# Coulomb Effects and the $O^{14}$ $\beta$-Decay Matrix Element*

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The change of the Fermi matrix element of the $\beta$ decay of $O^{14}$ to the first excited state of $N^{14}$ from its value $\sqrt{2}$ due to Coulomb corrections is reestimated. It is shown, that for pure $1p$-shell states the reduction is between 0.02 and 0.20%. Configuration mixing in the $1p$ shell may increase these figures considerably.

Instead of arbitrary admixtures of higher configurations, a collective model for the presumably most important of such admixtures is considered: The model of a "breathing state" (radial density oscillation). It is shown that for sufficiently low values of the energy of the first excited breathing state compatible with experimental data for the level spectrum of the mass-14 system, the $\beta$-decay matrix element may be reduced by as much as five percent. In view of the lack of present knowledge about the breathing state and nuclear compressibility for light nuclei, it is concluded that the discrepancy between the measured values of the vector and $\mu$-meson decay coupling constants of $\beta$ decay may well be due to such an effect.

## I. INTRODUCTION

THE observed near equality of the $\mu$-meson decay coupling constant $G_\mu$ and the vector coupling constant $G_V$ as measured from the $O^{14}-N^{14*}$ (2.312-Mev) $\beta$ decay forms the basis of the Conserved-Vector-Current theory.[1,2] Recent experiments[3,4] have indicated the validity of this theory. A refined measurement of the lifetime of $O^{14}$ [5,6] and of the transition energy[5] has yielded an improved value for $G_V$, $G_V=1.4145\times10^{-49}$ erg cm$^3$ without radiative corrections,[5] and $G_V=1.4025\times10^{-49}$ erg cm$^3$ if one applies the radiative corrections[5] as calculated by Kinoshita and Sirlin[7] and Berman.[8] (These numbers do not include the Coulomb corrections.) This has to be compared with the $\mu$-meson decay coupling constant[5] $G_\mu=1.4312\times10^{-49}$ erg cm$^3$. The difference between the two constants amounts to $\sim1\%$ without the radiative corrections, and to as much as 2% if one includes the radiative corrections, and is well outside the quoted experimental error. This error is 0.1% for $G_\mu$ (uncorrected), and 0.2% for $G_V$ (uncorrected).

How can one account for this difference if the conserved-vector-current theory holds? MacDonald[9] has considered the effect of the Coulomb interaction on the $\beta$-decay matrix element in $O^{14}$. This matrix element has the value $\sqrt{2}$ under the assumption that isotopic spin is a good quantum number. He obtained an upper limit of 0.18% for the change of the matrix element. Oneda and Pati[10] and Lee and Yang[11] have studied the effect of an intermediate boson. At present, there is some evidence for the existence of such an intermediate boson from the $\rho$ value of the $\mu$-meson decay. A boson of about the same mass as the $K$ meson could explain both the difference in $G_\mu$ and $G_V$ and the $\rho$ value. Blin-Stoyle and Le Tourneux[12] have estimated the effect of the $\pi^\pm-\pi^0$ mass difference on the $O^{14}$ $ft$ value and did obtain a correction for the coupling constant of about 1% which could explain the difference. If one excludes the possibility of a sizeable error in the electromagnetic corrections[7,8] and uses estimates for the contribution of second-forbidden $\beta$ matrix elements which show them to be small,[13] the investigation of Blin-Stoyle and Le Tourneux and the proposal by Lee and Yang and Oneda and Pati seem to offer the only possibilities to explain the difference in the two coupling constants in the framework of a conserved-vector-current theory.[14]

In this paper, we reconsider the estimate of MacDonald[9] of the Coulomb effects. The reasons for this are threefold. MacDonald did not completely antisymmetrize the wave functions with which he calculated the estimate. It is conceivable that proper antisymmetrization and angular-momentum coupling may change his estimate. Furthermore, he calculated the radial integrals of the Coulomb interaction by using harmonic-oscillator wave functions. It has been suggested, however, that a calculation with square-well radial wave functions may give a bigger estimate. (The repulsive Coulomb interaction will succeed better

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[11] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 410 (1960).

[12] R. J. Blin-Stoyle and J. Le Tourneux, Phys. Rev. **123**, 627 (1961).

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[14] While this manuscript was being revised, the author obtained a preprint of a paper by A. Altman and W. M. MacDonald (University of Maryland Technical Report No. 241) in which it is shown that a determination of the parameters of a charge-dependent potential as proposed by Blin-Stoyle and Le Tourneux from experimental data in the mass-14 system is found to give a reduction of the Fermi matrix element of $O^{14}$, which is very much too small to account for the observed discrepancy.

in separating the protons from each other if they are bound in a finite square well than if they are bound in an infinite harmonic oscillator.) Finally, MacDonald did not consider the effect of configuration mixing of the $p$-shell states in his estimate. Such mixing is indicated by recent experiments[15,16] and somewhat corroborated by the results of a subsequent paper.[17] Configuration mixing has a non-negligible effect on the $\beta$-decay matrix element: For pure $1p$-shell states, the Coulomb interaction in $N^{14*}$ and $O^{14}$ is the same, as we are dealing with a proton hole and a neutron hole in one, and with two neutron holes in the other case. The only, and very small, difference in the two wave functions originates from the difference in spin-orbit coupling of the neutron hole and the proton hole, from the difference in magnetic moment interaction, from the difference in binding energies, and from admixtures of other configurations via the Coulomb force. It is because of this fact, the equality of the Coulomb interaction in $N^{14*}$ and $O^{14}$ within the $p$ shell, that the estimated change of the matrix element is so small. This equality does not hold as soon as other configurations are mixed in, since one can then no longer use the concept of a hole state.

