Complex Angular Momentum in Relativistic S-Matrix Theory*

A. O. BARUT† AND D. E. ZWANZIGER
Department of Physics, University of California, Berkeley, California

(Received February 27, 1962; revised manuscript received April 11, 1962)

On the basis of unitarity and the Mandelstam representation we discuss the analyticity and threshold behavior of the positions and residues of the poles of the scattering amplitude in angular momentum as a function of energy, separate the right and left cuts of the partial wave amplitude for complex l, and extend, in a model theory in which elastic unitarity holds, the domain of analyticity in the l plane to the region $\mathrm{Re}l>1$.

I

THE importance of regarding the scattering amplitude in strong interactions as an analytic function of both energy and angular momentum has been recently emphasized.[1,2,3] A number of interesting experimental consequences of the hypothesis of poles in angular momentum l, in analogy to the Regge poles in nonrelativistic potential scattering,[4] have been obtained.[5–10] The purpose of this note is to establish, in the relativistic case, on the basis of the Mandelstam representation and unitarity, the following: (a) analyticity of the positions α_n and residues β_n of the poles of the scattering amplitude in the angular momentum plane as a function of energy; (b) threshold behavior of α_n and β_n for the purpose of an effective-range approximation; (c) separation of the left- and right-hand cuts in energy of the partial wave amplitude which extends the N/D method to complex l; and (d) extension of the domain of analyticity in l to $\mathrm{Re}l>1$ in a model theory in which the elastic unitarity condition holds, and a discussion of the role of unitarity and inelastic processes in establishing the analyticity in the l plane.[11]

II

We consider the two-body elastic scattering amplitude for equal mass spinless particles. We use the usual variables s, t, and u: $s=4(q^2+m^2)$, $t=-2q^2(1-z)$, $u=-2q^2(1+z)$, $z=\cos\theta$, and $\nu=q^2$; q and θ are the center-of-mass momentum and scattering angle in the s channel, respectively. For integral values of the angular momentum l, one obtains, from a fixed-energy dispersion relation with N subtractions, the partial wave amplitudes[12]

$$A(s,l)=\frac{1}{\pi}\int_{z_0}^{\infty}dz\,Q_l(z)[A_t(s,z)+(-1)^lA_u(s,z)],\qquad l\geqq N,\quad(1)$$

where $z_0=1+(4m^2/2q^2)$, A_t and A_u are the absorptive parts of the amplitude in the t and u channels, respectively, and $Q_l(z)$ is the Legendre function of the second kind. Equation (1) can be used to define an analytic function $A(s,l)$ of s and l which coincides with the partial wave amplitudes for physical values of l. From the asymptotic behavior of A_t and A_u, it follows immediately that $A(s,l)$ is holomorphic in l for $\mathrm{Re}l>N$ for all s, since $Q_l(z)$ behaves as z^{-l-1} for large z. Furthermore, a weak form of the unitarity condition in the crossed channels shows that for $s<0$ (physical region of the t channel) the bounds of A_t and A_u are such that $A(s,l)$ is holomorphic in l for $\mathrm{Re}l>1$.[12]

According to Eq. (1), the partial wave amplitude $A(q^2,l)$ generally has cuts along the whole real q^2 axis. Part of this cut is kinematical and will be separated as follows. Using a relation connecting $Q_l(z)$ to the hypergeometric function $F(a,b;c;z)$[13], we rewrite Eq. (1) in the form

$$A^{\pm}(\nu,l)=\frac{1}{\pi}\frac{\Gamma^2(l+1)}{\Gamma(2l+2)}(4\nu)^l\int_{4m^2}^{\infty}dw\frac{D^{\pm}(\nu,w)}{w^{l+1}}\times F(l+1,l+1;2l+2;-4\nu/w);\quad \mathrm{Re}l\geqq N\quad(2)$$

where $D^{\pm}(\nu,w)=A_t(\nu,w)\pm A_u(\nu,w)$ and we have introduced the $A^{\pm}$ partial wave amplitudes for which the unitary relation is diagonal. The function F has a cut

* Supported in part by the Air Force Office of Scientific Research under contract No. AF 49(638)-327 and by the Alfred P. Sloan Foundation.

