Sound Absorption in Liquid Helium II, $T<0.5°$K

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The expression $\Gamma=A\Omega T^4/\rho c_0^3$ for the absorption of sound in liquid He^4 II for $T<0.5°$K, where T is the absolute temperature, is shown to be a simple consequence of the principles of the excitation theory of a Bose liquid and a few other reasonable postulates. (Γ is the absorption, $\Omega/2\pi$ is the frequency and c_0 the velocity of the sound, and ρ is the density.) As in the work of Woodruff and Ehrenreich on sound absorption in insulating crystals, the analysis is based on the linearized Boltzmann transport equation and the Blount formula for the energy loss from a system of interacting excitations driven by a sound wave. It is valid for frequencies such that $\Omega\tau\gg1$ and $\Omega\ll KT/\hbar$, where τ is the relaxation time for the thermal phonons and K is Boltzmann's constant. The coefficient A is related to the rate of change of the sound velocity with density. An attempt is made to determine the exact magnitude of A within the framework of the present considerations by postulating a relative motion of the normal and superfluid components, but this approach leads to new difficulties.

I. INTRODUCTION

As a result of the work of many investigators during the past 24 years, the properties and behavior of bulk superfluid helium are now largely understood.[1] The theoretical structure on which this understanding is based will be referred to as the excitation theory of a Bose liquid; it is in the main an enlargement of Landau's two-fluid theory of liquid helium.[2–5]

Among the experimental observations which have not previously been explained in terms of this theory are those of Chase and Herlin,[6] Whitney,[7] and Dransfeld, Newell, and Wilks[8] on the absorption of first sound (in the 10-Mc/sec frequency range) for temperatures less than that of the maximum in the absorption ($\sim1°$K). In this temperature region they found an absorption much greater than could be accounted for by the analysis of Khalatnikov.[9] Dransfeld, Newell, and Wilks[8] noted that the data of Chase and Herlin[6] for the range below 0.5°K were well described by an expression [their Eq. (5)] equivalent to

$$\Gamma=A\Omega T^4/\rho c_0^3, \tag{1}$$

with $A=A_{\rm exp}=0.65\times10^6$ dyn cm^{-2} deg^{-4}. (Γ is in cm^{-1}; α as defined in reference 8 is $\Gamma/2$.) In what follows, Eq. (1) is deduced from the principles of the excitation theory of a Bose liquid and a few additional reasonable postulates. These principles and postulates are enumerated and discussed in the next section. The conclusions are drawn in the third section, and the demonstration as a whole is criticized and its significance appraised in Sec. IV.

II. PRINCIPLES OF THE EXCITATION THEORY OF A BOSE LIQUID AND ADDITIONAL POSTULATES USED IN THE DEMONSTRATION

The derivation of Eq. (1) is based on four of the fundamental principles of the excitation theory of a Bose liquid, each of which is now described.

(1) Every excitation of the liquid is characterized by a linear momentum $\mathbf{p}$. The energy ϵ required to produce an excitation of momentum $\mathbf{p}$ in the liquid at rest is a well-defined function $\epsilon(\mathbf{p})$, which is a property of the liquid. For superfluid helium at zero temperature and pressure it can be represented with sufficient accuracy[4] by

$$\begin{aligned}&\epsilon(\mathbf{p})=c_0p, \quad 0\leq p\leq P_1,\\ &\epsilon(\mathbf{p})=\Delta+(p-P_0)^2/2M_r, \quad P_1\leq p,\end{aligned} \tag{2}$$

with $\Delta/K=9.6°$ (K is Boltzmann's constant), $c_0=240$ m sec^{-1}, $P_0/\hbar=2.3$ Å^{-1}, and $M_r=0.40$(atomic mass of helium). P_1 is the value of p for which

$$c_0p=\Delta+(p-P_0)^2/2M_r.$$

The excitations with $p\leq P_1$ are phonons, quanta of density fluctuation waves. The excitations with $p>P_1$ are called rotons. For the temperatures of interest in this paper ($T<0.5°$K), roton energies are much greater than KT; hence, so few are excited that they have no measurable effect on the sound absorption, and we need consider only the phonon branch of the excitation spectrum. This branch is more completely described by

$$\begin{aligned}&\epsilon(\mathbf{p};\rho)=cp,\\ &c\cong c_0[1+(\gamma\Delta\rho/\rho_0)],\end{aligned} \tag{3}$$

where c is the velocity of first sound in the liquid at rest at $T=0°$K under a pressure such that the density $\rho=\rho_0+\Delta\rho$. The measurements of Atkins and Stasior[10]