In Sec. II, we give an outline of the procedure for the calculation of Coulomb effects for pure $p$-shell states; in Sec. III we describe the results. Section IV deals with the estimate of corrections due to configuration mixing. This estimate will turn out to be very unreliable. In view of the importance of the question and the difficulty to obtain a reliable estimate of the influence of configuration mixing, we shall give an alternate discussion of the Coulomb effect in Sec. V. Section VI summarizes the results of the paper.

## II. CALCULATION OF COULOMB CORRECTIONS FOR PURE $P$-SHELL STATES

### 1. Coulomb Effects and Other Causes for a Change of the Matrix Element within the $1p$ Shell

We make the following assumptions. Without the presence of any Coulomb forces, the ground and first excited states of $N^{14}$ and the ground states of $O^{14}$ and $C^{14}$ can be described as linear combinations of $p$-shell states. The ground state of $N^{14}$ has $J=1^+$, $T=0$; its wave function written in terms of the $L$-$S$ coupled wave functions $^{2S+1}L_J$ is of the form

$$|J=1^+,\ T=0\rangle = C_S\,{}^3S_1 + C_P\,{}^1P_1 + C_D\,{}^3D_1. \quad (1)$$

Correspondingly, the isotopic triplet consisting of the $C^{14}$ ground state, the first excited state of $N^{14}$, and the $O^{14}$ ground state can be described as the $T_z=-1, 0, +1$ components of a state with spin $J=0^+$, $T=1$, respectively, which has the wave function

$$|J=0^+,\ T=1\rangle = C_S'\,{}^1S_0 + C_P'\,{}^3P_0. \quad (2)$$

Approximate values for $C_S$, $C_P$, $C_D$, $C_S'$, $C_P'$ have been given by Ferrell and Visscher[18] (see, however, the discussion in reference 17). If the constants $C_S'$ and $C_P'$ in Eq. (2) are exactly the same for $O^{14}$ and for the first excited state of $N^{14}$ (henceforth denoted by $N^{14*}$), and if the functions $^1S_0$ and $^3P_0$ have exactly the same radial dependence for both nuclei, then the Fermi matrix element for the $O^{14}$ decay has the value $\sqrt{2}$:

$$M_F = \langle (C_S'\,{}^1S_0 + C_P'\,{}^3P_0)_{N^{14*}} | \tfrac{1}{2}(T_x - iT_y) | \times (C_S'\,{}^1S_0 + C_P'\,{}^3P_0)_{O^{14}} \rangle = \sqrt{2}. \quad (3)$$

Here, $T_x$ and $T_y$ are the $x$ and $y$ components of the total isotopic spin operator.

The Coulomb interaction between all the protons,

$$V = \sum_{i<j} \tfrac{1}{4}(1+t_z^{(i)})(1+t_z^{(j)})\frac{e^2}{r_{ij}}, \quad (4)$$

may change the value of the Fermi matrix element in three different ways: it may cause the constants $C_S'$ and $C_P'$ to be different in $O^{14}$ and $N^{14*}$, it may result in a different radial dependence of $^1S_0$ and $^3P_0$ in $O^{14}$ and $N^{14*}$, and it may cause configuration mixing. The state (2) will interact with higher lying non-$p$-shell states via the interaction (4), and the resulting eigenfunction will contain admixtures of these excited states. Since the expression (4) does not commute with $T^2$, the square of the total isotopic spin operator, these admixtures will be different for $O^{14}$ and $N^{14*}$. We shall discuss the change of the matrix element due to this type of configuration mixing in the following subsections of this section and in the next section; at the moment, we confine our attention to the Coulomb interactions within the $1p$ shell.

The difference in the constants $C_S'$ and $C_P'$ in $O^{14}$ and $N^{14*}$ can be estimated in the following way. In their analysis of the mass-14 system, Visscher and Ferrell[18] determined the constants $C_S'$ and $C_P'$ for $C^{14}$ from a nuclear Hamiltonian. They also considered the difference in the $ft$ value for the $\beta$ decays of $O^{14}$ and $C^{14}$ to the ground state of $N^{14}$. It is known that the $\beta$ decay of $O^{14}$ to the ground state of $N^{14}$ is not as highly forbidden as the $C^{14}$ decay. Whereas in $C^{14}$ the experimental value of the square of the Gamow-Teller matrix element is roughly $|\int\sigma|^2_{C^{14}} = 4.1\times10^{-6}$, we have for $O^{14}$ the experimental value $|\int\sigma|^2_{O^{14}} = 2.4\times10^{-4}$.[18] This difference is directly related to the difference of $C_S'$ and $C_P'$ in $O^{14}$ and $C^{14}$. The matrix element of the $\beta$ decay, $C_S C_S' - (1/\sqrt{3})C_P C_P'$, changes by roughly 0.005 when going from $O^{14}$ to $C^{14}$. From this change, Visscher and Ferrell could deduce the constants $C_S'$ and $C_P'$ for $O^{14}$. We overestimate the change of overlap in the matrix element (3) by equating $C_S'$ and $C_P'$ for $N^{14*}$ with $C_S'$ and $C_P'$ as determined by Visscher and Ferrell for $C^{14}$, and by using their values for $O^{14}$. This yields a reduction of (3) by 0.02%.

