Electromagnetic Properties of a Plasma-Beam System

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There are two electromagnetic modes in a plasma-beam system: the unstable "hybrid" mode characterized by the occurrence of a longitudinal component of the electric intensity **E**, and the transverse mode which does not show an instability and is characterized by the occurrence of a longitudinal component of the magnetic induction **B**. The behavior of this system is described in terms of macroscopic quantities such as the electric displacement **D** and the magnetic intensity **H**. The unstable hybrid mode in a plasma-beam medium is compared to related hybrid instabilities produced by a beam in a medium comprising harmonic oscillators. It is thus shown that the hybrid instability produced in a plasma is of type "*l*," as defined in Part I. Applying the appropriate criterion, it is found that the hybrid instability and the associated longitudinal instability are convective for $\theta \neq 0$, where θ is the angle formed by the direction of the beam and the direction of the growing waves resulting from these instabilities. For θ approaching $\pi/2$ the instability produced by the beam becomes aperiodic. The plasma-beam system is electromagnetically anisotropic and its anisotropy is investigated with reference to transverse and hybrid waves. For transverse waves the anisotropy is described in terms of a dependence between ϕ and θ, where ϕ is the angle formed by the vectors **H** and **B**. For hybrid waves the anisotropy is investigated in the "region of transparency" and in the "region of instability." In the region of transparency there is a dependence between ψ and θ, where ψ is the angle formed by the vectors **D** and **E**. In the region of instability there is a similar angular dependence and also a phase difference between the longitudinal and transverse components of the electric intensity **E**. Some of the effects produced by a beam in a thermal plasma are investigated in hydrodynamic representation. A formal analogy is established between the Vavilov-Cherenkov effect produced by a single particle passing through a thermal plasma and the longitudinal instability produced by a beam having the same velocity as the particle. This analogy does not apply to hybrid waves.

INTRODUCTION

An electron beam passing through a plasma produces an instability which may be either "electrostatic" or "electromagnetic." The electrostatic ("longitudinal") instability (due to the collective effect of Coulomb forces) has been extensively discussed in the literature.[1] The electromagnetic instability reported by Neufeld and Doyle[2] is characterized by growing waves in which the electric intensity **E** has a component in the direction of the wave vector **k**. This instability is designated as "hybrid," i.e., it is neither "longitudinal" nor "transverse." The plasma-beam medium comprises also a transverse mode in which **E** is perpendicular to **k**. This mode does not exhibit an instability.[3]

Part I of this investigation deals with properties of a medium in which an electron beam passes through a cold plasma. This part is essentially a continuation of the work reported by Neufeld and Doyle[2] (referred to hereafter as paper I), by Neufeld[4] (referred to as paper II), and by Neufeld and Wright[5] (referred to as paper III).

The behavior of the plasma-beam medium is described phenomenologically in terms of macroscopic field quantities such as the electric displacement **D** and the magnetic field strength **H**. The phenomenological representation is of interest since in current literature these macroscopic field quantities have not been defined in accordance with the theories of Maxwell and Lorentz. There exists at present a difference of opinion as to what constitutes an "electric displacement" or a "magnetic-field intensity" in a gaseous medium characterized by a nonisotropic velocity distribution. According to paper II, such commonly used concepts as the "electric displacement" or the "dielectric constant" do not designate a purely electrical entity since they are intrinsically related to the magnetic properties of the medium. In the present investigation magnetic properties of a structurally nonisotropic medium have been taken into account, and the macroscopic field quantities such as **D** and **H** have been redefined in accordance with the formulation given in paper II.

The phenomenological representation is particularly adapted to the study of electromagnetic anisotropies. Various angular relationships are established between vectors representing the macroscopic quantities **D** and **H** and the corresponding vectors representing microscopic quantities **E** and the magnetic induction **B**.

An investigation is made of the hybrid instability produced in a plasma by a beam. This instability is compared with similar instabilities produced in a medium containing harmonic oscillators.

Part II of this investigation deals with properties of a medium in which an electron beam interacts with a thermal plasma. Particular attention is directed to a similarity between the instability in a beam passing through a thermal plasma and the Vavilov-Cherenkov effect produced by a particle moving in the plasma with the same velocity as the beam. This similarity is investigated with reference to an appropriate formulation of the "beam problem" and of the "particle problem." It is shown that if the intensity of the

* Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

[1] See, for instance, A. I. Akhiezer and IA. B. Fainberg, Zhur. Eksp. i Teoret. Fiz. **21**, 1262 (1951); A. V. Haeff, Phys. Rev. **74**, 1532 (1948); J. R. Pierce, J. Appl. Phys. **20**, 1060 (1940); D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851; 1864 (1949); **79**, 992 (1950).

[2] Jacob Neufeld and P. H. Doyle, Phys. Rev. **121**, 654 (1961).

[3] See, for instance, S. A. Bludman, K. M. Watson, and M. N. Rosenbluth, Phys. Fluids **3**, 747 (1960), or reference 2.

[4] Jacob Neufeld, Phys. Rev. **123**, 1 (1961).

[5] Jacob Neufeld and Harvel Wright, Phys. Rev. **124**, 1 (1961).

beam tends to zero, or if the velocity of the beam approaches the velocity of light, the parameter representing the growth of the longitudinal instability decreases, and in the limit the wave representing this instability becomes identical with the Vavilov-Cherenkov wave. The longitudinal instability may, therefore, be considered as an "extension" of the Vavilov-Cherenkov effect. A similar situation does not apply to the hybrid instability.

I. INTERACTION OF A BEAM WITH A COLD PLASMA

1. Electromagnetic Waves in a Plasma-Beam Medium

(a) General Properties of a Plasma-Beam Medium

Some of the most general properties of nonisotropic dispersive media were investigated by Tellegen,[6] who assumed that the electric polarization **P** may be caused not only by the electric field but also by the magnetic field. He also assumed that the magnetic polarization **M** may be caused not only by the magnetic field but also by the electric field. Thus, in such a medium one has

$$\mathbf{P}=\chi_e\mathbf{E}+\alpha_e\mathbf{B}, \tag{1}$$

and

$$\mathbf{M}=\chi_{e\mu}\mathbf{E}+\alpha_{e\mu}\mathbf{B}, \tag{2}$$

where χ_e, α_e, $\chi_{e\mu}$, and $\alpha_{e\mu}$ are appropriate tensors.

This investigation deals with a dispersive medium in which an electron beam passes with velocity $\mathbf{V}=\boldsymbol{\beta}c$ through a substance having capacitivity ϵ. Such a medium may be considered from a phenomenological point of view as a special case of "Tellegen's medium" in which $\alpha_e=\alpha_{e\mu}=0$. Therefore, the behavior of this medium may be described in terms of an electric susceptibility

$$\chi_e\equiv\chi_e(\omega,\mathbf{k}), \tag{3}$$

and an "electromagnetic susceptibility"

$$\chi_{e\mu}\equiv\chi_{e\mu}(\omega,\mathbf{k}), \tag{4}$$

where ω designates the frequency.

The expressions (3) and (4) depend on the characteristics of the beam and also on the characteristics of the substance traversed by the beam. The substance traversed by the beam may comprise harmonic oscillators of binding frequency ω_a uniformly distributed with density n. In such case the capacitivity of this substance is expressed as

$$\epsilon\equiv\epsilon_d(\omega)=1-\omega_1^2/(\omega^2-\omega_a^2), \tag{5}$$

where

$$\omega_1=(4\pi n_1e^2/m)^{\frac{1}{2}}. \tag{6}$$

On the other hand, the substance traversed by the beam may be a stationary and charge-equilibrated plasma and in such case n_1 represents the density of electrons in the plasma. One has then

$$\epsilon\equiv\epsilon_p(\omega)=1-\omega_1^2/\omega^2. \tag{7}$$

Consider a rectangular x, y, z coordinate system in which the z axis is aligned in the direction of **k**, and $\boldsymbol{\beta}$ is contained in the plane formed by the z and x axes. Thus, $\beta_x=\beta_1=\beta\sin\theta$; $\beta_y=\beta_2=0$; $\beta_z=\beta_3=\beta\cos\theta$; $k_x=k_1=0$; $k_y=k_2=0$; $k_z=k_3=k$. Using the formulation of paper II, one can express the components of the tensor χ_e as follows:

$$\begin{aligned}
(\chi_e)_{11}&=\frac{1}{4\pi}\left[\epsilon-1+\frac{\omega_0^2g\beta^2\sin^2\theta}{(\omega-ck\beta\cos\theta)^2}-\frac{\omega_0^2g}{\omega(\omega-ck\beta\cos\theta)}\right],\\
(\chi_e)_{13}&=\frac{1}{4\pi}\frac{\omega_0^2g\beta^2\sin\theta\cos\theta}{(\omega-ck\beta\cos\theta)^2},\\
(\chi_e)_{22}&=\frac{1}{4\pi}\left[\epsilon-1-\frac{\omega_0^2g}{\omega(\omega-ck\beta\cos\theta)}\right],\\
(\chi_e)_{31}&=\frac{1}{4\pi}\left[\frac{\omega_0^2g\beta^2\sin\theta\cos\theta}{(\omega-ck\beta\cos\theta)^2}-\frac{\omega_0^2gck\beta\sin\theta}{\omega(\omega-ck\beta\cos\theta)^2}\right],\\
(\chi_e)_{33}&=\frac{1}{4\pi}\left[\epsilon-1+\frac{\omega_0^2g}{(\omega-ck\beta\cos\theta)^2}(\beta^2\cos^2\theta-1)\right],\\
(\chi_e)_{12}&=(\chi_e)_{21}=(\chi_e)_{23}=(\chi_e)_{32}=0,
\end{aligned} \tag{8}$$

where

$$g=(1-\beta^2)^{\frac{1}{2}}, \tag{9}$$

$$\omega_0=(4\pi n_0e^2/m)^{\frac{1}{2}}, \tag{10}$$

and n_0 represents the electron density in the beam.

The components of the tensor $\chi_{e\mu}$ are as follows:

$$\begin{aligned}
(\chi_{e\mu})_{12}&=-\frac{1}{4\pi}\frac{\omega_0^2g\beta\cos\theta}{\omega(\omega-ck\beta\cos\theta)},\\
(\chi_{e\mu})_{21}&=\frac{1}{4\pi}\left[\frac{\omega_0^2g\beta\cos\theta}{\omega(\omega-ck\beta\cos\theta)}-\frac{\omega_0^2gck\beta^2\sin^2\theta}{\omega(\omega-ck\beta\cos\theta)^2}\right],\\
(\chi_{e\mu})_{23}&=-\frac{1}{4\pi}\frac{\omega_0^2g\beta\sin\theta}{(\omega-ck\beta\cos\theta)^2},\\
(\chi_{e\mu})_{32}&=\frac{1}{4\pi}\frac{\omega_0^2g\beta\sin\theta}{\omega(\omega-ck\beta\cos\theta)},\\
(\chi_{e\mu})_{11}&=(\chi_{e\mu})_{13}=(\chi_{e\mu})_{22}=(\chi_{e\mu})_{31}=(\chi_{e\mu})_{33}=0.
\end{aligned} \tag{11}$$

It is assumed that the electromagnetic field varies with time and space as $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$.

Consider a plane electromagnetic wave passing through the medium described by the parameters (8) and (11). The wave satisfies Maxwell's equations,

$$\mathbf{k}\cdot\mathbf{D}=0, \tag{12}$$

$$\mathbf{k}\cdot\mathbf{B}=0, \tag{13}$$

[6] B. D. H. Tellegen, Philips Research Repts. **3**, 81 (1948).

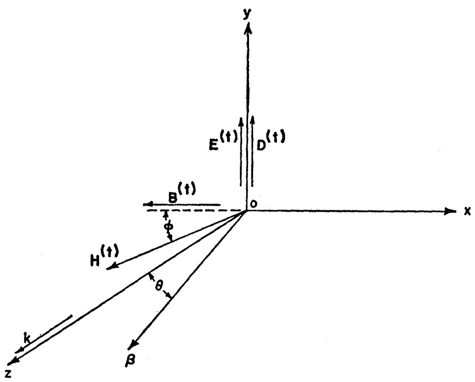

FIG. 1. System of vectors representing a transverse wave.

and, consequently, $\mathbf{k}\perp\mathbf{D}$ and $\mathbf{k}\perp\mathbf{B}$. Using the two other Maxwell equations, it will be shown that $\mathbf{B}\perp\mathbf{D}$ and, therefore that $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{k}$ are mutually perpendicular. Figures 1 and 2 show two systems of vectors which represent, respectively, two types of electromagnetic waves that may be transmitted through such a medium. In the rectangular system of Fig. 1 the vector $\boldsymbol{\beta}$ is contained in the plane formed by $\mathbf{B}$ and $\mathbf{k}$, whereas in the rectangular system of Fig. 2 the vector $\boldsymbol{\beta}$ is contained in the plane formed by $\mathbf{D}$ and $\mathbf{k}$. It will be shown that an electromagnetic wave represented in Fig. 1 is transverse, i.e., $\mathbf{E}$ is perpendicular to $\mathbf{k}$. Various field quantities associated with this wave will be designated by a superscript "t." An electromagnetic wave represented in Fig. 2 is hybrid, i.e., $\mathbf{E}$ has nonzero components in the direction of $\mathbf{k}$ and in the direction perpendicular to $\mathbf{k}$. Various field quantities associated with a hybrid wave shall be designated by a superscript "h." There also exists a third type of wave motion which is represented by a longitudinal wave and is characterized by a vector $\mathbf{E}$ aligned along the $\mathbf{k}$ direction. The longitudinal wave represents a particular case of the vector system shown in Fig. 2 in which $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ shrink to zero and $\mathbf{E}$ becomes parallel to $\mathbf{k}$.

(*b*) *Dispersion Equation*

The dispersion equation for a plasma-beam medium is given in the form of an expression operating on the vector $\mathbf{D}$ (instead of the previous formulation[2,5] in which a corresponding expression was operating on the vector $\mathbf{E}$). This is done in order to point out explicitly the relationship between the vector $\mathbf{D}$ and other quantities characterizing the electromagnetic field (see Figs. 1 and 2).

One can express Maxwell's equations in terms of $\mathbf{B}$ and $\mathbf{D}$ as follows:

$$\mathbf{k}\times\mathbf{E}=(\omega/c)\mathbf{B}, \tag{14}$$

$$\mathbf{k}\times\mathbf{B}-4\pi\mathbf{k}\times\chi_{e\mu}\epsilon_{\mathrm{pb}}^{-1}\mathbf{D}=-(\omega/c)\mathbf{D}. \tag{15}$$

The expression ϵ_{pb} in (15) represents the capacitivity of the plasma-beam medium, i.e.,

$$\epsilon_{\mathrm{pb}}=1+4\pi\chi_e. \tag{16}$$

Eliminating $\mathbf{B}$ from (14) and (15), one obtains

$$(c/\omega)\mathbf{k}\times(\mathbf{k}\times\epsilon_{\mathrm{pb}}^{-1}\mathbf{D})-4\pi\mathbf{k}\times\chi_{e\mu}\epsilon_{\mathrm{pb}}^{-1}\mathbf{D}+(\omega/c)\mathbf{D}=0. \tag{17}$$

Combining (8), (11), and (16) with (17), one obtains

$$\begin{bmatrix} R/ST & 0 & U/ST \\ 0 & W/S & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{vmatrix} D_1 \\ D_2 \\ D_3 \end{vmatrix}=0, \tag{18}$$

where

$$R=-\epsilon\left[\epsilon-\frac{c^2k^2}{\omega^2}\right]+\frac{\omega_0^2g}{(\omega-ck\beta\cos\theta)^2}\times\left[\epsilon\left(1-\beta^2+\frac{c^2k^2\beta^2\sin^2\theta}{\omega^2}\right)-\frac{c^2k^2}{\omega^2}(1-\beta^2\cos^2\theta)\right], \tag{19}$$

$$T=\epsilon-\omega_0^2g^3/(\omega-ck\beta\cos\theta)^2, \tag{20}$$

$$U=\frac{\omega_0^2gck\beta\sin\theta}{\omega(\omega-ck\beta\cos\theta)^2}\epsilon-\frac{\omega_0^4g^4ck\beta\sin\theta}{\omega^2(\omega-ck\beta\cos\theta)^3}-\frac{\omega_0^2gck^2\beta^2\sin\theta\cos\theta}{\omega^2(\omega-ck\beta\cos\theta)^2}, \tag{21}$$

$$W=c^2k^2/\omega^2-\epsilon+\omega_0^2g/\omega^2, \tag{22}$$

$$S=\epsilon-\omega_0^2g/\omega(\omega-ck\beta\cos\theta). \tag{23}$$

Expression (18) is equivalent to the following three equations:

$$(W/S)D_2=0, \tag{24}$$

and

$$(1/ST)(RD_1-UD_3)=0, \tag{25}$$

$$D_3=0. \tag{26}$$

Substituting (26) in (25), one obtains

$$(R/ST)D_1=0. \tag{27}$$

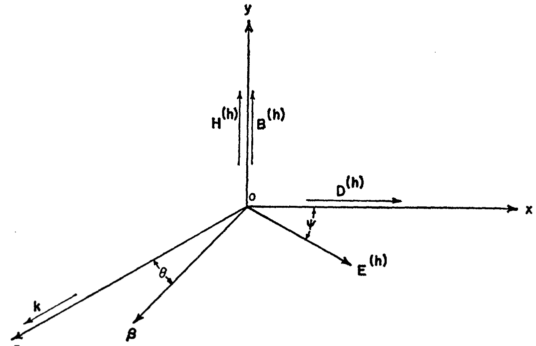

FIG. 2. System of vectors representing a hybrid wave.