† On leave from Syracuse University, Syracuse, New York.

[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1961).

[3] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1201 (1962).

[4] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960).

[5] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[6] V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 667 (1961) [translation: Soviet Phys.—JETP, **14**, 478 (1962)].

[7] B. M. Udgaonkar, Phys. Rev. Letters **8**, 142 (1962).

[8] S. C. Frautschi, M. Gell-Mann and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

[9] D. Wong, Phys. Rev. **126**, 1221 (1962).

[10] A. O. Barut, Phys. Rev. **126**, 1873 (1962).

[11] After the completion of this work we learned that the result (d) has also been obtained independently by K. Bardakci, University of Minnesota, 1962 (to be published).

[12] M. Froissart, Phys. Rev. **123**, 1053 (1961) and unpublished report to the La Jolla Conference on Weak and Strong Interactions, 1961. By using the limit $P_n(\cosh\xi)\to e^{(n+\frac{1}{2})|\mathrm{Re}\xi|}$, n real, for large l and by going through the same steps as in Froissart's paper, one obtains the result that for s in the neighborhood of the real negative axis the amplitude $A(s,l)$ is holomorphic for $\mathrm{Re}l>1+\epsilon$. (G. Prosperi, private communication). We make use of this result later in Eq. (13).

[13] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 133.

in $\nu=q^2$ from $-\infty$ to $-w/4$, and $D^{\pm}$ has the spectral cuts on the right from $\nu=0$ to ∞ and on the left from $\nu=-\infty$ to the boundary of the double spectral function. Thus, $A(\nu,l)/\nu^l$ is a real analytic function of ν, real in the gap between $\nu=-m^2$ and $\nu=0$. (We omit the superscripts $\pm$ unless necessary.) Thus, an N/D method is possible for complex l for the quantity.

$$B(\nu,l)=A(\nu,l)/\nu^l. \tag{3}$$

In addition, the representation (2) has the advantage of displaying the asymptotic behavior of $Q_l(z)$ for large z [since $F(a,b;c;z)$ behaves as unity for large z] and will allow us to separate the singular part of the amplitude. For this purpose we rewrite F in Eq. (2) as the sum of two terms by subtracting the first R terms in the power series expansion of F:

$$F(l+1,\,l+1;\,2l+2,\,-4\nu/w)$$

$$=\Big[F-\sum_{r=0}^{R} c_r(l)(\nu/w)^r\Big]+\sum_{r=0}^{R} c_r(l)(\nu/w)^r,$$

which give rise, respectively, to two terms in $B(\nu,l)$:

$$B(\nu,l)=B_0(\nu,l)+B_1(\nu,l). \tag{4}$$

The term $B_0(\nu,l)$ is holomorphic in l for $\mathrm{Re}l>N-R$ while the term $B_1(\nu,l)$ contains all the singularities in l of A and in ν, only the cuts of D which are due to the spectral functions. Since the terms of $B_1(\nu,l)$ are of the form

$$\int_{4m^2}^{\infty} dw\,\frac{D(\nu,w)}{w^{l+r+1}},$$

we have the result that the analyticity domain in l of $A(\nu,l)$ is closely connected with that of the Mellin transform of the absorptive part $D(\nu,t)$. This provides a useful criterion for the analyticity in l of a given amplitude.

III

We now make use of the unitarity condition. For $\nu>0$ but below the inelastic threshold the $A^{\pm}$ amplitudes separately satisfy the analytically continued unitarity relation,[14]

$$A(\nu,l)=(\nu+m^2/\nu)^{\frac{1}{2}}e^{i\delta(\nu,l)}\sin\delta(\nu,l), \tag{5}$$

or, with (3),

$$B^{-1}(\nu,l)=(\nu+m^2)^{-\frac{1}{2}}\nu^{l+\frac{1}{2}}(\cot\delta-i). \tag{6}$$