[1] For a survey of this work, see, e.g., K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959).
[2] L. Landau, J. Phys. U.S.S.R. **5**, 71 (1941); **11**, 91 (1947).
[3] R. B. Dingle, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis, Ltd., London, 1952), Vol. 1, p. 112.
[4] R. P. Feyman, *Progress in Low-Temperature Physics*, edited by C. J. Gorter (Interscience Publishers, Inc., New York, 1955), Vol. 1, p. 17.
[5] I. M. Khalatnikov, Uspekhi Fiz. Nauk **59**, (1956); **60**, 69 (1956). These two articles are translated into German in Fortschr. Phys. **5**, 211 (1957); **5**, 287 (1957).
[6] C. E. Chase and M. A. Herlin, Phys. Rev. **97**, 1447 (1955).
[7] W. M. Whitney, Phys. Rev. **105**, 38 (1957).
[8] K. Dransfeld, J. A. Newell, and J. Wilks, Proc. Roy. Soc. (London) **A243**, 500 (1958).
[9] I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 243 (1950); **23**, 8, 21 (1952).
[10] K. R. Atkins and R. A. Stasior, Can. J. Phys. **31**, 1156 (1953).

lead to the estimate $\gamma \cong (\rho/c)(\partial c/\partial\rho) \cong 3.0$ for the Grüneisen constant γ.

Landau[2] showed that the energy required to produce an excitation of momentum $\mathbf{p}$ in the superfluid when it is moving with velocity $\mathbf{v}_s$ is given by

$$H(\mathbf{p};\rho)=\epsilon(\mathbf{p};\rho)+\mathbf{p}\cdot\mathbf{v}_s. \tag{4}$$

(2) Any motion of the superfluid liquid can be analyzed into a motion of the ground state with velocity $\mathbf{v}_s$ and a drifting motion of the gas of excitations with velocity $\mathbf{v}_n$. It is possible in a consistent way to associate a part ρ_s of the density of the superfluid at any point with the motion of the ground state, and a part ρ_n with the drift of the excitations, so that

$$\rho=\rho_s+\rho_n, \tag{5}$$

and

$$\partial\rho/\partial t=-\mathrm{div}(\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n). \tag{6}$$

Equation (6) is equivalent to the statement that in the composite motion of the liquid, mass is conserved.

(3) In a steady motion of the system with velocities $\mathbf{v}_s$ and $\mathbf{v}_n$, the distribution of excitations[4] is given by

$$N_0[\epsilon(\mathbf{p})]=\{\exp[(\epsilon-\mathbf{p}\cdot\mathbf{v}_n+\mathbf{p}\cdot\mathbf{v}_s)/KT]-1\}^{-1}, \tag{7}$$

where $N_0[\epsilon(\mathbf{p})]$ is the total number of Bose excitations of momentum $\mathbf{p}$.

(4) The kinetic effects of a disturbance of wavelength λ on the excitations in the liquid are adequately described by that distribution function $N(\mathbf{p})$ which satisfies the Boltzmann equation,

$$\frac{\partial N}{\partial t}+\frac{\partial N}{\partial \mathbf{r}}\cdot\frac{\partial H}{\partial \mathbf{p}}-\frac{\partial N}{\partial \mathbf{p}}\cdot\frac{\partial H}{\partial \mathbf{r}}=I[N], \tag{8}$$

where $\mathbf{r}$ is the position vector and $I[N]$ is the collision integral,[5] provided $\lambda \gg h/p$ for all or nearly all of the modes $\mathbf{p}$ which are excited. This condition leads to the restriction $\Omega \ll KT/\hbar$ on the validity of the demonstration.

The four additional postulates needed for the demonstration are quite different in character from the preceding principles. Equations (5) and (7) are reasonable approximations to be made in order to simplify the analysis. Equation (6) states a special property of pure superfluid helium for $T<0.5°K$ which determines the whole character of the analysis. The eighth postulate is Blount's formula for the rate of energy dissipation in a sound wave.[11]