The expression (24) describes a wave in which the electric displacement vector **D** is aligned along the y axis. This is shown in Fig. 1. Using relationship (16), one obtains the corresponding electric intensity which is also aligned along the y axis. Therefore, this wave is transverse and its dispersion equation is expressed as

$$W=0. \tag{28}$$

The expression (27) describes a wave in which the electric displacement vector **D** is aligned along the x axis. This is shown in Fig. 2. Using the relationship $\mathbf{D}=(1+4\pi\chi_e)\mathbf{E}$, one obtains the corresponding electric field strength **E** which has components in the direction of **k** and in the direction perpendicular to **k**. Therefore, this wave is hybrid and its dispersion equation is represented as

$$R=0. \tag{29}$$

The expression (29) was obtained in nonrelativistic form in paper I.

2. Longitudinal and Hybrid Instability

(a) Comparison of Instabilities in a Plasma with Those Occurring in a Dielectric Substance Comprising Harmonic Oscillators

In the study of instabilities produced by a beam, it is convenient to use the "small κ approximation."[5] This approximation is valid when

$$\kappa=\omega_0 g^{\frac{1}{3}} \ll |\omega|. \tag{30}$$

The solution of the dispersion equation is represented then in the form

$$\omega=\tilde{\omega}+\delta, \tag{31}$$

where the "characteristic frequency" $\tilde{\omega}$ is expressed as

$$\tilde{\omega}=ck\beta\cos\theta. \tag{32}$$

The term δ represents the "frequency increment" due to the presence of the beam. One has

$$|\delta|\rightarrow 0 \quad \text{when} \quad \kappa\rightarrow 0. \tag{33}$$

The instability occurs when δ is complex and the term $\mathrm{Im}(\delta)$, when positive, represents the "excitation coefficient."

Figure 3(a) describes the behavior of a dielectric substance comprising harmonic oscillators and it illustrates graphically various instabilities produced in such a substance as a result of its interaction with the beam.[5] The beam produces two instability centers and there are two nonoverlapping frequency ranges comprising hybrid instabilities. Thus a distinction is made between a hybrid instability of type "l" and a hybrid instability of type "t." The hybrid instability of type "l" represents the "continuation" of the longitudinal (electrostatic) instability, and the hybrid instability of type "t" represents the "continuation" of the transverse (Vavilov-Cherenkov) instability. There are two

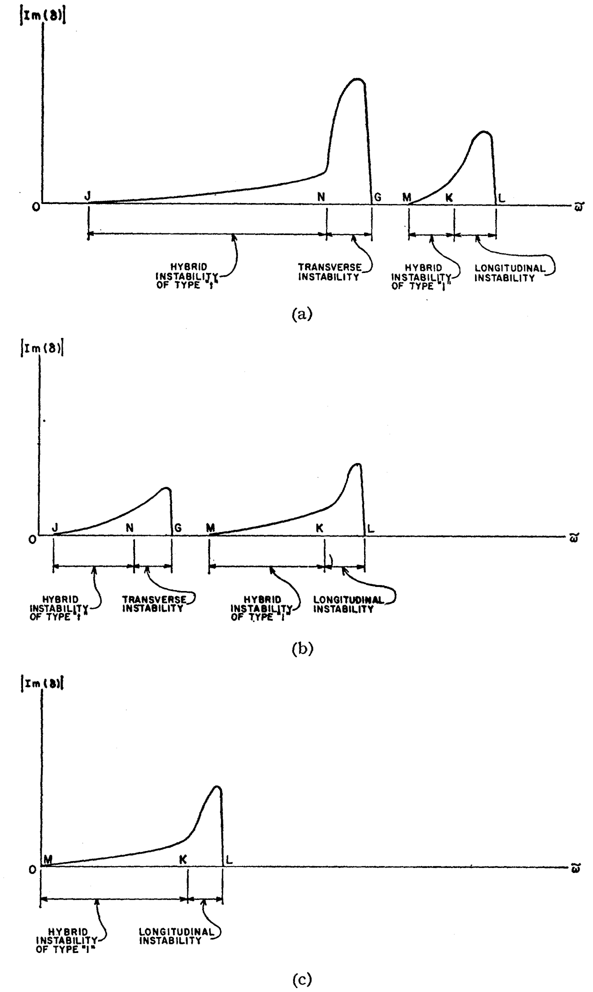

FIG. 3. Dependence between $|\mathrm{Im}(\delta)|$ and $\tilde{\omega}$; (a) for waves resulting from the passage of a beam through a dielectric substance (ω_a is relatively large); (b) for waves resulting from the passage of a beam through a dielectric substance (ω_a is relatively large); (c) in a plasma-beam system.

significant frequency ranges: JG and ML. The lower frequency range JG contains the transverse instability and the hybrid instability of type "t." The upper frequency range ML contains the longitudinal instability and the hybrid instability of type "l."

The occurrence of a transverse instability in the frequency range NG is directly related to the presence of harmonic oscillators in the dielectric medium. In a plasma the electrons are free and, therefore, such an instability does not occur. One has only a longitudinal instability and a hybrid instability. A question arises as to the proper identification of the hybrid instability in a plasma and its relationship to the two hybrid instabilities shown in Fig. 3(a).

In order to resolve this question and to investigate the behavior of instabilities produced in a plasma by a beam, a graphical representation is used. This representation is given in Figs. 3(a), 3(b), and 3(c), and

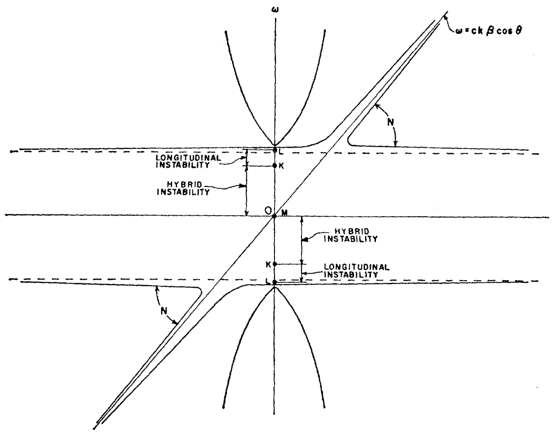

FIG. 4. "ω–k" diagram for hybrid and longitudinal oscillations in a plasma-beam system ($\theta\neq 0$).

it illustrates a "transition" from a dielectric comprising harmonic oscillators to a plasma. In such a "transition" the binding frequency gradually decreases, and for $\omega_a\rightarrow 0$ the expression (5) becomes identical with the expression (7) representing the capacitivity of a plasma. In Fig. 3(a) the binding frequency ω_a is relatively large, whereas in Fig. 3(b) it is smaller, and in Fig. 3(c), $\omega_a=0$. Comparing Figs. 3(a) and 3(b), it is noted that in Fig. 3(b) the region JG comprising the transverse and the related hybrid instability is shifted toward lower frequencies and occupies a considerably narrower range. The limiting case of $\omega_a=0$ shown in Fig. 3(c) represents the instability resulting from the interaction of a beam with a plasma. The region JG vanished and the region ML comprising the longitudinal and the hybrid waves of type "l" occupies a considerably wider frequency range than in Fig. 3(a) or 3(b). The only hybrid instability that remains in a plasma is, therefore, of type "l." This instability extends from $\tilde{\omega}=0$ into higher frequencies and gradually changes into a longitudinal instability. The slope of the curve shown in Fig. 3(c) is positive, i.e., the excitation coefficient for the hybrid instability increases with $\tilde{\omega}$.