Since B, and hence B^{-1}, is a real analytic function of ν and l for $\mathrm{Re}l>N$, its discontinutiy across the physical cut is twice the imaginary part and can therefore be written in the form

$$B^{-1}(\nu,l)=(\nu+m^2)^{-\frac{1}{2}}(\cos\pi l)^{-1}\{Y(\nu,l)$$
$$+\exp[(l+1/2)\ln(-\nu-i\epsilon)]\}, \tag{7}$$

where $Y(\nu,l)$ is a real meromorphic function of ν for $\mathrm{Re}l\geqq N$ that has no elastic cut but only the left-hand cut of $B(\nu,l)$ and a right-hand cut beginning at the inelastic threshold. It can be verified that the discontinuity of the second term of Eq. (7) is that of Eq. (6).

Equation (7) has a direct nonrelativistic counterpart. By a slight modification of a relation given by Bottino, Longoni, and Regge[15] the nonrelativistic amplitude can be written in the form

$$B^{-1}(\nu,l)=(\cos\pi l)^{-1}\{Y(\nu,l)+\exp[(l+1/2)\ln(-\nu-i\epsilon)]\},$$

where Y has again only the left cut from $\theta=-\infty$ to $\nu=-m^2/4$. For physical values of l we have

$$Y^{\pm}(\nu,l)=\pm\nu^{l+\frac{1}{2}}\cot\delta_l^{\pm},$$

and consequently $Y(\nu,l)$ is the analytic continuation of the inverse reaction matrix, K^{-1}.

IV

The relation (7) is exact and the fact that Y does not have the elastic cut allows us to discuss the analyticity and the threshold behavior of the Regge pole parameters, $\alpha_n(\nu)$ and $\beta_n(\nu)$.

Let us assume for the moment that $Y(\nu,l)$ is meromorphic in l in the region $\mathrm{Re}l>1$. Then the only singularities of the amplitude in the l plane allowed by unitarity are poles given by

$$Y(\nu,l)+\exp[(l+\tfrac{1}{2})\ln(-\nu-i\epsilon)]=0. \tag{8}$$

Let us denote the solutions of Eq. (8) in l by $\alpha_n(\nu)$ and the residues of $B(\nu,l)$ at these poles by $b_n(\nu)=\beta_n(\nu)/\nu^{\alpha(\nu)}$, where $\beta_n(\nu)$ are the residues of the amplitudes $A(\nu,l)$. It follows from the separation (4) that $\alpha_n(\nu)$ is obtained as the solution of $B_1^{-1}(\nu,l)=0$. Consequently, $\alpha_n(\nu)$ is an analytic function of ν in the domain of analyticity of $B_1(\nu,l)$ except, possibly, for isolated singularities at the points where B_1^{-1} has multiple poles. We recall that $B_1(\nu,l)$ has only the spectral cuts of D. Wherever $\alpha_n(\nu)$ is analytic and distinct from all other $\alpha_m(\nu)$, $b_n(\nu)$ is analytic since

$$b_n(\nu)=(2i\pi)^{-1}\oint B_1(\nu,l)dl$$

for an appropriate contour about the pole α_n.

It is clear in potential scattering, because of the uniqueness of the solution to the Schrödinger equation (second-order differential equation with the boundary condition $\varphi(0)=0$ for $\mathrm{Re}l>-\frac{1}{2}$), that the poles are distinct so that $\alpha_n(\nu)$ and $b_n(\nu)$ have only the right spectral cut, and are real analytic functions. If this holds

[14] See for exzmple, E. J. Squires, University of California Radiation Laboratory Report UCRL-10033 (unpublished) and G. Prosperi, University of California Radiation Laboratory Report UCRL-10116 (unpublished).

[15] A. Bottino, A. M. Longoni and T. Regge, University of Turin, 1961 (to be published).

also in the relativistic case then $\alpha_n(\nu)$ and $b_n(\nu)$ have, in the left, at most the spectral cut of $D(\nu,l)$ and will be real analytic functions. Even this spectral cut may not be there as can be seen from the following consistency argument.[16] For if a pole term of the form $\beta(s)P_{\alpha(s)}(-1-t/2q^2)/\sin\pi\alpha(s)$ should dominate for large t and $s<0$ the discontinuity of such a term in t must be real which implies that $\alpha(s)$ and $\beta(s)$ must be real for all $s<0$.

