a process involving a phonon with momentum $\pm$**q** is denoted by V_q; the unprimed and primed quantities f, ϵ, and E designate arguments **p** and **p**$-$**q**, respectively; the abbreviation $\bar{f}'$ stands for f_{-p+q}. If one sets $f=f_0+\Delta f$, where f_0 is the equilibrium distribution and Δf is a small deviation from equilibrium, and if one neglects higher than first order terms in Δf, one obtains two different contributions to $(\partial f_p/\partial t)_{\text{coll}}$ from the right-hand side of Eq. (3.6). One verifies easily that the first contribution is equal to $-2\Gamma_p^{\text{ph}}\Delta f_p$, where $2\Gamma_p^{\text{ph}}$ is identical to the expression which has been obtained here from first principles and is presented in Eq. (3.5). The second contribution is given by an integral over d^3q containing the deviation Δf_{p-q} as a factor in the integrand. If initially Δf_p is different from zero only around a point **p** in momentum space, then the second contribution will always be negligible, and one is left with the differential equation

$$(\partial f_p/\partial t)_{\text{coll}}=-2\Gamma_p^{\text{ph}}\Delta f_p. \quad (3.7)$$

Hence, according to this consideration, one must identify the quantity $2\Gamma_p^{\text{ph}}$ as the inverse decay time for a quasi-particle. We can now see the origin of the statistical factor $(1-f_p)^{-1}$. Consider, for instance, the contributions to $(\partial f_p/\partial t)_{\text{coll}}$ coming from the terms with delta functions $\delta(E'-E+\Omega)$ in the integrand of Eq. (3.6). First we have the ordinary term, proportional to $-\Delta f_p(1-f_{p-q})(1+N_q)$, which corresponds to the scattering from the state **p** into the state **p**$-$**q** with the emission of a phonon **q**. But there is also a second term, proportional to $-\Delta f_pf_{p-q}N_q$, which corresponds to the scattering from the state **p**$-$**q** into the state **p** with the absorption of a phonon **q**. The reason for the negative sign of this second contribution is that for a positive Δf_p the number of empty states around **p** and accordingly the backscattering is reduced in comparison to the equilibrium value. By making use of the delta function which multiplies both contributions one can transform the sum according to

$$-\Delta f_p[(1-f_{p-q})(1+N_q)+f_{p-q}N_q]$$
$$=-\Delta f_p(1-f_{p-q})(1+N_q)(1-f_p)^{-1}. \quad (3.8)$$

[10] L. P. Kadanoff (private communication). See also L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

We now see that the enhancement of the decay due to the reduced backscattering is included in the factor $(1-f_p)^{-1}$ which is greater than one. It is interesting to note that this result has been obtained quite automatically by employing thermal Green's functions.

The expression for $2\Gamma_p{}^{\mathrm{ph}}$, presented in Eq. (3.5), is evaluated by approximating the energy gap parameter ϕ_p by the constant BCS value[11] ϵ_0 valid for the given temperature, and further, by setting $\bar{\epsilon}_p=\epsilon_p$, $E_{rp}=E_p$, and $Z_p=1$ in a lowest order approximation. The integrations over the wave number $|\mathbf{q}|$ and the polar angle $\vartheta=\sphericalangle(\mathbf{p},\mathbf{q})$ are transformed into integrations over $d\Omega=d\Omega_q$ and $dE'=dE_{p-q}$, respectively.

Let us consider separately the three contributions which arise from the three delta functions in the integrand of Eq. (3.5):

(1) One can show from the argument of the first delta function, $\delta(E_{p-q}+\Omega_q-E_p)$, that $\Omega_q<(E_p-\epsilon_0)$ is a necessary and $\Omega_q>2(c/v_F)(E_p-\epsilon_0)$ is a sufficient condition for the integral over E' to be different from zero. Here we have denoted by c the velocity of sound, and by v_F the velocity of electrons at the Fermi surface. We shall neglect the second condition since c/v_F is much smaller than one. Accordingly the limits of the $d\Omega$ integration, which remains after the dE' integration has been carried out with the help of the delta function, are set equal to 0 and $(E_p-\epsilon_0)$.

(2) A rough estimate of the limits of the arguments of the second delta function, $\delta(E_{p-q}-\Omega_q-E_p)$, reveals that for all Ω_q which meet the condition $\Omega_q>4(c/v_F)E_p$ the dE' integral will be different from zero. Neglecting also this restriction put on Ω_q, we integrate over Ω from 0 to Ω_m, where Ω_m denotes the maximum phonon frequency.

(3) In the case of the delta function $\delta(E_{p-q}+E_p-\Omega_q)$ one finds that $\Omega_q>(E_p+\epsilon_0)$ is a necessary and $\Omega_q>2(c/v_F)(E_p-\epsilon_0)$ is a sufficient condition for the integral over E' to be different from zero. Thus, we have to integrate over Ω from $(E_p+\epsilon_0)$ to Ω_m.

