Two-Parameter Statistical Model

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A statistical model is described which empirically relates the volume (characterized by λ_i) associated with each particle to its mass (m_i). The functional dependence used in $m_i^\alpha\lambda_i=\text{const}$, where α is some constant. The two parameters λ_π and α, in addition to the masses m_i, determine all the other λ_i. The model is applied to antiproton-proton annihilation at low and moderate energies. The annihilation is assumed to proceed through all possible intermediate states consisting of all combinations of the five mesons, π, K (or $\bar{K}$), K^* (or $\bar{K}^*$), ρ, and ω. The experimental multiplicities and fraction of $K\bar{K}m\pi$ ($m=0, 1, 2, \ldots$) states determine the two parameters λ_π and α to be 4.0 ± 0.4 and 1.94 ± 0.06, respectively. These values of λ_π and α predict the extent to which the K^* (or $\bar{K}^*$), ρ, and ω take part in the annihilation. These predictions are found to be in reasonable agreement with recent experimental data on the role of the K^* (or $\bar{K}^*$), ρ, and ω in $\bar{p}p$ annihilation. The η meson is also considered.

INTRODUCTION

STATISTICAL models associate a volume with each particle in the final state. Previous models have ascribed different volumes to different types or sets of particles.[1] Each volume has been varied independently in order to fit the experimental data. The model described here empirically relates the volume associated with each particle to its mass. The large number of particles recently recognized can be incorporated in the statistical model by means of only two parameters.

MASS-VOLUME RELATION

We denote the mass of the ith-type particle as m_i and its volume Ω_i as a multiple of the volume Ω_0 by means of the parameter $\lambda_i=\Omega_i/\Omega_0$, where $\Omega_0=(4\pi/3)(\hbar/m_\pi c)^3$. We need, then, a relation between m_i and λ_i. The functional dependence used in this model is $m_i^\alpha\lambda_i=\text{const}$. The two parameters λ_π and α, in addition to the masses m_i, determine all the other λ_i. This functional dependence was chosen for the following reasons. The simplest version of the statistical model ascribes the same volume to all the particles, i.e., $\lambda_i\approx\text{const}$. In statistical models that use the Lorentz-invariant phase space, the volume is introduced in the combination $m_i\Omega_i=m_i\lambda_i\Omega_0$. A simple version of such a model might require $m_i\lambda_i=\text{const}$. A third possibility is that the volume is proportional to the cube of the particle's Compton wavelength, $\Omega_i\propto m_i^{-3}$ or $m_i^3\lambda_i=\text{const}$. The obvious generalization is $m_i^\alpha\lambda_i=\text{const}$. No physical interpretation is given to the parameters λ_i and α.

APPLICATION OF THE MODEL TO ANTIPROTON-PROTON ($\bar{p}p$) ANNIHILATION

Five mesons are currently recognized which will be used in this analysis. They are the π, K (and $\bar{K}$), K^* (and $\bar{K}^*$),[2] ρ,[3] and ω[4,5] mesons. The masses and quantum numbers (known or probable) and decay modes used are given in Table I.

All annihilation processes are considered:

$$\bar{p}p\to\text{all intermediate states involving all combinations of the above mesons}$$
$$\to n\pi\ (n=2,3,4,\cdots)\ \text{or}\ K\bar{K}m\pi(m=0,1,2,\cdots).$$

We have calculated the transition rates using the Lorentz-invariant phase space, with the volume introduced after the manner of Desai.[6] The transition rate R_n to n intermediate-state particles is given as

$$R_n\propto G(I)G(S)\{\textstyle\prod_j[2m_j\Omega_j/(2\pi)^3]^{n_j}/(n_j!)\}\rho_n,$$

where $G(S)$ is the number of spin states, $G(I)$ is the number of isotopic spin states, n_j is the number of particles of the jth type, $\Omega_j=\lambda_j\Omega_0$ is the volume associated with the jth particle, m_j is its mass, and ρ_n is the number of states in the Lorentz-invariant phase space. This phase space has been discussed by Srivastava and Sudarshan[7] and others.[8]

TABLE I. The five mesons used in the $\bar{p}p$ annihilation analysis.

Particle	Mass[a]	Spin	Isotopic spin	Decay mode[b]
π	0.138	0	1	$\cdots$
K (or $\bar{K}$)	0.496	0	$\frac{1}{2}$	$\cdots$
K^* (or $\bar{K}^*$)	0.883	1	$\frac{1}{2}$	$K^*\to K+\pi$
ρ	0.770	1	1	$\rho\to 2\pi$
ω	0.790	1	0	$\omega\to 3\pi$

[a] The masses are an average over the various charge states.
[b] Other possible decay modes are ignored.

