

Chamber's Solution of the Boltzmann Equation*

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It is shown that Chamber's kinetic integral is an exact solution of the Boltzmann equation in the relaxation time approximation.

STARTING from a kinetic argument, Chambers¹ has suggested that the following is a solution of the time-independent Boltzmann equation:

$$f = \int_{-\infty}^t f_0(E - \Delta E(t')) \exp\left(-\int_{t'}^t \frac{ds}{\tau(\mathbf{p}(s))}\right) \frac{dt'}{\tau(\mathbf{p}(t'))}, \quad (1)$$

where

$$\Delta E = \int_{t'}^t \mathbf{F} \cdot \mathbf{V}(t'') dt''$$

is the energy gain of an electron between time t' and t in the absence of collisions, τ is the relaxation time, and $f_0 = \exp(-E/kT)$ is the equilibrium distribution. Expanding the integrand to first order in E and integrating by parts gives

$$f = f_0 - \frac{df_0}{dE} \int_{-\infty}^t \mathbf{F} \cdot \mathbf{V}(t') \exp\left(-\int_{t'}^t \frac{ds}{\tau(\mathbf{p}(s))}\right) dt'. \quad (2)$$

Equation (2) has been derived from the linearized Boltzmann equation by Suzuki,² while Heine³ has argued that this is a valid solution to all orders in applied fields.

It will be shown that (1) is an exact solution of the Boltzmann equation and that (2) solves only the linearized equation.⁴

Let us consider the simple case of a spherical band, $\tau = \tau(p)$ and a constant force field $\mathbf{F} = F\mathbf{I}_z$. In this case:

$$\mathbf{V}(t') = [\mathbf{p} + \mathbf{F}(t' - t)]/m. \quad (3)$$

Integrating (1) by parts, using (3) and setting $t' - t = y$,

$$f = f_0 \left[1 + \int_0^\infty \exp\left(-\frac{\mathbf{F} \cdot \mathbf{p}y - F^2 y^2/2}{mkT}\right) \times \exp\left(-\int_0^y \frac{ds}{\tau(\mathbf{p} - \mathbf{F}s)}\right) \left(\frac{\mathbf{F}}{m} \cdot \left(\frac{\mathbf{p} - \mathbf{F}y}{kT}\right)\right) dy \right]. \quad (4)$$

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¹ R. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).

² H. Suzuki, J. Phys. Soc. Japan **16**, 2347 (1961).

³ V. Heine, Phys. Rev. **107**, 431 (1957).

⁴ In disagreement with Heine.

The Boltzmann equation in the relaxation time approximation is simply

$$\mathbf{F} \cdot \nabla_p f = -(f - f_0)/\tau. \quad (5)$$

Putting (4) into (5) and noticing that

$$\mathbf{F} \cdot \nabla_p \frac{1}{\tau(\mathbf{p} - \mathbf{F}s)} = -\frac{d}{ds} \left(-\frac{1}{\tau(\mathbf{p} - \mathbf{F}s)} \right),$$

we have

$$-\frac{\mathbf{F} \cdot \mathbf{p}}{mkT} + \int_0^\infty \exp\left(\frac{\mathbf{F} \cdot \mathbf{p}y - F^2 y^2/2}{mkT}\right) \times \exp\left(-\int_0^y \frac{ds}{\tau(\mathbf{p} - \mathbf{F}s)}\right) \left[-\frac{\mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)\mathbf{F} \cdot \mathbf{p}}{(mkT)^2} + \frac{F^2}{mkT} + \frac{F^2 y \mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)}{(mkT)^2} + \frac{\mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)}{mkT} \frac{1}{\tau(\mathbf{p} - \mathbf{F}y)} \right] dy = 0. \quad (6)$$

Integrating the last term of (6) by parts, one finds an exact cancellation of all terms; thus, (1) is an exact solution of (5).

We can do a similar thing with (2). In this case, one finds in place of (6)

$$-\frac{\mathbf{F} \cdot \mathbf{p}}{mkT} + \int_0^\infty \exp\left(-\int_0^y \frac{ds}{\tau(\mathbf{p} - \mathbf{F}s)}\right) \left[-\frac{\mathbf{F} \cdot \mathbf{p}\mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)}{(mkT)^2} + \frac{F^2}{mkT} + \frac{\mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)}{\tau(\mathbf{p} - \mathbf{F}y)mkT} \right] dy = 0. \quad (7)$$

Integrating the last term by parts, one finds that all terms cancel except

$$\frac{\mathbf{F} \cdot \mathbf{p}}{mkT} \int_0^\infty \exp\left(-\int_0^y \frac{ds}{\tau(\mathbf{p} - \mathbf{F}s)}\right) \frac{\mathbf{F} \cdot (\mathbf{p} - \mathbf{F}y)}{mkT} dy, \quad (8)$$

which is at least a second-order term.

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