All the terms with ϵ_{p-q} in the coherence factors drop out in the result for the E' integration because both signs of ϵ_{p-q} are encountered. The absolute square of the dielectric constant in the integrand of Eq. (3.5) can be approximated by the square of the Fermi-Thomas value, $[1+(\omega_{\mathrm{pl}}/\Omega_q)^2]^2$, and this expression can be approximated further by $(\omega_{\mathrm{pl}}/\Omega_q)^4$ if the conditions $E_p\ll\Omega_m$ and $k_BT\ll\Omega_m$ are fulfilled. The first condition results from the fact that the $d\Omega$ integral (1) has an upper limit $(E_p-\epsilon_0)$, and the second condition arises from the fact that the integrands of the $d\Omega$ integrals (2) and (3) contain the statistical factors $N(\beta\Omega)$ and $f[\beta(\Omega-E_p)]$, respectively. Also, we can replace the upper limits Ω_m of the $d\Omega$ integrals (2) and (3) by ∞ if these conditions are fulfilled. Then the final expression for $2\Gamma_p{}^{\mathrm{ph}}$ turns out to be

$$2\Gamma_p{}^{\mathrm{ph}}=\left(\frac{e^2m}{p\omega_{\mathrm{pl}}{}^2}\right)[f(-\beta E_p)]^{-1}\int_0^{E_p-\epsilon_0}\frac{d\Omega\,\Omega^2(E_p-\Omega)}{[(E_p-\Omega)^2-\epsilon_0{}^2]^{1/2}}\left(1-\frac{\epsilon_0{}^2}{E_p(E_p-\Omega)}\right)f[-\beta(E_p-\Omega)][1+N(\beta\Omega)]$$
$$+\int_0^{\infty}\frac{d\Omega\,\Omega^2(E_p+\Omega)}{[(E_p+\Omega)^2-\epsilon_0{}^2]^{1/2}}\left(1-\frac{\epsilon_0{}^2}{E_p(E_p+\Omega)}\right)f[-\beta(E_p+\Omega)]N(\beta\Omega)$$
$$+\int_{E_p+\epsilon_0}^{\infty}\frac{d\Omega\,\Omega^2(\Omega-E_p)}{[(\Omega-E_p)^2-\epsilon_0{}^2]^{1/2}}\left(1+\frac{\epsilon_0{}^2}{E_p(\Omega-E_p)}\right)f[\beta(\Omega-E_p)][1+N(\beta\Omega)]. \quad (3.9)$$

It is convenient to consider $\Gamma_p{}^{\mathrm{ph}}$ as a function of the parameters x and y defined by

$$x=E_p/\epsilon_0,\quad y=\epsilon_0\beta, \quad (3.10)$$

and to set

$$\Gamma_p{}^{\mathrm{ph}}=\left(\frac{e^2m}{2p\omega_{\mathrm{pl}}{}^2}\right)\epsilon_0{}^3\Gamma_s(x,y). \quad (3.11)$$

From Eqs. (3.9) and (3.11) one obtains, after some transformations have been carried out,

$$\Gamma_s(x,y)=\int_0^{x-1}\frac{dt\,t^2(x-t-x^{-1})}{[(x-t)^2-1]^{1/2}}\{(1+e^{-y(x-t)})^{-1}+(e^{yt}-1)^{-1}\}$$
$$+\int_0^{\infty}\frac{dt\,t^2(x+t-x^{-1})}{[(x+t)^2-1]^{1/2}}\{(1-e^{-yt})^{-1}-(1+e^{-y(x+t)})^{-1}\}$$
$$+\int_{x+1}^{\infty}\frac{dt\,t^2(t-x+x^{-1})}{[(t-x)^2-1]^{1/2}}\{(1+e^{y(t-x)})^{-1}+(e^{yt}-1)^{-1}\}. \quad (3.12)$$

[11] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957), referred to as BCS.

The normal state values for the damping, denoted by $\Gamma_{p,n}{}^{\mathrm{ph}}$, are obtained from the expression in Eq. (3.9) if ϵ_0 is set equal to zero and E_p is replaced by ϵ_p. The result can be written in the form

$$\Gamma_{p,n}{}^{\mathrm{ph}}=\left(\frac{e^2m}{2p\omega_{\mathrm{pl}}{}^2}\right)\beta^{-3}\Gamma_n(z), \tag{3.13}$$

where the function $\Gamma_n(z)$ is given by

$$\Gamma_n(z)=(8/3)F_2(0)+F_2(z)+F_2(-z), \tag{3.14}$$

with

$$z=\epsilon_p\beta \tag{3.15}$$

and

$$F_2(\eta)=\int_0^\infty dt\, t^2(1+e^{t-\eta})^{-1}. \tag{3.16}$$

The function $F_2(-\eta)$ has been tabulated by Rhodes.[12]