[1] E. Fermi, Progr. Theoret. Phys. Kyoto **5**, 570 (1950); R. Hagedorn, Nuovo cimento **15**,434 (1960); Z. Koba and G. Takeda, Progr. Theoret. Phys. Kyoto **19**, 269 (1958); F. Cerulus, Nuovo cimento **14**, 8271 (1959).

[2] M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, and S. G. Wojcicki, Phys. Rev. Letters **6**, 300 (1961).

[3] The ρ is the $T=J=1$ π-π resonant state, which has been so widely discussed and investigated. A thorough bibliography on this subject has been constructed [M. L. Stevenson, Lawrence Radiation Laboratory Report UCRL-9999, 1961 (unpublished)].

[4] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961).

[5] N. H. Xuong and G. R. Lynch, Phys. Rev. Letters **7**, 327 (1961).

[6] B. R. Desai, Lawrence Radiation Laboratory Report UCRL-9024, 1960 (unpublished).

[7] P. P. Srivastava and G. Sudarshan, Phys. Rev. **110**, 765 (1958).

[8] T. E. Kalogeropoulous, thesis, Lawrence Radiation Laboratory Report UCRL-8677, 1959 (unpublished).

TABLE II. Experimental data and the model's predictions: $\bar{p}p \to n\pi$ and $K\bar{K}m\pi$.

$P_{\bar{p}}$ (BeV/c)	$E_{\bar{p}p}$ c.m. (BeV)	Experimental data $\langle n\rangle_{av}$	$\%K\bar{K}m\pi$	$\langle m+2\rangle_{av}$	Two-parameter solutions[a] λ_π	α	Predictions from[b] λ_π^* and α^* $\langle n\rangle_{av}$	$\%K\bar{K}m\pi$	$\langle m+2\rangle_{av}$
rest	1.88	4.65±0.15	4 ±1	...	5.2 ±1.4	2.0 ±0.11	4.46±0.1	5.6±0.6	3.71±0.05
1.05	2.10	4.95±0.22	8 ±2	4.4±0.5	4.7 ±1.1	1.90±0.15	4.8 ±0.1	7.6±0.8	4.14±0.05
1.61	2.29	5.2 ±0.2	10.3±1.1	4.4±0.1	3.87±0.4	1.94±0.06	5.10±0.1	9.5±1.0	4.47±0.05
1.99	2.43	5.4 ±0.35	13 ±3	4.6±0.2	3.8 ±0.9	1.88±0.13	5.29±0.1	10.9±1.1	4.70±0.05

[a] See Figs. 1 through 4.
[b] Values of λ_π^* and α^* are 4.0±0.4 and 1.94±0.06, respectively.

The calculations were performed for antiprotons at rest and 1.05, 1.61, and 1.99 BeV/c incident on protons.[9] The average multiplicity in all $n\pi$ final staes (and in all $K\bar{K}m\pi$ final states) and the fraction of $K\bar{K}m\pi$ in the annihilation are the experimental quantities used to determine the two parameters λ_π and α. In this analysis the λ_i and α are assumed to be independent of the total energy in the $\bar{p}$-p c.m. system. Contours of the multiplicities and of the fraction of $K\bar{K}m\pi$ in the annihilation are plotted vs λ_π and α for each of the four energies (see Figs. 1 through 4).

The experimental data[9,10] and the solutions (λ_π and α) for the four energies are given in Table II. The four sets of λ_π and α are seen to be consistent. A weighted average of these four values gives $\lambda_\pi^*=4.0\pm0.4$ and $\alpha^*=1.94\pm0.06$. The predictions of the model at each of the four energies is also given in Table II for this average set of values.

The predictions in Table II, of course agree with the data from which they were derived. However, many additional predictions are made based on this model which may be checked against other experimental data. Some detailed results are given in Table III for 1.61-BeV/c antiprotons. Button *et al.*[11] find ~50%ρ, and Maglić *et al.*[4] find (10±2)%ω in $\bar{p}p\to 2\pi^+ + 2\pi^- + \pi^0$ (part of the $\bar{p}p\to 5\pi$ channel), in approximate agreement with the predictions of this model.[12] Xuong *et al*[5] find (33±8)%ω in $\bar{p}p\to 3\pi^+ + 3\pi^- + \pi^0$ (part of the $\bar{p}p\to 7\pi$ channel), also in agreement with the predictions of this model.[13] The unpublished results of Kalbfleisch[14] on $\bar{p}p\to K\bar{K}m\pi$ indicate that a large fraction of these reactions involve the K^* (a crude lower limit of ~25% K^* was found), as is predicted by this model.