[15] W. E. Moore, J. N. McGruer, and A. I. Hamburger, Phys. Rev. Letters **1**, 29 (1958).
[16] E. Baranger and S. Meshkov, Phys. Rev. Letters **1**, 30 (1958).
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This number—0.02%—is probably an overestimate of the change of the matrix element for the following reason. For pure $p$-shell states, the interaction (4) has vanishing nondiagonal matrix elements for $N^{14*}$ and $O^{14}$. This means that the other $J=0^+$ $p$-shell state [the linear combination of $^1S_0$ and $^3P_0$ which is orthogonal to (2)] does not mix with the state (2), independently of the values of the parameters $C_S'$ and $C_P'$. In an unprecise manner of speaking, one may say that the two neutron holes in $O^{14}$ and the neutron hole and the proton hole in $N^{14*}$ have no Coulomb interactions. The nondiagonal matrix elements for $C^{14}$, however, do not vanish, since in $C^{14}$ the proton holes interact. Therefore, we expect the wave function of $C^{14}$ to differ from the wave functions for $O^{14}$ and $N^{14}$, which we expect to be equal. If the difference of $C_S'$ and $C_P'$ in $O^{14}$ and $C^{14}$ were due to Coulomb forces only, we would expect no change in the Fermi matrix element from its value (3).

Visscher and Ferrell have pointed out, however, that the Coulomb interaction in $C^{14}$ can account only for one-third of the observed difference in the $\beta$-decay matrix elements of $O^{14}$ and $C^{14}$ to the ground state of $N^{14}$. The other two-thirds must be ascribed to the difference in spin-orbit coupling of a neutron hole and a proton hole, or other causes. It is only this part of the difference of the constants $C_S'$ and $C_P'$ for $O^{14}$ and $C^{14}$ that we have to consider as we will expect a corresponding difference in the wave functions for $N^{14*}$ and $O^{14}$. The above estimate, 0.02%, will therefore have to be multiplied by $(\frac{2}{3})^2$, taking out the part due to Coulomb interaction, and by $(\frac{1}{2})^2$, taking into account that the difference in wave functions between $N^{14*}$ and $O^{14}$ will be only half the difference between $C^{14}$ and $O^{14}$. This yields $\frac{1}{8}\times 0.02\%$, a negligible number. MacDonald[9] obtained instead 0.13%. This figure contains a numerical error.[19]

Having ruled out the difference of the constants $C_S'$ and $C_P'$ for $O^{14}$ and $N^{14*}$ as a possible cause for a sizeable change of the Fermi matrix element (3), we shall now consider the radial dependence of the functions $^1S_0$ and $^3P_0$ in $O^{14}$ and $N^{14*}$. The last neutron in $N^{14*}$ has a binding energy of 8.283 MeV, the last proton in $O^{14}$ has a binding energy of 5.403 MeV.[20] (In these numbers, we have already included the neutron-proton mass difference.) Therefore, the radial wave functions of these two particles will be asymptotically different, and their radial overlap integral will be different from 1. We have tried to estimate this effect. We have constructed several square-well potentials of different radii which give a binding energy of 8.283 MeV for a $1p$-shell neutron. As the characteristic parameter of a square well potential is $VR^2$, well depth times square of the radius, each of these potentials actually includes a whole set of binding energies for a $1p$-shell neutron.

[19] W. M. MacDonald (private communication).

[20] F. Ajzenberg-Selove and T. Lauritsen, in *New Series*, edited by Landolt-Börnstein (Springer-Verlag, Berlin, 1961), Group 1, Vol. 1.

TABLE I. Reduction of radial overlap because of different binding energy ($V_0$=well depth, $R$=radius of the well, $I$=radial overlap integral).

| $R$ (in $10^{-13}$ cm) | $V_0$ (in Mev) | $I$ |
|---|---|---|
| 2.7 | 41.59 | 0.99819 |
| 3.5 | 29.14 | 0.99837 |
| 5.0 | 19.43 | 0.99980 |

In order to describe the different binding of a $1p$-shell proton, we used the wave function of the neutron inside the well and a wave function with the correct binding energy (5.403 MeV) outside the well. This wave function does not have a continuous derivative at $r=R$, of course. We normalized the wave functions and calculated the overlap integral.

The results of such a calculation are shown in Table I. We give the depth of the well, the radius, and the radial overlap integral. We see that for increasing radius, the overlap integral approaches one, as it should from the construction. A good value of the nuclear radius is 2.7 F (see the discussion in Sec. III); here we obtain a reduction in overlap of 0.18%. This may overestimate the effect however, since we have assumed that the proton wave functions assumes its asymptotic value already at $r=R$. Therefore, the use of a bigger radius seems more "realistic," and we conclude that the reduction in overlap is probably less than 0.10%.

After the discussion of the Coulomb effects within the $1p$ shell, we turn now our attention to the Coulomb interaction between the state (2) and non-$p$-shell states.