To discuss the threshold behavior[17] of $\alpha_n(\nu)$ and $b_n(\nu)$, we introduce two constants which are real since Y is real at the threshold:

$$\partial Y/\partial\nu(0,\alpha(0))\equiv Y_\nu,\quad \partial Y/\partial l(0,\alpha(0))\equiv Y_l. \tag{9}$$

In Eq. (8), we expand $Y(\nu,l)$ about the point $(0,\alpha(0))$ and obtain in the region $\mathrm{Re}\alpha_n>-1/2$ for each pole[18] (we omit the index n)

$$\alpha(\nu)=\alpha(0)-(Y_\nu/Y_l)\nu+O(\nu^2) \\ -Y_l^{-1}\nu^{\alpha(\nu)+\frac{1}{2}}e^{-i\pi[\alpha(\nu)+\frac{1}{2}]}. \tag{10}$$

Only the last term contributes to the imaginary part of $\alpha(\nu)$, everything else is real. Below threshold, ν real and negative, we see immediately that $\alpha_n(\nu)$ is real, and above threshold we can make the following useful approximation:

$$\mathrm{Im}\alpha(\nu)\approx Y_l^{-1}\nu^{\alpha(0)+\frac{1}{2}}\sin\pi[\alpha(0)+\tfrac{1}{2}] \tag{11}$$

and

$$\mathrm{Re}\alpha(\nu)\approx\alpha(0)-(Y_\nu/Y_l)\nu+O(\nu^2) \\ -Y_l^{-1}\nu^{\alpha(0)+\frac{1}{2}}\cos\pi[\alpha(0)+\tfrac{1}{2}].$$

If, in $A^\pm$, $\alpha(0)$ becomes very close to an even or odd integer l, a Regge pole term approximates very well the partial wave amplitude for this l. Then Eq. (10) is the basis for an effective-range approximation.[10] For $\alpha(0)\geqq 1/2$ the slope of the curve $\mathrm{Re}\alpha_n(\nu)$ is continuous at threshold. In the q plane, for real l, the trajectories of the poles come down along the positive imaginary axis, make a right-angle turn at $q=0$ for $l>1/2$, and make a turn through an angle

$$\varphi=\pi[1-1/(2l+1)]$$

for $l\leqq 1/2$. Thus, an s-wave trajectory shows a distinct behavior from other physical partial waves.

The residues are given by

$$b(\nu)=\frac{(\nu+m^2)^{\frac{1}{2}}\cos\pi\alpha(\nu)}{[(\partial Y/\partial l)(\nu,\alpha(\nu))+\ln(-\nu-i\epsilon)(-\nu)^{\alpha(\nu)+\frac{1}{2}}]} \tag{12}$$

or, at threshold, simply by

$$b(\nu)\approx(\nu+m^2)^{\frac{1}{2}}Y_l^{-1}\cos\pi\alpha(\nu).$$

We note that although Y has a left-hand cut in ν from $-m^2$ to $-\infty$, $(\partial Y/\partial l)(\nu,\alpha(\nu))$ has at most the left-hand spectral cut. We note the important fact that $b(\nu)$, $\mathrm{Im}\alpha(\nu)$ have the same sign at threshold.