Both longitudinal and transverse instabilities are centered in the immediate neighborhood of the corresponding resonance frequencies. The frequency range occupied by each of these two instabilities is a function of κ which tends to zero with κ. For small values of κ such a frequency range is relatively small when compared to the frequency range of the corresponding hybrid instability. In the graphical representation, the widths of the frequency ranges for the longitudinal and the hybrid instabilities are, for the sake of clarity, considerably enlarged.

(b) *Convective and Aperiodic Instabilities*

According to Sturrock,[7] one can ascertain from the behavior of the dispersion equation in the $\omega-k$ plane whether an instability is convective or nonconvective. In a convective instability, a disturbance increases as it is carried along the system and remains finite at each point. In a nonconvective instability, a disturbance which originated in a limited region of space at any instant of time grows indefinitely for $t\rightarrow\infty$ in this region. It was pointed out in several recent publications[8–11] that the longitudinal instability produced by a beam in a plasma is convective. It is of interest to determine whether or not a hybrid instability is convective. This can be done by means of an appropriate diagram, shown in Fig. 4. Figure 4 illustrates graphically the dispersion equation for a plasma-beam system when the growing wave is aligned at angle $\theta\neq 0$ with respect to the direction of the beam. By applying the criterion formulated by Sturrock, one can ascertain that both the longitudinal instability occurring within the frequency region KL and the hybrid instability occurring within the frequency region OK are convective. This "convective" feature is associated with the branch of the $\omega-k$ diagram designated in Fig. 4 as "N."

The excitation coefficient has different orders of magnitude for a hybrid and a longitudinal instability.[2] The hybrid instability is relatively weak and thus

$$|\mathrm{Im}(\delta)|\sim|\delta|=O(\kappa), \tag{34}$$

whereas the longitudinal instability is relatively intense and one has

$$|\mathrm{Im}(\delta)|\sim|\delta|=O(\kappa^{\frac{1}{3}}). \tag{35}$$

Using the relationship $|\delta|\ll\tilde{\omega}$ and the equalities (32) and (34), one finds that for a hybrid instability, the small κ approximation applies to those values of θ which satisfy the inequality

$$k\cos\theta\gg\kappa/c\beta. \tag{36}$$

Similarly, using the relationship $|\delta|\ll\tilde{\omega}$ and equalities (32) and (35), one finds that for a longitudinal instability one needs an inequality

$$k\cos\theta\gg\kappa^{\frac{1}{3}}/c\beta. \tag{37}$$

The $\omega-k$ diagram of Fig. 4 is applicable to the values of θ that are not in the neighborhood of $\pi/2$. For $\theta=\pi/2$, the dispersion equation (29) yields solutions for ω which satisfy $\omega^2<0$. Therefore, the frequencies are represented by pure imaginary numbers.[2] Consequently, an initial disturbance which at $t=0$ was distributed in space in accordance with a function

[7] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).

[8] J. E. Drummond and D. B. Chang, Bull. Am. Phys. Soc. **6**, 411 (1958).

[9] P. A. Sturrock, Phys. Rev. **117**, 1426 (1960).

[10] I. F. Kharchenko, IA. B. Fainberg, P. M. Nikolayev, E. A. Kornilov, E. A. Lutzenko, and N. S. Pedenko, *Proceedings of the Fourth International Conference on Ionization Phenomena in Gasses, Uppsala, Sweden* (North-Holland Publishing Company, Amsterdam, 1960).

[11] R. A. Demirkhanov, A. K. Gevorkov, and A. F. Popov, Abstract CN-10/235/A, IAE Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, Austria, 1961 (unpublished).

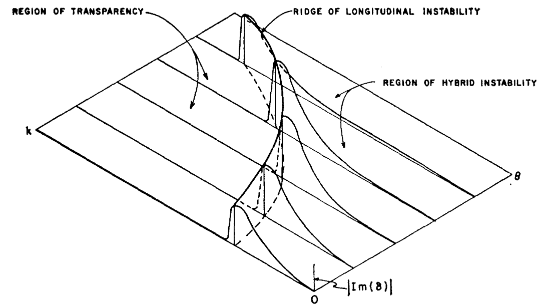

FIG. 5. Perspective view of $|\mathrm{Im}(\delta)|=\phi(k,\theta)$.

$M(\mathbf{r})$ will not oscillate and will progressively increase with time in accordance with a function $M(\mathbf{r})\times\exp|\mathrm{Im}(\omega)|t$. The terms "convective" and "absolute" instability are generally applied to an oscillatory motion. For $\theta=\pi/2$ the instability is nonoscillatory and following the terminology of Vvedenov, Velikhov, and Sagdeev,[12] it is designated as "aperiodic."

(c) *Three-Dimensional Representation of Instabilities in a Plasma*

Figure 5 illustrates diagrammatically an instability for $\theta\neq 0$ in the small κ approximation. The vertical coordinate represents the excitation coefficient $|\mathrm{Im}(\delta)|$, and the horizontal plane contains rectangular coordinates θ and k. The three-dimensional surface shown in this figure is characterized by a "longitudinal instability ridge" which divides the surface into two regions; the region of "hybrid instability" and the "transparency region." The ridge slopes gradually in the hybrid instability region and has a very steep slope into the transparency region. In the region of transparency $\mathrm{Im}(\delta)=0$ and therefore the electromagnetic waves in this region neither grow nor decay.

3. Electromagnetic Anisotropy in a Plasma-Beam Medium

Consider a charge-equilibrated gaseous medium characterized by an electron-velocity distribution $f(\mathbf{v})$. In describing such a medium one may differentiate between a structural and an electromagnetic (optical) anisotropy.[4] A medium is structurally isotropic if $f(\mathbf{v})$ is spherically symmetrical. On the other hand, it is electromagnetically isotropic if one has $\mathbf{D}\|\mathbf{E}$ and $\mathbf{H}\|\mathbf{B}$ for all directions of $\mathbf{k}$. Otherwise, the medium is electromagnetically nonisotropic. A structurally isotropic medium may be electromagnetically nonisotropic.

The plasma-beam medium is both structurally and electromagnetically nonisotropic. Its electrical anisotropy is expressed by two functions: $\phi=F_1(\theta)$ and $\psi=F_2(\theta)$. The angle ϕ is formed by the vectors $\mathbf{H}$ and $\mathbf{B}$, and the angle ψ is formed by the vectors $\mathbf{D}$ and $\mathbf{E}$.

[12] A. A. Vvedenov, E. P. Velikhov, and R. Z. Sagdeev, Uspekhi Fiz. Nauk. **73**, 701 (1961).

The electromagnetic anisotropy of the plasma-beam medium will be considered with reference to transverse and hybrid waves.

(a) *Transverse Waves*

A vector system representing various field quantities associated with a transverse wave is shown in Fig. 1. Let l_1, l_2, and l_3 designate unit vectors aligned along the positive x, y, and z directions, respectively. Using Maxwell's equations and the macroscopic parameters χ_e and $\chi_{e\mu}$, one obtains for transverse waves the following relationships:

$$\mathbf{D}^{(t)}=\mathbf{l}_2 D_y, \tag{38}$$

$$\mathbf{E}^{(t)}=\mathbf{l}_2 D_y/[1+4\pi(\chi_e)_{22}], \tag{39}$$

$$\mathbf{B}^{(t)}=-\mathbf{l}_1 ckD_y/\omega[1+4\pi(\chi_e)_{22}], \tag{40}$$

$$\mathbf{l}_1\cdot\mathbf{M}^{(t)}=(\chi_{e\mu})_{12}D_y/[1+4\pi(\chi_e)_{22}], \tag{41}$$

$$\mathbf{l}_3\cdot\mathbf{M}^{(t)}=(\chi_{e\mu})_{32}D_y/[1+4\pi(\chi_e)_{22}], \tag{42}$$

$$\mathbf{l}_1\cdot\mathbf{H}^{(t)}=-\frac{[ck+4\pi\omega(\chi_{e\mu})_{12}]D_y}{\omega[1+4\pi(\chi_e)_{22}]}, \tag{43}$$

$$\mathbf{l}_3\cdot\mathbf{H}^{(t)}=-4\pi(\chi_{e\mu})_{32}D_y/[1+4\pi(\chi_e)_{22}]. \tag{44}$$

Using Eqs. (43) and (44), one can express the angle ϕ between $\mathbf{B}^{(t)}$ and $\mathbf{H}^{(t)}$ as follows:

$$\tan\phi=\frac{\mathbf{l}_3\cdot\mathbf{H}^{(t)}}{\mathbf{l}_1\cdot\mathbf{H}^{(t)}}=-\frac{4\pi(\chi_{e\mu})_{12}\omega}{ck-4\pi\omega(\chi_{e\mu})_{12}}. \tag{45}$$

Substituting in (45) the expressions for $(\chi_{e\mu})_{12}$ and $(\chi_{e\mu})_{32}$ as given in (11), one obtains

$$\tan\phi=-\frac{\kappa^2\beta\sin\theta}{ck(\omega-\tilde{\omega})+\kappa^2\beta\cos\theta}. \tag{46}$$

The quantities ω and k should satisfy the dispersion equation (28) for transverse waves.