## 2. Basic Formulas

Besides mixing the two $J=0^+$ $p$-shell states, the interaction (4) introduces admixtures to (2) from higher-lying $J=0^+$ states. $V$ can be written as the sum of a scalar, the $z$ component of a vector, and the $zz$ component of a second rank tensor in isotopic spin space,

$$V=V_0+V_1+V_2, \tag{5a}$$

where

$$V_0=\sum_{i<j}\tfrac{1}{4}[1+\tfrac{1}{3}(\mathbf{t}^{(i)}\cdot\mathbf{t}^{(j)})]\frac{e^2}{r_{ij}},$$

$$V_1=\sum_{i<j}\tfrac{1}{4}(t_z^{(i)}+t_z^{(j)})\frac{e^2}{r_{ij}}, \tag{5b}$$

$$V_2=\sum_{i<j}\tfrac{1}{4}[t_z^{(i)}t_z^{(j)}-\tfrac{1}{3}(\mathbf{t}^{(i)}\cdot\mathbf{t}^{(j)})]\frac{e^2}{r_{ij}}.$$

Therefore, the admixtures will have isotopic spin 0, 1, 2, and 3. As $V_0$ is a scalar, we may add it to the nuclear potential and discard it for the purposes of the following discussion.

We calculate the influence of the Coulomb interaction with non-$p$-shell states in perturbation theory. This is

presumably good since non-$p$-shell states have higher excitation energies than the other $J=0^+$ $p$-shell state (probably to be identified with the observed $J=0^+$, $T=1$ state in $N^{14}$ at 8.62 MeV). We denote the wave function of the $O^{14}$ ground state by $\psi_i$, the wave function of the $N^{14}$ first excited state by $\psi_f$. Perturbation theory then yields

$$\psi_i=N_i^{-1}\Big(\psi_i^0+\sum_{n\neq i}\frac{\langle\psi_n(T_z=+1)\,|\,V_1+V_2\,|\psi_i^0\rangle}{E_i-E_{in}}\times\psi_n(T_z=+1)\Big), \quad (6)$$

$$\psi_f=N_f^{-1}\Big(\psi_f^0+\sum_{n\neq f}\frac{\langle\psi_n(T_z=0)\,|\,V_1+V_2\,|\psi_f^0\rangle}{E_f-E_{fn}}\psi_n(T_z=0)\Big),$$

where $N_i$ and $N_f$ are normalization factors, $E_{in}$ and $E_{fn}$ the energies of the states with wave functions $\psi_n(T_z=+1)$ and $\psi_n(T_z=0)$, and $\psi_i^0$ and $\psi_f^0$ are the $T_z=+1$ and $T_z=0$ components of the wave function (2). Using the Wigner-Eckart theorem, we get

$$\psi_i=N_i^{-1}[\psi_i^0+\sum_{n\neq i}\sum_{T'=1}^{2}C(1T'T(n);+1,0)\times\alpha_i(T',n)\psi_n(T_z=+1)], \quad (7)$$

$$\psi_f=N_f^{-1}[\psi_f^0+\sum_{n\neq f}\sum_{T'=1}^{2}C(1T'T(n);0,0)\times\alpha_f(T',n)\psi_n(T_z=0)],$$

where

$$\alpha_{i,f}(T',n)=\frac{\langle\psi_n\|V_{T'}\|\psi_{i,f}\rangle}{E_{i,f}-E_{i,fn}}. \quad (8)$$

The summation over $T'$ extends over the two terms in the potential, $V_1$ and $V_2$. Under the assumption $E_i-E_{in}=E_f-E_{fn}$, which seems justified except for higher order Coulomb effects, we get $\alpha_i(T',n)=\alpha_f(T',n)=\alpha(T',n)$. The values for the normalization factors are

$$N_f^2=1+\sum_{T'T''n}C(1T'T(n);0,0)C(1T''T(n);0,0)\times\alpha(T',n)\alpha^*(T'',n), \quad (9)$$

and correspondingly for $N_i^2$. With (7) and (9), the Fermi matrix element changes by (we keep only terms of second order in $\alpha$):

$$\tfrac{1}{2}\delta=1-\frac{1}{\sqrt{2}}(\psi_f\,|\,T^-\,|\psi_i)=\sum_{T'T''n}\alpha(T',n)\alpha^*(T'',n)\beta(T',T'';T(n)). \quad (10)$$

Here, the parameters $\beta$ are of geometrical origin and defined by

$$\beta(T',T'',T(n))=\tfrac{1}{2}C(1T'T(n);0,0)C(1T''T(n);0,0)+\tfrac{1}{2}C(1T'T(n);+1,0)C(1T''T(n);+1,0)-\frac{1}{2\sqrt{2}}\{T(n)[T(n)+1]\}^{\frac{1}{2}}[C(1T'T(n);0,0)\times C(1T''T(n);+1,0)+C(1T'T(n);+1,0)\times C(1T''T(n);0,0)]. \quad (10a)$$

The problem consists in calculating the quantities $\alpha(T',n)$. The definition of $\delta$ agrees with the definition given by MacDonald[9]: positive $\delta$ gives the amount of reduction of the square of the matrix element.