V

Finally, we discuss the question of the extension of the domain of analyticity in l plane to the region $\mathrm{Re}l>1$ on the basis of elastic unitarity condition (5) or (7). Let us consider a model theory in which the elastic unitarity condition holds. Or, let us assume that the phenomenological elasticity factor $\eta(\nu,l)$ in the uintarity relation

$$A(\nu,l)=(\nu+m^2/\nu)^{\frac{1}{2}}\{\eta(\nu,l)\exp[2i\delta(\nu,l)]-1\}/2i$$

is analytic in l in the region $\mathrm{Re}l>1$ so that the inelastic problem can be formally transformed into an elastic one.[19] Under these conditions we can prove the analyticity of $A(\nu,l)$ in $\mathrm{Re}l>1$. For this purpose we write a dispersion relation for $Y(\nu,l)$ in the ν-plane with subtractions and pole terms which we combine into a single rational function[20]:

$$Y(\nu,l)=\frac{1}{\pi}\nu^M\int_{-\infty}^{-m^2}d\nu'\left[\frac{\mathrm{Im}Y(\nu',l)}{\nu'^M(\nu'-\nu)}\right. \\ \left.+\Big(\sum_{n=0}^{N_0}a_n(l)\nu^n\Big)\Big/\Big(\sum_{m=0}^{M_0}b_m(l)\nu^m\Big)\right]. \tag{13}$$

This expression is written first for $\mathrm{Re}l\geqq N$. However, for some domain which includes real ν, $\nu<-m^2$, $Y(\nu,l)$ is a meromorphic function of l in the region $\mathrm{Re}l>1$.[12] Consequently, the functions $a_n(l)$ and $b_m(l)$ are meromorphic functions of l for $\mathrm{Re}l>1$, as they are the solutions of the system of linear equations obtained by writing Eq. (13) for N_0+M_0 fixed values of $\nu<-m^2$. The right-hand side of Eq. (13) thus provides a representation of $Y(\nu,l)$ which is meromorphic in l for $\mathrm{Re}l>1$ and all ν. Fixed cuts in $\mathrm{Re}l>1$, due to the endpoint or pinching poles from the term containing the integral, cannot arise since they are known to be absent for $\nu<-m^2$.

Analyticity in angular momentum does not follow from Mandelstam representation and crossing symmetry alone. The unitarity condition is also essential. In fact, one can construct amplitudes satisfying Mandelstam representation which are not meromorphic in the l plane. (For example, take the spectral function

$$\rho(s,t)=(t/M^2)^{-p}[\ln(t/M^2)]^{q-1}(s/M^2)^{-p} \\ \times[\ln(s/M^2)]^{q-1};\quad s,\,t>M^2 \\ =0,\ \text{otherwise}$$

[16] P. Burke (private communication).

[17] We are indebted to Marcel Froissart for pointing out to us the usefulness of Eq. (8) in the study of the threshold behavior of the amplitude.

[18] If $\mathrm{Re}\alpha_n(0)<-1/2$, one may rewrite Eq. (8) in the form

$$Y^{-1}(\nu,l)+\exp[-(l+\tfrac{1}{2})\ln(-\nu-i\epsilon)]=0$$

and expand $I(\nu,l)\equiv Y^{-1}(\nu,l)$, and obtain instead of Eq. (10):

$$\alpha(\nu)=\alpha(0)-(I_\nu/I_l)\nu+0(\nu^2)-I_l^{-1}\nu^{-[\alpha(\nu)+\frac{1}{2}]}e^{i\pi[\alpha(\nu)+\frac{1}{2}]}.$$

[19] M. Foissart, Nuovo cimento 22, 191 (1961).

[20] The validity of this equation rests upon assumptions about the asymptotic behavior of Y and would not be rigorous if there were an infinite number of poles.

which gives for Mellin transform of the discontinuity function

$$\frac{1}{\pi}M^{-2l}\Gamma(q)/(l+p)^q\int_{M^2}^{\infty}\frac{ds'}{s'-s}(s/M^2)^{-p}[\ln(s/M^2)]^{q-1},$$

which has a cut at $l=-p$ for nonintegral q.)

Even for the proof of holomorphy for $\mathrm{Re}l \geqq 1$, $\nu < -m^2$, a weak form of the unitarity condition has been used. It is also clear that the poles in the l plane for one channel are the result of highly inelastic processes in the crossed channels. This can be seen from the fact that these poles dominate the high-energy behavior of the amplitude in crossed channels and correctly predict a purely imaginary forward scattering amplitude. Therefore, a complete proof of analyticity in l must, we feel, make use of the full unitarity condition. Because the unitarity condition couples all channels, such a proof must await further developments in the S-matrix theory of multiparticle processes. Also perturbation theory is no guide in establishing the analyticity in l, since a finite number of diagrams does not give energy-dependent poles. The proof given in this section may indicate the direction along which a future proof should proceed.