The above expressions describe the magnetic behavior of transverse waves. The term "transverse" refers to the relationship between $\mathbf{E}$ and $\mathbf{k}$, i.e., the wave is transverse with respect to $\mathbf{E}$. In one special case when $\mathbf{k}$ is parallel to $\boldsymbol{\beta}$ (i.e., $\theta=0$), the wave is also transverse with respect to $\mathbf{H}$, i.e., $\mathbf{H}$ is perpendicular to $\mathbf{k}$. For all other directions of $\mathbf{k}$ the wave is hybrid with respect to $\mathbf{H}$, i.e., there is a longitudinal and a transverse component of $\mathbf{H}$. In some instances the transverse wave may be purely longitudinal with respect to $\mathbf{H}$ and in such case there is no transverse component of the magnetic intensity. This occurs when

$$\beta\cos\theta=ck\omega/(c^2k^2-\omega^2)\quad(\omega=\tilde{\omega}). \tag{47}$$

If ω is in the neighborhood of the characteristic frequency $\tilde{\omega}$, one obtains $\phi \sim -\theta$.

Consider small values of κ that satisfy for $\omega > \omega_1$ the inequality

$$\kappa^2 \ll 2(\omega^2 - \omega_1^2). \quad (48)$$

Using the binomial expansion and retaining the first two terms, one obtains from the dispersion equation (28)

$$k = [2(\omega^2 - \omega_1^2) - \kappa^2]/2c(\omega^2 - \omega_1^2)^{\frac{1}{2}}, \quad (49)$$

and substituting (49) in (46), one obtains

$$\tan\phi = -\frac{2\kappa^2\beta \sin\theta\, S^{\frac{1}{2}}}{(2S-\kappa^2)[2\omega S^{\frac{1}{2}} - (2S-\kappa^2)\beta\cos\theta] - \kappa^2\beta\cos\theta}, \quad (50)$$

where

$$S = \omega^2 - \omega_1^2. \quad (51)$$

(*b*) *Hybrid Waves*

A vector system representing various field quantities associated with a hybrid wave is shown in Fig. 2. One obtains for hybrid waves the following relationships:

$$\mathbf{D}^{(h)} = \mathbf{l}_1 D_x, \quad (52)$$

$$\mathbf{l}_1 \cdot \mathbf{E}^{(h)} = \frac{[1+4\pi(\chi_e)_{33}]D_x}{[1+4\pi(\chi_e)_{11}][1+4\pi(\chi_e)_{33}] - 16\pi^2(\chi_e)_{13}(\chi_e)_{31}}, \quad (53)$$

$$\mathbf{l}_3 \cdot \mathbf{E}^{(h)} = -\frac{4\pi(\chi_e)_{31}D_x}{[1+4\pi(\chi_e)_{11}][1+4\pi(\chi_e)_{33}] - 16\pi^2(\chi_e)_{13}(\chi_e)_{31}}, \quad (54)$$

$$\mathbf{B}^{(h)} = \frac{\mathbf{l}_2 ck[1+4\pi(\chi_e)_{33}]D_x}{\omega[1+4\pi(\chi_e)_{11}][1+4\pi(\chi_e)_{33}] - 16\pi^2(\chi_e)_{13}(\chi_e)_{31}}, \quad (55)$$

$$\mathbf{M}^{(h)} = \frac{\mathbf{l}_2\{(\chi_{e\mu})_{21}[1+4\pi(\chi_e)_{33}] - 4\pi(\chi_{e\mu})_{23}(\chi_e)_{31}\}D_x}{[1+4\pi(\chi_e)_{11}][1+4\pi(\chi_e)_{33}] - 16\pi^2(\chi_e)_{13}(\chi_e)_{31}}. \quad (56)$$

Therefore,

$$\mathbf{H}^{(h)} = \frac{\mathbf{l}_2\left\{\left[\frac{ck}{\omega} - 4\pi(\chi_{e\mu})_{21}\right][1+4\pi(\chi_e)_{33}] + 16\pi^2(\chi_{e\mu})_{23}(\chi_e)_{31}\right\}D_x}{[1+4\pi(\chi_e)_{11}][1+4\pi(\chi_e)_{33}] - 16\pi^2(\chi_e)_{13}(\chi_e)_{31}}. \quad (57)$$

Using Eqs. (53) and (54), one can express the angle ψ between $\mathbf{E}^{(h)}$ and $\mathbf{D}^{(h)}$ as follows:

$$\tan\psi = \mathbf{l}_3 \cdot \mathbf{E}_3/\mathbf{l}_1 \cdot \mathbf{E}_1 = -4\pi(\chi_e)_{31}/[1+4\pi(\chi_e)_{33}]. \quad (58)$$

Substituting the expression (8) for $(\chi_e)_{31}$ and $(\chi_e)_{33}$ in (58), one obtains

$$\tan\psi = \frac{\kappa^2\omega\beta \sin\theta(ck - \omega\beta\cos\theta)}{(\omega^2-\omega_1^2)(\omega-\tilde{\omega})^2 - \kappa^2\omega^2(1-\beta^2\cos^2\theta)}. \quad (59)$$

The quantities ω and κ in the above expressions should satisfy the dispersion Eq. (29) for hybrid waves. Putting in (59) $\omega = \tilde{\omega} + \delta$ subject to the condition $|\delta| \ll \tilde{\omega}$, one obtains

$$\tan\psi = \frac{\kappa^2\beta^2c^2k^2 \sin\theta\cos\theta(1-\beta^2\cos^2\theta)}{(\tilde{\omega}^2-\omega_1^2)\delta^2 - \kappa^2\tilde{\omega}^2(1-\beta^2\cos^2\theta)}. \quad (60)$$

The quantity δ may be expressed as[5]

$$\delta = \kappa F^{\frac{1}{2}}, \quad (61)$$

where

$$F = \frac{[1-\epsilon_p(\tilde{\omega})\beta^2](1-\beta^2\cos^2\theta)}{\epsilon_p(\tilde{\omega})[1-\epsilon_p(\tilde{\omega})\beta^2\cos^2\theta]}. \quad (62)$$

There is an instability if F is negative, and consequently

$$\delta = i\kappa|F|^{\frac{1}{2}}. \quad (63)$$

Substituting (63) in (59), one can express $\tan\psi$ in the form:

$$\tan\psi = Re^{i\xi}. \quad (64)$$

The terms R and ξ may be expressed as:

$$R = \frac{[(A_1C_1+B_1S_1)^2 + (A_1S_1-B_1C_1)^2]^{\frac{1}{2}}}{(C_1^2+S_1^2)}, \quad (65)$$

$$\tan\xi = -(A_1S_1 - B_1C_1)/(A_1C_1+B_1S_1), \quad (66)$$

where

$$A_1 = \sin\theta(\beta^2\cos^2\theta - 1)/|F|\cos\theta, \quad (67)$$

$$B_1 = \kappa\sin\theta/|F|^{\frac{1}{2}}\tilde{\omega}\cos\theta, \quad (68)$$

$$C_1 = 1 - \omega_1^2/\tilde{\omega}^2 + (1-\beta^2\cos^2\theta)/|F|, \quad (69)$$

$$S_1 = 2\omega_1^2\kappa|F|^{\frac{1}{2}}/\tilde{\omega}^3. \quad (70)$$

Using equalities (58) and (64), one can express the relationship between the longitudinal component E_3 and the transverse component E_1 of a hybrid wave in the following form:

$$E_3 = RE_1e^{i\xi}. \quad (71)$$

This relationship is applicable within the region of instability, and it shows the existence of a phase displacement ξ between the components of electrical intensity that are, respectively, perpendicular and parallel to **k**.