### 3. Calculation of the Quantities $\alpha(T,n)$

As we did not succeed in reducing the expression (10) by means of closure, we tried to estimate $\delta$ by calculating it within the framework of the shell model. The reduced matrix elements $\langle\psi_n\|V_{T''}\|\psi_{i,f}\rangle$ occurring in (8) depend on the form of $\psi_{i,f}$. According to (2), these integrals consist of two terms, multiplied by $C_S'$ and $C_P'$, respectively. For the purposes of the present estimate, we assumed $C_S'=C_P'=1/\sqrt{2}$. This is in rough agreement with the results of Visscher and Ferrell.[18] Among the excited states $\psi_n$ that enter into the expression $\alpha(T',n)$ (8), those corresponding to two-quantum excitations from the ground-state configuration certainly have the smallest energy denominator, and the largest overlap. They were the only configurations taken into account in the calculation. The reduced matrix elements also depend on the form of the $\psi_n$, i.e., on the coupling scheme used for the description of $\psi_n$. We constructed all wave functions $\psi_n$ with spin $J=0$ which resulted from the following two quantum excitations:

$$\begin{array}{llll} 1p\to 2p, & (1p)^2\to(1d)^2, & 1s\to 1d, & \\ 1p\to 1f, & (1p)^2\to(2s,1d), & (1s)^2\to(1p)^2, & \\ (1p)^2\to(2s)^2, & 1s\to 2s. & & \end{array} \quad (11)$$

These wave functions were subject to the condition of complete antisymmetry and proper angular momentum coupling. In most cases, there are many wave functions belonging to one of the excitation modes in (11) and differing in the way the angular momenta are coupled. We obtained an upper limit for the sum (10) in the following way: We took only the term in the summation over $n$, the $\psi_n$ of which gave the largest overlap integral. (The other overlap integrals are normally smaller by a factor of three at least.) The energy denominator was chosen to be $E_n-E_i=10$ MeV. From the $N^{14}$ spectrum,[20] one infers that the lowest non-$p$-shell level with spin $0^+$ has an excitation energy of at least 11.6 MeV. This corresponds to $E_n-E_i=9.3$ MeV justifying the assumed value of 10 MeV. We conclude that the

TABLE II. Values of the radial integrals (in $F^{-1}$) for the Coulomb interaction for various excitation modes and for different choices of radial wave functions as described in the text. ($V$ in MeV, $R$ in F.)

| | | H.O. | $VR^2=933$ $R=4.829$ | $VR^2=835$ $R=4.569$ | $VR^2=835$ $R=2.892$ | $VR^2=670$ $R=4.093$ | $VR^2=670$ $R=2.892$ |
|---|---|---|---|---|---|---|---|
| 1. | $\langle 2p\ 1p\|L=0\|1p\ 1p\rangle$ | 0.0575 | 0.0350 | −0.0033 | 0.0533 | ··· | ··· |
| 2. | $\langle 2p\ 1p\|L=2\|1p\ 1p\rangle$ | 0.0319 | 0.0124 | −0.0122 | 0.0220 | ··· | ··· |
| 3. | $\langle 2p\ 1s\|L=0\|1p\ 1s\rangle$ | 0.0894 | 0.0515 | 0.0145 | 0.0771 | ··· | ··· |
| 4. | $\langle 2p\ 1s\|L=1\|1s\ 1p\rangle$ | 0.1150 | 0.0678 | 0.0555 | 0.1009 | ··· | ··· |
| 5. | $\langle 2s\ 2s\|L=1\|1p\ 1p\rangle$ | 0.0707 | 0.0483 | 0.0585 | 0.0792 | 0.0269 | 0.0381 |
| 6. | $\langle 2s\ 1d\|L=1\|1p\ 1p\rangle$ | −0.1313 | −0.0547 | −0.0771 | −0.0895 | −0.0290 | −0.0411 |
| 7. | $\langle 1d\ 1d\|L=1\|1p\ 1p\rangle$ | 0.2222 | 0.1887 | 0.1960 | 0.3096 | 0.2100 | 0.2973 |
| 8. | $\langle 1d\ 1d\|L=3\|1p\ 1p\rangle$ | 0.1414 | 0.1256 | 0.1308 | 0.2061 | 0.1391 | 0.1969 |
| 9. | $\langle 2s\ 1p\|L=0\|1s\ 1p\rangle$ | 0.0495 | 0.0338 | 0.0275 | 0.0561 | 0.0476 | 0.0674 |
| 10. | $\langle 2s\ 1p\|L=1\|1p\ 1s\rangle$ | −0.0495 | −0.0362 | −0.0560 | −0.0589 | −0.0127 | −0.0180 |
| 11. | $\langle 2s\ 1s\|L=0\|1s\ 1s\rangle$ | 0.0990 | 0.0593 | 0.0622 | 0.0982 | 0.0703 | 0.0995 |
| 12. | $\langle 1d\ 1p\|L=2\|1s\ 1p\rangle$ | 0.1564 | 0.1403 | 0.1396 | 0.2307 | 0.1568 | 0.2219 |
| 13. | $\langle 1d\ 1p\|L=1\|1p\ 1s\rangle$ | 0.2190 | 0.1847 | 0.1812 | 0.3038 | 0.2072 | 0.2933 |
| 14. | $\langle 1p\ 1p\|L=1\|1s\ 1s\rangle$ | 0.2424 | 0.1899 | 0.1758 | 0.3138 | 0.2157 | 0.3053 |
| 15. | $\langle 1p\ 1p\|L=0\|1p\ 1p\rangle$ | 0.3636 | 0.2692 | 0.2604 | 0.4446 | 0.3060 | 0.4331 |
| 16. | $\langle 1f\ 1p\|L=2\|1p\ 1p\rangle$ | 0.1543 | ··· | ··· | ··· | ··· | ··· |
| | $\langle r^2\rangle_{1p^{\frac{1}{2}}}$ | 2.63 | 3.80 | 3.93 | 2.49 | 3.68 | 2.60 |

estimate for $\delta$ obtained in this manner gives definitely an upper limit on $\delta$, the actual value of which will presumably be much smaller.