(5) The collision integral $I[N]$ in Eq. (8) is in general a function of the occupations $N(\mathbf{p})$ of all modes $\mathbf{p}$ and of all the couplings between these modes. The couplings are both intrinsic (arising from the nonlinearities in the equations used to define the modes) and determined by boundary and impurity conditions. A better feeling for the meaning of $I[N]$ is obtained by imagining a situation in which at time $t=0$ there is no dependence on $\mathbf{r}$ of N or H, but there is a nonequilibrium distribution $N(\mathbf{p};t=0)$. For this situation Eq. (8) becomes

$$\partial N/\partial t=I[N(\mathbf{p})].$$

Experience suggests that for a complex system such as a macroscopic container full of superfluid helium, $N(\mathbf{p},t)\rightarrow N_0[\epsilon(\mathbf{p})]$ in a relatively smooth way. The simplest imaginable smooth approach is obtained if $I[N(\mathbf{p})]$ is replaced by $[\tau(\mathbf{p})]^{-1}\{N(\mathbf{p};t)-N_0[\epsilon(\mathbf{p})]\}$. In the demonstration, it will be assumed that under the perturbation caused by a sound wave, the approach of the function $N(\mathbf{p};t)$ to $N_0[\epsilon(\mathbf{p})]$ is not significantly different from this simplest kind of approach function. That is, the relation

$$I[N(\mathbf{p})]=-\{N(\mathbf{p})-N_0[\epsilon(\mathbf{p})]\}/\tau(\mathbf{p}), \tag{9}$$

will be assumed, and usually $\tau(\mathbf{p})$ will be replaced by a constant τ. Thus a possible explicit dependence of $I[N(\mathbf{p})]$ on $\mathbf{r}$, such as might come from boundary effects, is not admitted.

(6) The ratio of the relaxation time τ introduced under (5) to the period of the sound wave plays an important role in the analysis. The frequency of the sound wave multiplied by 2π will be symbolized by Ω. For the experiments under discussion ($\Omega\sim 10^8$ and $T<0.5°K$) it is assumed that $\Omega\tau\gg 1$. Evidence for the appropriateness of this assumption is provided by the thermal conductivity experiments of Fairbank and Wilks[12] and the calculations of various collision probabilities by Khalatnikov.[5]

(7) As already indicated under (5), all boundary effects will be neglected. Attention will be focused on a point or small region in the medium far removed from the walls or impurities, and the rate at which energy is removed from the sound wave and transferred to the surroundings in this region will be taken as equal to the average rate of energy removal for the whole medium. The possibility that the sound wave may induce motions of the liquid such that $\mathbf{v}_n$ is very different from $\mathbf{v}_s$ will be exploited, but the problems associated with determining a complete set of equations of motion for the case $\Omega\tau\gg 1$ and solutions of them which satisfy the boundary conditions are beyond the scope of this paper.

(8) It will be assumed that the rate of energy dissipation in a sound wave $\mathbf{u}=\mathbf{u}_0\exp[i(\sigma z-\Omega t)]$ is described by

$$Q=\tfrac{1}{2}\,\mathrm{Re}\left[\sum_{\mathbf{p}}\{N(\mathbf{p})-N_0[\epsilon(\mathbf{p})]\}\left(\frac{\partial H}{\partial t}\right)^*\right], \tag{10}$$

where H is given by Eqs. (3) and (4), and the instantaneous occupation $N(\mathbf{p})$ and the instantaneous equilibrium occupation $N_0[\epsilon(\mathbf{p})]$ both have parts which follow the sinusoidal oscillations of the sound

[11] E. I. Blount, Phys. Rev. **114**, 418 (1959).

[12] H. A. Fairbank and J. Wilks, Proc. Roy. Soc. (London) **A231**, 545 (1955).

wave. This formula was derived by Blount[11] and was applied by Woodruff and Ehrenreich[13] to the absorption by insulating crystals.

III. DEMONSTRATION OF FORMULA FOR SOUND ABSORPTION IN LIQUID HELIUM II

The passage of the sound wave through the medium is described by the displacements $\mathbf{u}_s$ and $\mathbf{u}_n$ of the superfluid and normal components at each point:

$$\begin{aligned} \mathbf{u}_s &= (0,0,\mathbf{u}_{sz}), & \mathbf{u}_n &= (0,0,u_{nz}), \\ u_{sz} &= u_s \exp[i(\sigma z-\Omega t)], & u_{nz} &= u_n \exp[i(\sigma z-\Omega t)]. \end{aligned} \tag{11}$$