We have evaluated $\Gamma_s(x,y)$, as a function of x, numerically for a number of different parameter values y. We compare the decay rate of a quasi-particle, $2\Gamma_p{}^{\mathrm{ph}}$, with that of a normal-state excitation, $2\Gamma_{p,n}{}^{\mathrm{ph}}$, by plotting the ratio of these quantities vs $z=\epsilon_p\beta$. From the Eqs. (3.11), (3.13), and (3.10) we obtain for this ratio the expression

$$\Gamma_p{}^{\mathrm{ph}}/\Gamma_{p,n}{}^{\mathrm{ph}}=y^3\Gamma_s(y^{-1}(z^2+y^2)^{1/2};y)/\Gamma_n(z). \tag{3.17}$$

In Fig. 2 the results are shown for some parameter values of y. The corresponding reduced temperatures, T/T_c, are obtained by writing $y=[\epsilon_0(0)/k_BT_c][\epsilon_0(T)/\epsilon_0(0)](T_c/T)$ and by using the results of the BCS theory for the ratio $\epsilon_0(T)/\epsilon_0(0)$ and the BCS relation $2\epsilon_0(0)=3.52k_BT_c$.

IV. ELECTRONIC THERMAL CONDUCTIVITY OF A PURE SUPERCONDUCTOR

In this section we will use the results which have been obtained for the lifetime of a quasi-particle at finite temperatures in order to determine the thermal electronic conductivity in the case where the interaction with the phonons dominates over that with the impurities. In the earlier theory presented by Bardeen, Rickayzen, and Tewordt[4] this problem has been attacked by setting up the corresponding Boltzmann equation. The resulting integral equation which determines the deviation Δf_p from the equilibrium distribution $f_p=f(\beta E_p)$ of the quasi-particles in the presence of a temperature gradient has been written as

$$\left(\frac{p_z}{m}\frac{\epsilon_p}{E_p}\right)\frac{E_p}{T}\frac{df_p}{dE_p}\frac{dT}{dz}=\Lambda\chi_p. \tag{4.1}$$

The z axis is taken along the direction ∇T. The quantity χ_p is connected with Δf_p through the relation

$$\Delta f_p=-(df_p/dE_p)\chi_p. \tag{4.2}$$

[12] P. Rhodes, Proc. Roy. Soc. (London) **A204**, 396 (1950).

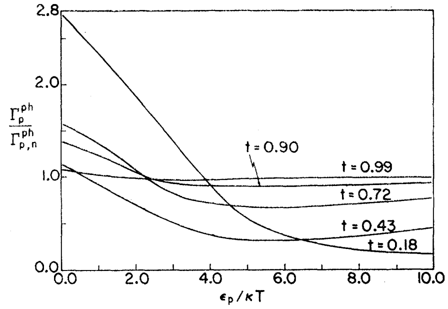

FIG. 2. The ratio of the inverse lifetime of a quasi-particle, $2\Gamma_p{}^{\mathrm{ph}}$, to that of a normal-state excitation, $2\Gamma_{p,n}{}^{\mathrm{ph}}$, vs ϵ_p/kT for different reduced temperatures $t=T/T_c$. The quantity ϵ_p is the energy of the normal-state excitation, measured from the Fermi energy.

The integral operator Λ is defined by the identity

$$\Lambda\chi_p=\int W(\mathbf{p},\mathbf{p}')(\chi_p-\chi_{p'})d^3p'. \tag{4.3}$$

From the definition of the Kernel $W(\mathbf{p},\mathbf{p}')$ in BRT one finds that the integral over d^3p' of this quantity is connected with the decay rate $2\Gamma_p{}^{\mathrm{ph}}$ [Eq. (3.5)] by the relation

$$\int W(\mathbf{p},\mathbf{p}')d^3p'=-\frac{df_p}{dE_p}2\Gamma_p{}^{\mathrm{ph}}. \tag{4.4}$$

Kadanoff and Martin[7] have argued that there is little point for our crude model of a superconductor to consider that instead of the scattering cross section the transport cross section enters the conductivity theory. The difference between these two cross sections is represented by the term $\chi_{p'}$ in the integrand of Eq. (4.3), and one sees that this term gives rise to a reduction of the forward scattering. We follow here the simplifying assumption made by KM and accordingly neglect the term with $\chi_{p'}$ in Eq. (4.3). Then the integral operator is replaced simply by a factor, and we can write with the help of Eqs. (4.4) and (4.2)

$$\Lambda\chi_p=2\Gamma_p{}^{\mathrm{ph}}\Delta f_p. \tag{4.5}$$

Using the formula $\mathbf{v}=\nabla_pE_p$ of BRT for the group velocity of quasi-particles one finds that the thermal conductivity, κ_s, is determined by the expression

$$\kappa_s=-\left(\frac{dT}{dz}\right)^{-1}2\int E_p\left(\frac{p_z}{m}\frac{\epsilon_p}{E_p}\right)\Delta f_pd^3p. \tag{4.6}$$