One additional meson, the η of Pevsner *et al.*,[15] should be included in the above analysis. However, only its mass[15,16] and its (neutral/charged) branching ratio[16] is established. By assuming some properties for this meson, we can make some predictions of the extent to which the η should enter the annihilation process.

We will assume that the mass of the η is 0.550 BeV/c^2 and that its spin and isotopic spin are zero (i.e., similar to Gell-Mann's predicted pseudoscalar χ^0 meson).[17]

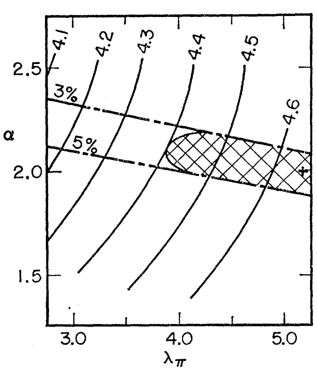

FIG. 1. Contours of the pion and $K\bar{K}m\pi$ multiplicities and of the percentage of $K\bar{K}m\pi$ vs λ_π and α for antiprotons at rest (1.88-BeV c.m. energy). The shaded area represents the region defined by the experimental data. Solid lines are for $\langle n\rangle_{av}$; dot-dash lines indicate percent $K\bar{K}m\pi$.

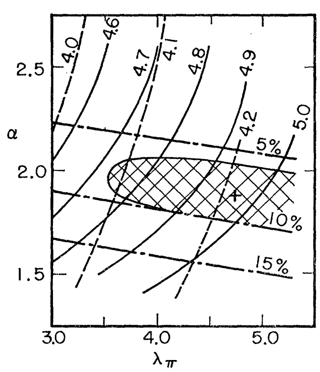

FIG. 2. Contours of the pion and $K\bar{K}m\pi$ multiplicities and of the percentage of $K\bar{K}m\pi$ vs λ_π and α for antiprotons at 1.05 BeV/c (2.10-BeV c.m. energy). The shaded area represents the region defined by the experimental data. Solid lines are for $\langle n\rangle_{av}$; dashed lines are for $\langle m+2\rangle_{av}$; dot-dash lines indicate percent $K\bar{K}m\pi$.

[9] These energies correspond to the experimental data available. The annihilation at rest is summarized by Kalogeropoulous (reference 8). The experiment at 1.05 BeV/c was performed by Goldhaber *et al.* [S. Goldhaber, G. Goldhaber, W. Powell, and R. Silberberg, Phys. Rev. **121**, 1525 (1961)]. The data at 1.61 BeV/c and at 1.99 BeV/c come from the experiments in the Lawrence Radiation Laboratory's 72-in. liquid-hydrogen bubble chamber (reference 10).

[10] G. R. Lynch, Revs. Modern Phys. **33**, 395 (1961).

[11] J. Button, G. R. Kalbfleisch, G. R. Lynch, B. C. Maglić, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. **126**, 1882 (1962).

[12] The statistical-model predictions for the branching ratios into the various charge states for 5π, for $\rho\,3\pi$, and for $\omega\,2\pi$ can be used to obtain the predictions for the percentage of ρ and ω in $2\pi^+ + 2\pi^- + \pi^0$ based on this model. The results are 39%ρ and 6%ω in $\bar{p}p\to 2\pi^+ + 2\pi^- + \pi^0$ at 1.61 BeV/c, in closer agreement with the experimental results. The branching ratios into the various charge states were taken from A. Pais, Ann. Phys. **9**, 548 (1960).

[13] The percentage of ω in $3\pi^+ + 3\pi^- + \pi^0$ at 1.61 BeV/c by this model is 33% (using the method in footnote 12).

[14] G. R. Kalbfleisch, thesis, Lawrence Radiation Laboratory Report UCRL-9597, 1961 (unpublished).

[15] A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961).

[16] P. L. Bastien, J. P. Berge, O. I. Dahl, M. Ferro-Luzzi, D. H. Miller, J. J. Murray, A. H. Rosenfeld, and M. B. Watson, Phys. Rev. Letters **8**, 114 (1962).

[17] M. Gell-Mann, California Institute of Technology Scientific Laboratory Report CTSL-20, 1961 (unpublished); Phys. Rev. **125**, 1067 (1962).