The velocities associated with this motion are given by the time derivatives

$$\begin{aligned} \mathbf{v}_s &= (0,0,v_{sz}), & v_n &= (0,0,v_{nz}), \\ v_{sz} &= \dot{u}_{sz} & v_{nz} &= \dot{u}_{nz} \\ &= v_s \exp[i(\sigma z-\Omega t)], & &= v_n \exp[i(\sigma z-\Omega t)], \end{aligned} \tag{12}$$

where

$$v_s \equiv -i\Omega u_s, \quad v_n = -i\Omega u_n. \tag{13}$$

If u_s is taken as real, then u_n, v_n, and v_s are complex numbers. It is convenient to introduce the complex velocity ratio F and the associated real numbers F_1 and F_2 as follows:

$$(v_n/v_s) \equiv F \equiv F_1 + iF_2.$$

At this stage in the argument F is completely undetermined and may depend on other quantities such as the temperature and pressure. The velocity of the sound wave may with sufficient accuracy be taken as c_0, so that

$$\Omega = c_0\sigma. \tag{14}$$

Postulate (2) as expressed in Eq. (6) and the expressions for $\mathbf{v}_s$ and $\mathbf{v}_n$ introduced in Eqs. (12) and (13) lead to

$$\partial\rho/\partial t = -i\sigma(\rho_s v_s + \rho_n v_n) \exp[i(\sigma z-\Omega t)],$$

or integrating once with respect to time,

$$\Delta\rho = c_0^{-1}(\rho_s v_s + \rho_n v_n) \exp[i(\sigma z-\Omega t)].$$

Thus

$$\frac{\Delta\rho}{\rho} = \frac{(\rho_s v_s + \rho_n v_n)}{c_0(\rho_s+\rho_n)} \exp[i(\sigma z-\Omega t)].$$

For $T < 0.5°\mathrm{K}$ it can be shown that $\rho_s \gg \rho_n$, so that to a very good approximation,

$$\Delta\rho/\rho = (v_s/c_0) \exp[i(\sigma z-\Omega t)] = -i\sigma u_s \exp[i(\sigma z-\Omega t)].$$

Inserting this relation in Eqs. (3) and (4) gives

$$H = \epsilon_0 + \tilde{\epsilon}_1, \tag{15}$$

with

$$\epsilon_0 \equiv c_0 p, \tag{16}$$

$$\tilde{\epsilon}_1 = [1+(\mu/\gamma)]\epsilon_1, \tag{17}$$

where μ is the angle between $\mathbf{p}$ and the z axis and

$$\epsilon_1 \equiv a\epsilon_0 \exp[i(\sigma z-\Omega t)], \tag{18}$$

$$a \equiv -i\gamma\sigma u_s. \tag{19}$$

Attention will be restricted to the case in which u_s is a small quantity, and terms of higher order in this quantity will be neglected throughout.

The distribution function $N(\mathbf{p})$ can be written as the sum of a time-independent part, $N_{00}(p)$, and a small part proportional to $\exp[i(\sigma z-\Omega t)]$, $N_1(p)$:

$$\begin{aligned} N(\mathbf{p}) &= N_{00}(\mathbf{p}) + N_1(\mathbf{p}), \\ N_{00}(\mathbf{p}) &= N_0[\epsilon_0(\mathbf{p})] \equiv [\exp(\epsilon_0/KT)-1]^{-1}. \end{aligned} \tag{20}$$

It should be noted that with these definitions, the quantity $N(\mathbf{p}) - N_0[\epsilon(p)] = -\tau I[N(\mathbf{p})]$, which appears in Eqs. (9) and (10), is given by

$$N - N_0 = N_1 - (N_0 - N_{00}),$$

$$\begin{aligned} N_0 - N_{00} &= N_{00}'\epsilon_1[1+(\mu/\gamma)(1-F)] \\ &= N_{00}'[\tilde{\epsilon}_1 - (\mu/\gamma)F\epsilon_1], \end{aligned}$$

where

$$N_{00}' \equiv \partial N_{00}/\partial\epsilon_0;$$

or

$$-\tau I[N(\mathbf{p})] = N_1 - N_{00}'[\tilde{\epsilon}_1 - (\mu/\gamma)F\epsilon_1].$$

The solution of the linearized form of the Boltzmann equation, Eq. (8), is now easily found to be

$$N_1(\mathbf{p}) = \left[1 + \frac{i\Omega\tau}{1-i\Omega\tau(1-\mu)}\right] N_{00}'\tilde{\epsilon}_1 - \frac{(\mu/\gamma)F}{1-i\Omega\tau(1-\mu)} N_{00}'\epsilon_1. \tag{21}$$