## III. RESULTS OF THE CALCULATION

### 1. Calculation with Harmonic Oscillator Wave Functions

Of all the excitations listed in (11), by far the largest contribution comes from the excitation $1p \rightarrow 2p$. For the harmonic oscillator constant $\omega$ we have used the value $(\omega/2\pi)^{\frac{1}{2}}=0.24236\times10^{13}$ cm$^{-1}$. This is the value used by MacDonald.[9] Visscher and Ferrell[18] used instead the value $0.23746\times10^{13}$ cm$^{-1}$. The difference in the result is unimportant. These figures are essentially in agreement with Coulomb energy differences and the analysis of electron-scattering data on neighboring nuclei. Meyer-Berkhout *et al.*[21] obtain a value for the root-mean-square radius which is slightly dependent on the method of analysis of the electron scattering data; it lies between 2.40 and $2.57\times10^{-26}$ cm$^2$. This is to be compared with the value corresponding to our choice for $(\omega/2\pi)^{\frac{1}{2}}$, $2.63\times10^{-26}$ cm$^2$. The difference is unimportant for the purpose of our discussion. With this value for $(\omega/2\pi)^{\frac{1}{2}}$ and the procedure described in Sec. II, we obtain

$$|\delta/2|\leq 0.05\%. \tag{12}$$

This contribution comes only from the $1p \rightarrow 2p$ excitation (all other contributions are significantly smaller), and for this excitation, $\delta$ turns out to be positive. Thus, we obtain the right sign, but an effect the upper limit of which is an order of magnitude too small. The estimate (12) agrees very well with MacDonald's result[9] which is $|\delta|\leq 0.09\%$. Actually, the upper limit given in (12) is more conservative than MacDonald's as we have used a smaller energy denominator.

In the estimate (12), we have used a particular $T=1$ level of the $1p \rightarrow 2p$ excitation which gives the largest contribution to $\delta/2$. There is a $T=2$ level of the same configuration, giving a *negative* value for $\delta$ of about one-half the above estimate. The possibility cannot be excluded that this $T=2$ level is lower in energy than the previously used $T=1$ level. In view of the smallness of the estimate (12) and since we do not consider it likely that the $T=2$ level will be below the $T=1$ level, we will not consider this possibility any further.

### 2. Calculation with Finite Square-Well Radial Wave Functions

It is interesting to compare the results of a calculation utilizing harmonic-oscillator wave functions with those using finite square-well wave functions. The difficulty with a square well is, of course, that with a reasonable value for the binding energy of the last $1p$ particle, the $2p$ state (or even the $2s$ and $1d$ states) will not be bound. We circumvented this difficulty by adopting values of the range and depth of the well in such a manner that the $2p$ state was bound. For comparison with a more reasonable choice of the potential, we calculated some of the integrals also for a more "realistic" nuclear potential of square-well character which gave a reasonable binding energy for the last bound nucleon. The calculations were done numerically; the results are shown in Table II.

In the first column, we define the radial integral in the form $\langle nln'l'|L=L_0|n''l''n'''l'''\rangle$. Here, $n$ and $l$ are the quantum numbers of the normalized bound-state

[21] U. Meyer-Berkhout, K. W. Ford, and A. E. S. Green, Ann. Phys. 8, 285 (1955).

radial wave function in question, and $L=L_0$ specifies the value of $L_0$ in the expression $r_<^{L_0}/r_>^{L_0+1}$ which occurs in the expansion of the Coulomb interaction in Legendre polynomials,

$$\frac{1}{r_{12}}=\frac{1}{(r_1^2+r_2^2-2r_1r_2\cos\theta_{12})^{\frac{1}{2}}}=\sum_{L=0}^{\infty}P_L(\cos\theta_{12})\frac{r_<^L}{r_>^{L+1}}, \quad (13)$$

where $r_<=r_1$, $r_>=r_2$ for $r_1<r_2$, and $r_<=r_2$, $r_>=r_1$ for $r_1>r_2$. In the second column, we give the values for these integrals for the harmonic oscillator potential in units of $10^{+13}$ cm$^{-1}$ using $(\omega/2\pi)^{\frac{1}{2}}=0.24236\times10^{13}$ cm$^{-1}$. In the following five columns, we give the values again in units of $10^{13}$ cm$^{-1}$ for five different choices of square-well potentials, indicated in the two top lines by the values of $VR^2$ in units MeV$\times10^{-26}$ cm$^2$, where $R$ = radius of the well, $V$ = well depth, and by the value of $R$ in $10^{-13}$ cm. The blanks correspond to integrals between states, one of which is unbound for the particular choice of parameters. We have not attempted to calculate the contribution of the $1p\rightarrow1f$ excitation, as here the reduction of the matrix element due to the angular part of the overlap integral is so large that the possible change in the radial integral is insignificant. In the last row, we give for comparison the root-mean-square radius for the $1p$-shell for the various potentials.