If one determines Δf_p from Eqs. (4.1) and (4.5) and inserts the result into Eq. (4.6), one obtains

$$\kappa_sT=\tfrac{1}{2}\beta\frac{n}{m}\int_0^\infty d\epsilon_p\frac{\epsilon_p{}^2\,\mathrm{sech}^2(\frac{1}{2}\beta E_p)}{2\Gamma_p{}^{\mathrm{ph}}}, \tag{4.7}$$

where n is the density and m the mass of the electrons. This is exactly the expression which has been derived

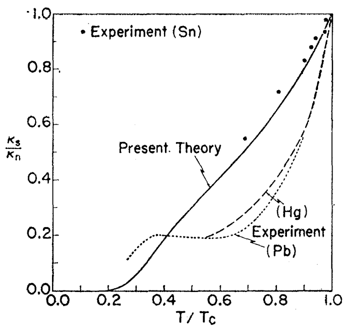

FIG. 3. The ratio of the electronic thermal conductivity in the superconductor, κ_s, to that in the normal metal, κ_n, when interaction of the quasi-particles with the phonons is predominant.

by KM by using thermal Green's functions. The normal state value of the thermal conductivity, κ_n, is obtained from the right-hand side of Eq. (4.7) by letting ϵ_0 go to zero.

While KM have neglected the dependence of the decay rates $2\Gamma_p^{\text{ph}}$ and $2\Gamma_{p,n}^{\text{ph}}$ on the excitation energies, and have even set these two quantities equal to each other, we insert the actual values for the decay rates which have been determined in the last section, and we evaluate the resulting integrals numerically. In order to bring these integrals into a convenient form we transform in Eq. (4.7) to the integration variable $x=E_p/\epsilon_0$ and in the corresponding expression for $\kappa_n T$ to the integration variable $z=\epsilon_p\beta$, and we introduce the functions $\Gamma_s(x,y)$ and $\Gamma_n(z)$ which are related to Γ_p^{ph} and $\Gamma_{p,n}^{\text{ph}}$ via Eqs. (3.11) and (3.13), respectively. Then we obtain for the ratio of the thermal conductivities, κ_s/κ_n, the final expression

$$\frac{\kappa_s}{\kappa_n}=\frac{\displaystyle\int_1^\infty dx\, x(x^2-1)^{1/2}\,\text{sech}^2(\tfrac{1}{2}xy)[\Gamma_s(x,y)]^{-1}}{\displaystyle\int_0^\infty dz\, z^2\,\text{sech}^2(\tfrac{1}{2}z)[\Gamma_n(z)]^{-1}}. \tag{4.8}$$

This expression depends on the temperature only through the parameter $y=\epsilon_0(T)/k_BT$. Therefore, the plot of the ratio κ_s/κ_n vs the reduced temperature depends on $\epsilon_0(T)/\epsilon_0(0)$ as a function of the reduced temperature and on the value of the ratio $\epsilon_0(0)/k_BT_c$. If we use the results of the BCS theory then we obtain the plot shown in Fig. 3. This curve has a positive limiting slope at $T/T_c=1$ and decreases monotonically with decreasing T/T_c. For comparison we have included in this figure measurements of κ_s/κ_n obtained for pure tin,[5] mercury,[6] and lead.

V. CONCLUDING REMARKS

Our calculation of the lifetime of a quasi-particle in a superconductor at finite temperatures provides one more example of the power of the thermodynamic Green's functions in describing the properties of many-particle systems. In this specific example it is especially satisfactory, in that this method leads quite automatically to a certain statistical correction factor to the expression for the inverse lifetime which is anticipated from a more naive calculation. This correction factor, $(1-f_p)^{-1}$, has been verified later with the help of the Boltzmann equation. The explicit calculations show that the lifetimes of a quasi-particle and a normal-state excitation depend strongly on the excitation energy, and that the ratio of these quantities differs markedly from one, for most excitation energies and for temperatures not to close to the transition temperature (see Fig. 2).

The application of the results for the lifetimes to the problem of electronic thermal conductivity, as limited by lattice scattering, leads to a fair agreement with the experimental data obtained for the "weak" superconductor tin (see Fig. 3). In future work we intend (a) to solve the Boltzmann integral equation numerically in order to make certain that the relaxation-time treatment of this equation is justified, and (b) to take into account the dependence of the energy gap parameter on the excitation energy in the case of the "strong" superconductors mercury and lead.

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