If one invokes a principle of maximal analyticity in angular momentum then the previous discussion shows that the only singularities required by unitarity are poles, throughout the entire l plane. However, in order to avoid the introduction of a new postulate, one would like to derive the principle of maximal analyticity in l as a consequence of maximal analyticity in linear momentum.

We would like to express our gratitude to Professor G. F. Chew for his helpful encouragement during the course of this work.

PHYSICAL REVIEW VOLUME 127, NUMBER 3 AUGUST 1, 1962

Rate Difference between a Clock in an Artificial Satellite and a Clock on the Earth*

SJUR REFSDAL
Institute for Theoretical Physics, University of Oslo, Blindern, Norway

(Received December 27, 1961; revised manuscript received April 3, 1962)

Taking account of the diurnal rotation and the oblateness of the earth, the average relative rate difference is calculated for an arbitrary inclination of the satellite orbit. For eccentricities less than 0.1, an accuracy of order 10^{-14} is obtained.

IT is well known that[1,2] due to the general theory of relativity, the rate of a clock on the earth will be different from the rate of a similar clock in an artificial satellite. For an elliptic satellite orbit the average relative rate difference is[3,4]

$$\bar{\Delta}\equiv\frac{\tau_s-\tau_E}{\tau_E}=\frac{GM_E}{c^2r_0}\left(1-\frac{3r_0}{2a}\right)=6.96\times10^{-10}\left(1-\frac{3r_0}{2a}\right). \quad (1)$$

τ_E and τ_s are the periodic times read on the earth clock and the satellite clock, respectively; G is the gravitational constant, M_E is the mass of the earth, r_0 is the radius of the earth, and $2a$ is the major axis of the satellite orbit. In this derivation the diurnal rotation and the oblateness of the earth are not taken into account. Hoffmann[5] has calculated the influence on $\bar{\Delta}$ from these two perturbations, and finds for a circular orbit in the equatorial plane corrections terms in $\bar{\Delta}$ of order 10^{-12}. In the present paper we will calculate the correction terms for an elliptic orbit with an arbitrary inclination. The rate of a standard clock in a satellite and on the earth, respectively, is given by[6,7]

$$dt_s=(1+2\chi_s/c^2-u_s^2/c^2)^{\frac{1}{2}}dt, \quad (2)$$

$$dt_E=(1+2\chi_E/c^2-u_E^2/c^2)^{\frac{1}{2}}dt, \quad (3)$$

t_E, t_s, and t being the times read on the earth clock, the satellite clock, and a coordinate clock, respectively u_E and u_s being the velocity of the earth clock and the satellite, respectively, relative to the center of the earth, and χ_E and χ_s being the scalar gravitational potentials on the earth and on the satellite, respectively. Neglecting terms of order χ^2/c^4, $\chi u^2/c^4$, and u^4/c^4, we get from Eqs. (2) and (3)

$$dt_s=[1+c^{-2}(\chi_s-\chi_E-\tfrac{1}{2}u_s^2+\tfrac{1}{2}u_E^2)]dt_E. \quad (4)$$

* Work supported by the Norwegian Research Council for Science and the Humanity.

1 F. Winterberg, Astronaut. Acta. **2**, 25 (1956).
2 S. F. Singer, Phys. Rev. **104**, 11 (1956).
3 C. Möller, Suppl. Nuovo cimento **6**, 381 (1957).
4 S. Refsdal, Phys. Rev. **124**, 996 (1961).
5 B. Hoffmann, Phys. Rev. **106**, 358 (1957).

6 C. Möller, *The Theory of Relativity* (Clarendon Press, Oxford, England, 1960), p. 247.
7 We have ignored the vector potential because the fractional error in $\bar{\Delta}$ due to this is of order u_E^2/c^2.