Two facts become obvious from Table II: the square-well radial integrals for values of $VR^2$ and $R$ for which the $2p$ state is bound do not tend to be significantly larger than the values calculated with harmonic-oscillator wave functions, even if one multiplies them with the ratio of the root-mean-square radius for the $1p$ shell divided by the same value for the harmonic oscillator to correct for the "wrong" radius. A comparison of the different values of the radial integrals corresponding to different choices of the square well shows that the dependence on the potential is slight except in the case of functions which have a node, where the integral may become very small. We conclude that our previous estimate for $\delta$, $|\delta|\leq0.10\%$, need not be altered, and that the admixture of excited states through the Coulomb interaction cannot be made responsible for the observed difference in the coupling constants $G_V$ and $G_\mu$.

The results of this perturbation-theoretic estimate agree very nicely with the results of Sec. II. The calculation using the argument of the reduced overlap of the wave functions for proton and neutron should give the same answer as perturbation theory for excitations of the $1p\rightarrow2p$ type. Since these are the only excitations actually used in the estimate for $\delta$ in Sec. III, we expect agreement of the two answers. (This was pointed out to the author by Dr. MacDonald.)

As to the contribution of all the other excitation modes listed in (11), inspection of the calculated matrix elements shows that their contribution is very much smaller than 0.05%, in spite of the very large number of states that can be constructed from them.

## IV. ESTIMATE OF THE EFFECTS OF CONFIGURATION MIXING ON THE MATRIX ELEMENT

It is clear that the aforementioned estimate of $\delta$ will be significantly altered by configuration mixing in the $1p$-shell. However, without any knowledge of the amount and type of admixture and the matrix elements involved, it seems to be quite impossible to give a reasonable estimate for the effect of this admixture. We therefore want to give a semiempirical discussion of this question.

As was mentioned in Sec. II.1, the matrix elements for the $\beta$ decay of $O^{14}$ and $C^{14}$ to the ground state of $N^{14}$ differ by 0.005. One may take this value of 0.5% as a measure of the effect of Coulomb distortion on the matrix element *also* for the $O^{14}\rightarrow N^{14*}$ $\beta$ decay in the presence of configuration mixing. If such a mixing were important, then the above number of 0.5% would partly represent the influence of this mixing. According to aforementioned considerations by Ferrell and Visscher, only one-third of the above number can be accounted for by Coulomb interactions within the $1p$-shell. One-half of the remaining 0.34%, that is,

$$\delta/2=0.17\%, \quad (14)$$

would be the estimate for the change of the $O^{14}\rightarrow N^{14*}$ $\beta$-decay matrix element obtained this way.

This estimate is very crude in many ways. Firstly, we have no absolute scale on which to correlate corrections to Fermi and to Gamow-Teller matrix elements. Furthermore, we do not know whether the Coulomb correction influences the Gamow-Teller matrix element in the same way as the Fermi matrix element. We expect that the Gamow-Teller matrix element is influenced more strongly as it depends—for pure $p$-shell states—on the difference $C_SC_S'-3^{-\frac{1}{2}}C_PC_P'$, whereas the Fermi matrix element depends on $C_S'(O^{14})C_S'(N^{14*})+C_P'(O^{14})C_P'(N^{14*})$. Because of normalization, the latter quantity is less subject to changes than the former. However, this very argument becomes invalidated by the presence of sizeable admixtures of non-$1p$-shell configurations.

In view of the difficulty of estimating the amount and importance of configuration mixing, we turn to the discussion of a collective description of the effects of the Coulomb interaction.

## V. THE EFFECT OF THE NUCLEAR COMPRESSIBILITY

From the fact that the Coulomb interaction repels the protons from each other, it is clear that $O^{14}$ will tend to be slightly bigger than $N^{14*}$. This will result in a reduction of the overlap integral. We can estimate the change in radius and the reduction of the $\beta$-decay matrix element as follows: The Coulomb energy is given by

$$E_c=\alpha Z(Z-1)e^2/R, \quad (15)$$

where $R$ is the nuclear radius. If $R$ changes by a small amount, $\Delta R$, $E_c$ changes by

$$\Delta E_c = -\alpha Z(Z-1)(e^2/R)(\Delta R/R) + \alpha Z(Z-1)e^2/R(\Delta R/R)^2 + \cdots. \quad (16)$$

Under volume changes, the volume energy changes by

$$\Delta E_v = \tfrac{1}{2}K(\Delta R/R)^2 + \cdots, \quad (17)$$

where $K$ is the nuclear compressibility. The equilibrium radius is given by $R_{eq} = R[1+(\Delta R/R)_{eq}]$, where

$$\left(\frac{\Delta R}{R}\right)_{eq} = \frac{\alpha Z(Z-1)e^2/R}{2\alpha Z(Z-1)e^2/R + K}. \quad (18)$$

The value of $\alpha Z(Z-1)e^2/R$ can be obtained directly from the $O^{14}-N^{14}$ (2.312-MeV) mass difference, and for $K$ we use $K=700$ MeV (see the discussion below). This yields

$$\left(\frac{\Delta R}{R}\right)_{eq,O^{14}} - \left(\frac{\Delta R}{R}\right)_{eq,N^{14}} = 0.38\%. \quad (19)$$

Assuming a constant density distribution within each nucleus, and a sharp cutoff for the density at the nuclear surface, this change in radius results in a reduction of the overlap integral (and therefore the $\beta$-decay matrix element) by $\delta/2 = 0.57\%$, which has the right order of magnitude. This estimate depends mainly upon two assumptions which we shall discuss presently: The value of $K$ that we used, and the assumed simple form of the density distribution.