Substituting this expression and

$$(\partial H/\partial t)^* = i\Omega\tilde{\epsilon}_1^*$$

in Eq. (10) yields

$$\begin{aligned} Q &= \tfrac{1}{2}\,\mathrm{Re}\left\{\sum_{\mathbf{p}} i\Omega\tilde{\epsilon}_1^* N_{00}'\left[\frac{i\Omega\tau\tilde{\epsilon}_1 - i\Omega\tau\epsilon_1(\mu/\gamma)(1-\mu)F}{1-i\Omega\tau(1-\mu)}\right]\right\} \\ &= \tfrac{1}{2}|a|^2\Omega^2 T \sum_{\mathbf{p}} \tau S(\mathbf{p}) \frac{[1+(\mu/\gamma)]^2 - (\mu/\gamma)[1-\mu][1+(\mu/\gamma)][F_1 - F_2\Omega\tau(1-\mu)]}{1+(\Omega\tau)^2(1-\mu)^2}, \end{aligned} \tag{22}$$

[13] T. O. Woodruff and H. Ehrenreich, Phys. Rev. **123**, 1553 (1961).

where $S(p)$, defined by

$$S(\mathbf{p})\equiv -T^{-1}N_{00}'\epsilon_0^2,$$

is the specific heat associated with the mode **q**, so that the total specific heat C_v of the undisturbed liquid is given by

$$C_v=\sum_{\mathbf{p}} S(\mathbf{p}).$$

The integrations over μ involved in the **p** summation are straightforward. With these performed, the expression for Q in the limit as $\Omega t\rightarrow\infty$ becomes

$$Q=\tfrac{1}{4}(\sigma u_s)^2\Omega TC_v[\tfrac{1}{2}\pi(\gamma+1)^2+\tfrac{2}{3}F_2].$$

The attenuation Γ is obtained by dividing Q by $c_0\rho\Omega^2u_s^2/2$:

$$\Gamma=[\tfrac{1}{4}\pi(\gamma+1)^2+\tfrac{1}{3}F_2]\Omega TC_v/\rho c_0^3. \tag{23}$$

From simple calculations based on a model of low-temperature liquid helium as a Debye solid with only longitudinal vibrational modes,[1] as well as from the experimental work of Wiebes, Niels-Hakkenberg, and Kramers,[14] it is known that

$$C_v=0.0205T^3\text{ J g}^{-1}\text{ deg}^{-1}. \tag{24}$$

Equations (21) and (22) yield the desired result, Eq. (1), with

$$A=A_{\text{calc}}=0.205[\tfrac{1}{4}\pi(\gamma+1)^2+\tfrac{1}{3}F_2]\times10^6\text{ dyn cm}^{-2}\text{ deg}^{-4}.$$

Since $\gamma\cong3.0$, as noted after Eq. (3), agreement between A_{exp} and A_{calc} is obtained with $F_2\cong-28$.

IV. DISCUSSION

The main point of the preceding is to show that the experimentally observed dependence of sound attenuation for $T<0.5°$K on the first power of frequency and the fourth power of temperature, as given in Eq. (1), can be understood within the framework of the excitation theory of a Bose liquid if F is independent of temperature and frequency.

A second point of considerable interest is that the magnitude of the constant A can be understood if $|F_2|$ is large, i.e., if the sound wave excites a large relative motion of normal fluid and superfluid at the low temperatures under consideration here. With respect to this relative motion the demonstration is admittedly incomplete in that no effort has been made to determine, much less to solve, a full set of equations of motion for the normal and superfluid components in a sound wave with $\Omega\tau\gg1$. It has not been shown how to set up a system of equations of motion and boundary conditions for this case and how to find a solution of them in which F depends little or not at all on temperature and frequency and differs from unity. The author has merely shown how such a motion, if it existed, would lead to an understanding based on the excitation theory of a Bose liquid and simple principles of kinetics of the magnitude of the low-temperature sound absorption in liquid helium, which is otherwise quite puzzling.

In the absence of a complete set of equations of motion it is difficult to show that the ratio F of the velocities of the normal and superfluid components is relatively insensitive to temperature, in contrast to the ratio of the densities of the two components. Furthermore, the relative motion of the two components in the sound wave might be expected to lead to something like temperature or entropy oscillations accompanying the first sound wave, and without a complete set of equations of motion it has not been possible to demonstrate that the coupling of these oscillations does not appreciably alter the velocity of the sound waves. These are among the most serious difficulties associated with the present effort to base a discussion of sound absorption with $\Omega\tau>1$ on the excitation theory of a Bose liquid and the use of the Boltzmann equation with a relaxation time.

It is interesting to note that when the possibility of relative motion of normal and superfluid components is not considered, F becomes unity, and Eq. (23) of this paper agrees with the formula obtained by Dransfeld[15] from very different considerations.

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[14] J. Wiebes, C. G. Niels-Hakkenberg, and H. C. Kramers, Physica **23**, 625 (1957).

[15] K. Dransfeld, Bull. Am. Phys. Soc. **3**, 225 (1958); and private communications.