Consider now hybrid waves in the region in which there is no growth or decay. The quantity F is positive, and one has

$$\delta = \kappa F^{\frac{1}{2}}. \tag{72}$$

Substituting (72) in (60), one obtains

$$\tan\psi = \frac{\tilde{\omega}^2(1-\beta^2\cos^2\theta)\tan\theta}{\tilde{\omega}^2(F-1+\beta^2\cos^2\theta)-F\omega_1^2}. \tag{73}$$

II. INTERACTION OF A BEAM WITH A THERMAL PLASMA

1. Description of a Thermal Plasma

Consider a charge-equilibrated medium in which a beam of electrons interacts with a thermal plasma. The beam is described in terms of its Langmuir frequency ω_0 and its velocity $\mathbf{V}=\boldsymbol{\beta}c$. The plasma is described by its Langmuir frequency ω_1 and by its temperature T. Instead of the temperature, one may use the mean thermal velocity s which is related to the temperature as follows:

$$s = (3K_BT/m)^{\frac{1}{2}}, \tag{74}$$

where K_B is the Boltzmann constant.

There are two formulations of the problem dealing with response of a thermal plasma to an electromagnetic field. In one formulation the plasma is assumed to be nonmagnetic[13,14] (i.e., the magnetic permeability $\mu=1$). Consequently, when an electromagnetic field is impressed on the plasma, one has

$$\mathbf{H}=\mathbf{B}. \tag{75}$$

In the other formulation[13,15] the thermal plasma is assumed to be magnetic. The relationship between the magnetic field and the magnetic induction is represented in the form

$$\mathbf{H}=[1/\mu(\omega,\mathbf{k})]\mathbf{B}, \tag{76}$$

where $\mu(\omega,\mathbf{k})$ is an appropriate scalar function of ω and **k**.

The above situation involves two different assumptions that are contradictory. It is also confusing, particularly in view of recent results obtained in paper II. According to these results a plasma which is not in thermal equilibrium is magnetically polarizable, whereas a thermal plasma is not magnetically polarizable. A brief discussion on this subject is given in Appendix A.

[13] J. Linhard, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 8 (1954).

[14] M. E. Gertzenshtein, Zhur. Eksp. i Teor. Fiz. **22**, 303 (1952).

[15] A. A. Rukhadze and V. P. Silin, Uspekhi Fiz. Nauk. **74**, 223 (1961); **75**, 79 (1962).

In the present investigation the macroscopic parameters of a thermal plasma are formulated in accordance with the theory outlined in paper II. Thus the electromagnetic susceptibility,

$$\chi_{e\mu}^{(\mathrm{th})}=0, \tag{77}$$

and the relationship between **P** and **E** is expressed as

$$\mathbf{P}=\chi_e^{\mathrm{th}}\mathbf{E}, \tag{78}$$

where χ_e^{th} is the electric-susceptibility tensor of a thermal plasma. Using the x, y, z representation shown in Fig. 1, one can formulate the relationship (78) as follows:

$$\begin{vmatrix}P_1\\P_2\end{vmatrix}=\begin{pmatrix}(\chi_e)_t & 0\\ 0 & (\chi_e)_l\end{pmatrix}\begin{vmatrix}E_1\\E_3\end{vmatrix}. \tag{79}$$

The terms $(\chi_e)_t$ and $(\chi_e)_l$ represent, respectively, the "transverse" and the "longitudinal" components of the electric-susceptibility tensor. A relativistic formulation of these terms in gas kinetic representation is given in paper II. The gas kinetic representation depends explicitly on an expression representing the electron velocity distribution in the plasma.

Using relationships (77) and (78), one can express the dispersion equation for a plasma as follows:

$$F_p\equiv F_p(\omega,\mathbf{k})=\epsilon_l[(c^2k^2/\omega^2)-\epsilon_t]=0, \tag{80}$$

where

$$\epsilon_l\equiv\epsilon_l(\omega,\mathbf{k})=1+4\pi(\chi_e)_l, \tag{81}$$

$$\epsilon_t\equiv\epsilon_t(\omega,\mathbf{k})=1+4\pi(\chi_e)_t. \tag{82}$$

In the subsequent discussion the behavior of a thermal plasma will be formulated in hydrodynamic representation. The hydrodynamical representation is based on Euler's equations of motion together with Maxwell's equations. In this representation the terms ϵ_t and ϵ_l may be expressed as:

$$\epsilon_t=1-(\omega_1^2/\omega^2), \tag{83}$$

$$\epsilon_l=1-\frac{\omega_1^2}{\omega^2-k^2(S^2/3)}, \tag{84}$$

(see Appendix B.)

The hydrodynamic representation fails to bring out some of the essential features of the plasma-beam medium such as the occurrence of Landau damping. However, the hydrodynamic representation formulates very clearly the effectiveness of the beam in producing instabilities as a result of its interaction with plasma.

2. "Particle Effect" and "Beam Effect"

The formulation of plasma-beam instabilities is related to the formulation of another problem which deals with the passage of a single particle through plasma. There is a very extensive literature dealing with the behavior of a single particle.[16] Among the early publications on this subject, those of Vlasov have pointed out the occurrence of a longitudinal Vavilov–Cherenkov effect produced by a particle moving through a plasma. Thus, when the velocity of the particle $\mathbf{V}_p=\beta_p c$ exceeds the mean thermal velocity of plasma, i.e., when

$$\beta_p c \gg s, \tag{85}$$

the particle radiates longitudinal (electrostatic) waves. These waves exist only within the Mach (Cherenkov) cone given by the angle θ_m for which

$$\sin\theta_m = s/\beta_p c. \tag{86}$$

When the plasma temperature is zero ($s=0$), the cone degenerates into a straight line and one obtains a series of oscillators aligned along the particle track.[17]

The passage of a beam through a thermal plasma was studied by Akhiezer and Fainberg,[18] under the assumption that the growing wave resulting from the instability is aligned in the direction of the beam. Under these conditions the interaction is electrostatic, i.e., the wave is longitudinal. When the velocity of the beam $\mathbf{V}=\beta c$ is below the thermal velocity s, the instability is relatively weak. It becomes relatively intense when

$$\beta c \gg s. \tag{87}$$

When comparing expressions (85) and (87) one sees that there is a formal analogy between the Vavilov-Cherenkov effect and the plasma-beam instability, provided

$$\beta_p \sim \beta. \tag{88}$$

Such an analogy was pointed out in a remark made by Kharchenko, Fainberg, Nikolaiev, Kornilov, Lutzenko, and Pedenko.[10] This remark needs, perhaps, clarification. One should point out that there are distinctive features that differentiate the Vavilov-Cherenkov effect from the plasma-beam instability. In the conventional formulation of the Vavilov-Cherenkov effect the wave radiated by the particle neither grows nor decays and the radiation is associated with the occurrence of a nonzero Poynting vector directed outwardly from the particle track at infinite lateral distances from the track. On the other hand, the conventional formulation of the plasma-beam interaction is based on stability considerations and the "beam effect" is expressed by the existence of a wave having an amplitude that grows indefinitely with time.

The subsequent discussion will deal with a more precise formulation of the analogy between the "beam effect" and the "particle effect." One needs to ascertain whether this analogy applies to the longitudinal instability, to the hybrid instability, or to both instabilities.

3. Waves "Concurrent with a Particle"

Some of the characteristic features of the Vavilov–Cherenkov radiation may, perhaps, be clarified by introducing the concept of waves that are "concurrent with a particle." The velocity of a concurrent wave is maintained at a fixed relationship to the velocity of the particle. Wave vectors associated with concurrent waves are aligned along a conical surface forming with the particle track an angle θ such that

$$v_{\text{ph}} = c\beta_p \cos\theta. \tag{89}$$

The quantity v_{ph} designates the phase velocity of the wave and can be represented as

$$v_{\text{ph}} = \omega/k. \tag{90}$$

A concurrent wave moves slower than the particle. The velocity of the concurrent wave is the same as the velocity of the component of the particle velocity along the direction of propagation of the wave.

It is noted from (89) and (90) that the frequency of a concurrent wave can be expressed as $\omega = ck\beta\cos\theta$. Therefore, using the terminology applied to the plasma-beam system, one has $\omega=\tilde{\omega}$, i.e., a concurrent wave has the "characteristic frequency" that is associated with a plasma-beam instability.