The value of $K$ used in the estimate given above, $K=700$ MeV, corresponds to the value obtained from an analysis of isotope shifts in heavy elements.[22] (Because of the large mass effect in light nuclei and their smaller volume, the compressibility for light nuclei cannot be obtained directly from such an analysis.) The experimental errors in the isotope shifts are comparable with values of $K$ as large as 1400 MeV.[22] The study of nuclear matter[23] yields a value of $K$, $K \cong 2500$ MeV, which is, however, incompatible with the analysis of isotope shift results.[22] These numbers refer to either very heavy nuclei or infinite nuclear matter. Very little is known about the compressibility of the light nuclei. In order to have a better understanding of the orders of magnitude involved, we prefer a discussion in terms of the excitation energy of a "breathing mode" (radial volume oscillation) rather than the compressibility.

We assume that the "breathing mode" can be described as a radial oscillation of the homogeneous density around its equilibrium value. If we restrict ourselves to quadratic terms (harmonic oscillator approximation), we obtain the following relationship between the compressibility $K$ and the excitation energy $E_B$ of the first excited state of the radial volume oscillation (breathing mode):

$$(3mAR^2/5\hbar^2)E_B^2 = K, \quad (20)$$

where $m$ is the nucleon mass. For $K=700$ MeV and $R=2.6$ F, this yields $E_B \cong 22$ MeV. In general, we would expect the breathing mode to have an energy $E_B$ of about twice the distance of levels in the shell model, i.e., 20 MeV $\leq E_B \leq$ 30 MeV, corresponding to $500 \leq K \leq 1300$ MeV. There is, however, some evidence that such a breathing mode may occur at lower energies for the region of the periodic table considered here: Ferrell and Visscher[24] suggested that the 6.06-MeV state in $O^{16}$ can be interpreted as a breathing mode, and a similar argument applies to the 7.6-MeV state[25] in $C^{12}$. Griffin[26] has criticized these considerations, and there seems to be general agreement that these two states are not pure breathing states, but contain a sizeable admixture of the breathing state wave function, together with other two-quantum excitations. A situation similar to the one encountered in the discussion of these two states exists in $N^{14}$: It is not possible to explain the small distance in energy between the two lowest $J=0^+$, $T=1$ states with the shell model.[17]

A low-lying breathing mode implies a reduced stiffness of the system against density oscillations. Since our estimate in Secs. II and III amounts practically to the estimate of this stiffness with the $p$-shell model, the above considerations must be interpreted in shell-model language as follows: For a low-lying breathing mode, the lowest $0^+$, $T=1$ state has a certain amount of the breathing-mode wave function mixed in (configuration mixing) which reduces its stiffness against volume deformations. This links the present arguments to those of the last section. In this connection it is

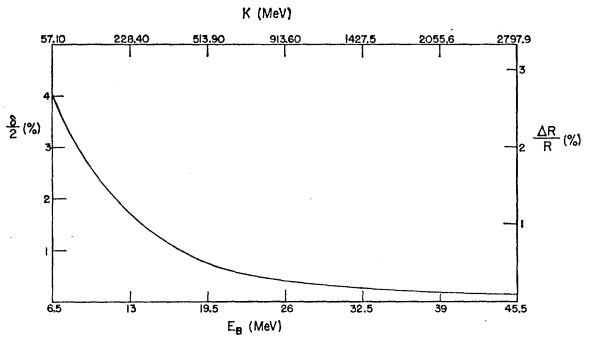


FIG. 1. The relative change of radius, $\Delta R/R = (\Delta R/R)_{Eq,O^{14}} - (\Delta R/R)_{Eq,N^{14}}$ as defined in Eqs. (18) and (19), and the resulting reduction of the Fermi matrix element, $\delta/2$, as defined in Eq. (10), both in percent, as functions of the nuclear compressibility $K$ and the energy of the breathing mode $E_B$.

[22] A. R. Bodmer, Compt. rend. Congr. intern. phys. nucléaire, Paris, 1958, 718 (1959) and references quoted therein.
[23] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).
[24] R. A. Ferrell and W. M. Visscher, Phys. Rev. **102**, 450 (1956).
[25] R. A. Ferrell and W. M. Visscher, Phys. Rev. **104**, 475 (1956).
[26] J. J. Griffin, Phys. Rev. **108**, 328 (1957).

gratifying to see that for an excitation energy $E_B=30$ MeV, the change of overlap ($\equiv\delta/2$) is reduced to 0.30% in rough agreement with the estimates of Secs. II and III. If, on the other hand, $E_B=6$ MeV (this means that we identify the second $0^+$, $T=1$ state with the breathing state), we obtain $\delta/2\cong 5\%$!

Since we do not know at what excitation energy the breathing mode occurs in the mass-14 system, we conclude this discussion by giving in Fig. 1 the relationship between the excitation energy of the breathing mode $E_B$ (measured, of course, from the lowest $0^+$, $T=1$ state), the compressibility $K$, the relative change in radius between $O^{14}$ and $N^{14*}$ in percent, and the change in overlap (and reduction in the Fermi-matrix element $\delta/2$) in percent estimated with the simple model discussed above.