TABLE III. Additional predictions of the model at 1.61 BeV/c for $\lambda_\pi^*=4.0\pm0.4$ and $\alpha^*=1.94\pm0.06$[a]

	$\bar{p}p\to n\pi$				$\bar{p}p\to K\bar{K}m\pi$[b]				
		Percent ω's or ρ's per channel				Percent K^*'s per channel		Percent ω's or ρ's per channel	
n	Percent of all $n\pi$ annihilation	ω	ρ	$m+2$	Percent of all $K\bar{K}m\pi$ annihilation	$K^*\bar{K}$ (and $\bar{K}^*K^*$)	$K^*\bar{K}^*$	ω	ρ
2	0.3	...	...	2	0.4	...	...	...	...
3	5	...	6	3	9	7	...	...	...
4	22	0.6	18	4	42	36	2	...	3
5	39	5	30	5	41	56	7	2	4
6	26	17	31	6	8	55	6	9	1
7	8	40	23	7	c	...	...	...	...
8	c	...	...	...	...	...	...	...	...
all		10	26	all		43			

[a] The errors on all the entries in the table due to the error in λ_π^* and α^* are approximately $\pm10\%$.
[b] This is $9.5\pm1.0\%$ of the annihilation.
[c] Not calculated.

Using the five-meson solution (λ_π^*,α^*) obtained above as a first approximation, we find the reaction $\bar{p}p\to$ (η with π, ρ, ω) to be 12% of $\bar{p}p\to$ and $\bar{p}p\to$ (η with K's) to be 3% of $\bar{p}p\to K\bar{K}m\pi$ at 1.61 BeV/c. If, in addition, we assume η decays on the average mainly to two particles (two pions), then a λ_π vs α graph can be constructed for a six-meson solution. Such a solution changes the value of λ_π from 4.0 ± 0.4 to 3.6 ± 0.5 and the value of α from 1.94 ± 0.06 to 1.90 ± 0.06. These values predict $\bar{p}p\to$ (η with pions) to be 13% of $\bar{p}p\to n\pi$, and $\bar{p}p\to$ (η with Kmesons) to be 4% of $\bar{p}p\to K\bar{K}m\pi$ at 1.61 BeV/c. These values are only slightly larger than obtained above. Thus the η participates in $\bar{p}p$ annihilation to a small extent according to this model.

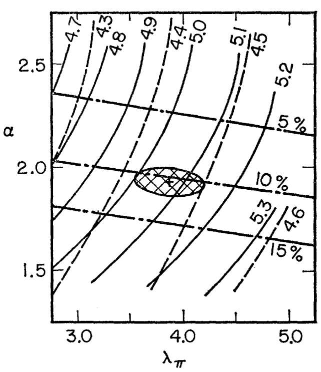

FIG. 3. Contours of the pion and $K\bar{K}m\pi$ multiplicities and of the percentage of $K\bar{K}m\pi$ vs λ_π and α for antiprotons at 1.61-BeV/c (2.29-BeV c.m. energy). The shaded area represents the region defined by the experimental data. Solid lines are for $\langle n\rangle_{av}$; dashed lines are for $\langle m+2\rangle_{av}$; dot-dash lines indicate percent $K\bar{K}m\pi$.

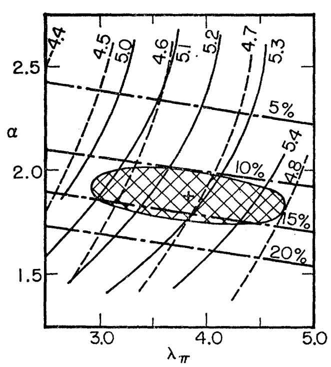

FIG. 4. Contours of the pion and $K\bar{K}m\pi$ multiplicities and of the percentage of $K\bar{K}m\pi$ vs λ_π and α for antiprotons at 1.99-BeV/c (2.43-BeV c.m. energy). The shaded area represents the region defined by the experimental data. Solid lines are for $\langle n\rangle_{av}$; dashed lines are for $\langle m+2\rangle_{av}$; dot-dash lines indicate percent $K\bar{K}m\pi$.

DISCUSSION

It is expected that the model can be applied to KN (or $\bar{K}N$) and πN interactions at sufficiently high energies (where the statistical model may have some reasonable success). In applying the model to KN or $\bar{K}N$ resonances and πN interactions, all the πN and πY resonances with well-defined quantum numbers and sufficiently long lifetimes should be included. Whether this two-parameter model can fit the data reasonably (as it did in the application above) in these reactions remains to be seen. It also remains to be seen if the predictions agree with the experimental data after more detailed results with better statistics are available. Better fits to the data may be possible if the parameters λ_i and α are allowed to have an energy dependence.

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