A concurrent wave is not necessarily a Vavilov–Cherenkov wave. The equalities (89) and (90) represent a necessary but not a sufficient condition for the occurrence of Vavilov–Cherenkov effect. A Vavilov–Cherenkov wave must also satisfy the dispersion equation $F_p=0$ as given by (80). Therefore, combining the equalities (89), (90), and (80), one obtains the necessary and sufficient condition for a Vavilov–Cherenkov wave. One obtains then

$$(F_p)_{\omega=\tilde{\omega}} = 0. \tag{91}$$

Using (80), one can represent (91) in the form of two equalities. One of these has the form

$$(c^2k^2/\tilde{\omega}^2) - (\epsilon_t)_{\omega=\tilde{\omega}} = 0, \tag{92}$$

and it represents the necessary and sufficient condition for the occurrence of a transverse Vavilov–Cherenkov wave. The other equality has the form

$$(\epsilon_l)_{\omega=\tilde{\omega}} = 0, \tag{93}$$

and it represents the corresponding condition for the occurrence of a longitudinal Vavilov–Cherenkov wave. The equality (92) cannot be satisfied for $\beta<1$, and

[16] A number of investigations dealing with the passage of a charged particle through plasma and published prior to 1955 are listed in a paper by J. Neufeld and R. H. Ritchie, Phys. Rev. **98**, 1632 (1955). Of particular significance are investigations by A. Vlasov, *Teoria Mnogikh Chastits* (Moscow, 1950), pp. 309–318, and by A. T. Akhiezer and A. G. Sitenko, Zhur. Eksp. i Teor. Fiz. **23**, 161 (1959). Results similar to those of Vlasov and of Akhiezer and Sitenko were obtained more recently by S. K. Majumdar-Proc. Phys. Soc. (London) **76**, 657 (1960), and by M. H. Cohen, Phys. Rev. **123**, 711 (1961).

[17] A. Bohr, Kgl. Danske. Videnskab. Selskab, Mat.-fys. Medd. **24**, No. 19 (1948).

[18] A. I. Akhiezer and IA. B. Fainberg, Zhur. Eksp. i Teor. Fiz. **21**, 1262 (1951).

therefore there is no transverse Vavilov–Cherenkov effect.[19] The only Vavilov–Cherenkov effect in a plasma appears in the form of longitudinal waves and is expressed by the equality (93).

4. Dispersion Equation

Consider the interaction of a beam with a thermal plasma. Applying the procedure outlined in paper III and taking into account the electromagnetic anisotropy of a thermal plasma, one obtains the following dispersion equation:

$$F_{pb} \equiv F_{pb}(\omega,k,\theta) = F_p - [\kappa^2/(\omega-\tilde{\omega})^2]F_b = 0, \quad (94)$$

where $F_p \equiv F_p(\omega,k,\theta)$ is given by the expression (80) and

$$F_b \equiv F_b(\omega,k,\theta) = [\epsilon_l\beta^2 \sin^2\theta(c^2k^2-\omega^2)/\omega^2 + \epsilon_t(1-\beta^2\cos^2\theta) - (c^2k^2/\omega^2)(1-\beta^2\cos^2\theta)]. \quad (95)$$

Expression (94) represents a generalized form of the dispersion equation (28). The equalities (94) and (28) are identical for $\epsilon = \epsilon_l = \epsilon_t$.

The dispersion Eq. (94) is investigated in the usual manner by assigning real values to **k** and finding the corresponding roots for ω. The expressions for ω may be represented in one of the following two forms:

$$\omega = \tilde{\omega} + \delta, \quad (96)$$

and

$$\omega = W + \delta, \quad (97)$$

where

$$|W - \tilde{\omega}| \gg \delta. \quad (98)$$

In formulating the expressions (96) and (97), no assumptions are made as to whether the quantities W and δ are real, imaginary, or complex. However, one has $\tilde{\omega} = ck\beta\cos\theta$ and, therefore, the quantity $\tilde{\omega}$ is real by definition. The term δ represents a small quantity that depends on κ and one has

$$|\delta| \to 0 \quad \text{for} \quad \kappa \to 0. \quad (99)$$

In the absence of an instability, i.e., when δ is real, the wave (96) is designated as a "stationary concurrent wave." When the instability occurs, i.e., when $\mathrm{Im}(\delta) > 0$, the expression (96) represents an "excited concurrent wave." The wave (97) is "nonconcurrent with the beam."

In studying the plasma-beam interaction, one generally ignores solutions of type (97) and considers only solutions of type (96). It will be shown that such a procedure is justified since both W and δ in the expression (97) are real. Therefore, nonconcurrent waves neither grow nor decay and are of no interest from the standpoint of stability considerations. On the other hand, in the expression (96) $\tilde{\omega}$ is real but δ may be complex and consequently concurrent waves may show an instability.

[19] The effect of temperature on the transverse field produced by a particle passing through a plasma is discussed in the gas kinetic representation by A. I. Akhiezer and A. G. Sitenko, Zhur. Eksp. i Teor. Fiz. **23**, 161 (1959), and by Jacob Neufeld, Phys. Rev. **116**, 1 (1959).

5. Nonconcurrent Waves

Substitute $\omega = W + \delta$ in (94) and assume that $\kappa \to 0$. Consequently, $\delta \to 0$, and one obtains in the limit

$$(F_{pb})_{\omega=W} = (F_p)_{\omega=W} = 0. \quad (100)$$

It is seen from (100) that the quantity W satisfies the dispersion equation for a thermal plasma in the absence of a beam. In the hydrodynamic representation the thermal plasma is "transparent," i.e., the waves that are transmitted through the plasma neither grow nor decay. Consequently, W is a real number.

Consider now the expression $\omega = W + \delta$ for $\delta \neq 0$ but small, and substitute this expression in (94). Applying Taylor's expansion to F_p in the neighborhood of $\omega = W$ and retaining the first two terms, one obtains

$$(F_p)_{\omega=W+\delta} = (F_p)_{\omega=W} + \delta(\partial F_p/\partial\omega)_{\omega=W}. \quad (101)$$

A similar expression is obtained for Taylor's expansion of F_b. These two expressions are applied to (94) which is thus formulated as an equation of the first degree in δ; one obtains

$$\delta = M/N, \quad (102)$$

where

$$M = [\kappa^2/(W-\tilde{\omega})^2](F_b)_{\omega=W} - (F_p)_{\omega=W}, \quad (103)$$

and

$$N = (\partial F_p/\partial\omega)_{\omega=W} - [\kappa^2/(W-\tilde{\omega})^2](\partial F_b/\partial\omega)_{\omega=W}. \quad (104)$$

The expression (102) for δ is always real. Consequently, the waves that are nonconcurrent with the beam neither grow nor decay.

6. Stationary and Excited Concurrent Waves

Waves represented by the expression (96) may show significant instabilities and, therefore, in investigating the plasma-beam interaction, they are often the only ones that are considered.

The possible occurrence of instabilities in the waves (96) may be shown by applying the same arguments that lead to the conclusion that the waves $\omega = W + \delta$ are of no interest. Thus, substituting $\omega = \tilde{\omega} + \delta$ in (94) and using Taylor's expansion for F_p and F_b in the neighborhood of $\tilde{\omega}$, one obtains the following equation from (94):

$$(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}\delta^3 + (F_p)_{\omega=\tilde{\omega}}\delta^2 - \kappa^2(\partial F_b/\partial\omega)_{\omega=\tilde{\omega}}\delta - \kappa^2(F_b)_{\omega=\tilde{\omega}} = 0. \quad (105)$$

This equation is of third degree and therefore the solution for δ may be complex. Consequently, there may be an instability. The solutions expressing an instability would represent excited concurrent waves.

The occurrence of an instability depends on the coefficients of the cubic equation (105) and these are determined by the particular numerical value that one chooses to assign to the characteristic frequency. This frequency may have any value from 0 to $ck\cos\theta$. It is expressed as $\tilde{\omega}=ck\beta\cos\theta$, and, therefore, for a beam of given velocity it depends on the values of k and θ.

Consider the inequality

$$\left(\frac{\partial F_p/\partial\omega}{F_p}\right)\delta\gg 1. \tag{106}$$

This inequality is satisfied for values of $\tilde{\omega}$ in the region of longitudinal oscillation

$$\tilde{\omega}\sim[\omega_1^2+k^2s^2/3]^{\frac{1}{2}}, \tag{107}$$

and also in two other regions

$$\tilde{\omega}\sim ks/\sqrt{3} \quad\text{and}\quad \tilde{\omega}\sim(c^2k^2+\omega_1^2)^{\frac{1}{2}}$$

which are of no interest here.

If $\tilde{\omega}$ is not close to one of the three values listed above, then for sufficiently small values of κ the inequality

$$\left(\frac{\partial F_p/\partial\omega}{F_p}\right)_{\omega=\tilde{\omega}}\delta\ll 1, \tag{108}$$

will hold.

Assuming (106) holds, one may neglect the second term of (105) when added to the first and obtain

$$(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}\delta^3-\kappa^2(\partial F_b/\partial\omega)_{\omega=\tilde{\omega}}\delta-\kappa^2(F_b)_{\omega=\tilde{\omega}}=0. \tag{109}$$

Assume now that

$$\frac{|\kappa^2(\partial F_b/\partial\omega)_{\omega=\tilde{\omega}}|}{|\delta^2(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}|}\ll 1. \tag{110}$$

One can then neglect the second term of (109) when compared with the first and obtain the solution

$$\delta=\kappa^{\frac{2}{3}}\left(\frac{F_b}{(\partial F_p/\partial\omega)_{\omega=\tilde{\omega}}}\right)^{\frac{1}{3}}. \tag{111}$$

Since $\delta=O(\kappa^{\frac{2}{3}})$ one sees that for sufficiently small values of κ the assumption (110) is valid. The solution (111) yields an instability and can be written

$$|\mathrm{Im}(\delta)|=\kappa^{2/3}(\sqrt{3}/2^{4/3})\omega_1^{1/3}(1-s^2/3c^2\beta^2\cos^2\theta)^{1/6}\times(1-\beta^2\cos^2\theta)^{1/3}, \tag{112}$$

$$\mathrm{Re}(\delta)=-\kappa^{2/3}(1/2^{4/3})\omega_1^{1/3}(1-s^2/3c^2\beta^2\cos^2\theta)^{1/6}\times(1-\beta^2\cos^2\theta)^{1/3}. \tag{113}$$

The expressions (112) and (113) represent a generalization of the results obtained by Akhiezer and Fainberg and in paper I.

If the inequality (108) is satisfied, the first term of (105) may be neglected and one obtains

$$(F_p)_{\omega=\tilde{\omega}}\delta^2-\kappa^2(\partial F_b/\partial\omega)_{\omega=\tilde{\omega}}\delta-\kappa^2(F_b)_{\omega=\tilde{\omega}}=0. \tag{114}$$

If one also assumes

$$\left|\frac{\kappa^2(\partial F_b/\partial\omega)_{\omega=\tilde{\omega}}}{(F_p)_{\omega=\tilde{\omega}}\delta}\right|\ll 1, \tag{115}$$

the solution of (114) may be written as

$$\delta=\kappa[(F_b)_{\omega=\tilde{\omega}}/(F_p)_{\omega=\tilde{\omega}}]^{\frac{1}{2}}. \tag{116}$$

Again, since $\delta=O(\kappa)$ the inequality (115) is satisfied for sufficiently small values of κ.

The solution (116) can be written

$$\mathrm{Im}(\delta)=\kappa\left\{\frac{[1-\epsilon_t(\tilde{\omega})\beta^2\cos^2\theta-\epsilon_l(\tilde{\omega})\beta^2\sin^2\theta](1-\beta^2\cos^2\theta)}{\epsilon_l(\tilde{\omega})[1-\epsilon_t(\tilde{\omega})\beta^2\cos^2\theta]}\right\}^{\frac{1}{2}}. \tag{117}$$

The expression (117) represents in a generalized form the excitation coefficient for hybrid waves obtained in paper I. For $s=0$, one has $\epsilon_t=\epsilon_l=1-\omega_1^2/\omega^2$ and the above expression agrees for $\beta\ll 1$ with the corresponding expression obtained in paper I.

7. Comparison of Excited Concurrent Waves with those which are Concurrent with a Particle

Waves concurrent with a particle have a frequency represented by a real quantity $\tilde{\omega}$. Therefore, these waves neither grow nor decay. On the other hand, excited concurrent waves show an instability which is characterized by an excitation coefficient $\mathrm{Im}(\delta)>0$.

Consider now the behavior of instabilities in a plasma-beam system when κ tends to zero. Since the quantity δ tends to zero, the frequency of an excited concurrent wave approaches the limiting value $\omega=\tilde{\omega}$. Consequently, the closer one gets to the limit, the smaller is the difference between an excited concurrent wave and the wave concurrent with a particle. One is interested in selecting the waves concurrent with a particle that satisfy the Vavilov–Cherenkov criterion and in determining those excited waves which for $\kappa\to 0$ approach the Vavilov–Cherenkov waves.

The dispersion equation for a plasma-beam system has the form $F_{pb}=0$ and it depends on the quantity κ. It is of interest to determine the limiting form of the equation $F_{pb}=0$ for $\kappa\to 0$ and compare it with the dispersion equation (91) for Vavilov–Cherenkov waves. Consider in that connection the expression

$$\lim_{\kappa\to 0} F_{pb}.$$

It will be shown that the form of this limiting expression is different for longitudinal and for hybrid instabilities.

The limiting form of the dispersion equation $F_{pb}=0$ for the longitudinal instability is obtained by substituting $\omega=\tilde{\omega}+\delta$ in (94) where δ is given by (111) and assumiug $\kappa \to 0$. Thus,

$$\lim_{\kappa\to 0} (F_{pb})_{\omega=\tilde{\omega}} = (F_p)_{\omega=\tilde{\omega}}. \quad (118)$$

Therefore, when κ is small the dispersion equation for the longitudinal instability in a plasma-beam system differs very little from the dispersion equation for a thermal plasma (in the absence of a beam). For $\kappa \to 0$ the difference tends to zero and in the limit both dispersion equations become identical. It is noted that the dispersion equation $(F_p)_{\omega=\tilde{\omega}}=0$ expresses the necessary condition for the occurrence of a Vavilov–Cherenkov wave. Consequently, those waves concurrent with the beam which represent the longitudinal instability become in the limit Vavilov–Cherenkov waves.

Consider hybrid waves. Substituting in (94) $\omega=\tilde{\omega}+\delta$ where δ is given by (116), one obtains

$$\lim_{\kappa\to 0} (F_{pb})_{\omega=\tilde{\omega}} \neq (F_p)_{\omega=\tilde{\omega}}. \quad (119)$$

Consequently, when $\kappa \to 0$ the dispersion equation for hybrid waves does not tend to a limiting form which expresses the necessary condition for a Vavilov–Cherenkov wave.

One may conclude, therefore, that when the intensity of the beam tends to zero, or if the velocity of the beam approaches the velocity of light ($\kappa \to 0$), the parameter representing the growth of a longitudinal instability decreases, and in the limit the longitudinal wave becomes identical with a Vavilov–Cherenkov wave. Consequently, the longitudinal instability may be considered an "extension" of the Vavilov–Cherenkov effect. The hybrid instability is not as directly related to the Vavilov–Cherenkov effect.

The transition from an unstable longitudinal wave into the Vavilov–Cherenkov wave for decreasing κ, or the opposite transition for increasing κ is characterized by a change in the phase velocity of the wave. The growing wave becomes retarded with respect to the Vavilov–Cherenkov wave. This change in the phase velocity is expressed by a quantity $\mathrm{Re}(\delta)/\omega$, where $\mathrm{Re}(\delta)$ is given by (113).

8. Amplified Concurrent Waves

The presence of an excited concurrent wave indicates that the plasma-beam system is unstable. However, it does not indicate whether the instability is convective or non-convective. In order to resolve this question, it may be useful to introduce the concept of an "amplified concurrent wave." In that connection one can use the following procedure:

Assign real values to ω in the dispersion equation (94) and assume that the quantity k may be expressed in two different forms. In accordance with one formulation, one has:

$$k=\tilde{k}+\alpha, \quad (120)$$

where

$$\tilde{k}=\omega/(v\cos\theta). \quad (121)$$

In accordance with the other formulation, one has

$$k=K+\alpha, \quad (122)$$

where

$$|K-\tilde{k}| \gg \alpha. \quad (123)$$

Furthermore,

$$\alpha \to 0 \quad \text{for} \quad \kappa \to 0. \quad (124)$$

Applying a method similar to the one that led to Eqs. (100) and (102), it can be shown that both K and α are real, and, therefore, the wave (122) neither grows nor decays with respect to space coordinates. A significant situation may occur in a wave of type (120) when $\mathrm{Im}(\alpha)<0$. In such case the term $\mathrm{Im}(\alpha)$ is designated as the amplification coefficient and the wave (120) will be referred to as "an amplified concurrent wave." In order to determine the possible occurrence of such a wave, one substitutes $k=\tilde{k}+\alpha$ in (94) and using Taylor's expansion for F_p and F_b in the neighborhood of $\tilde{k}$, one obtains the following equation for α:

$$(\partial F_p/\partial k)\alpha^3\beta^3c^3\cos^2\theta+(F_p)\alpha^2\beta^2c^2\cos^2\theta - \kappa^2(\partial F_b/\partial k)\alpha-\kappa^2(F_b)=0, \quad (125)$$

where the terms in parentheses are evaluated at $k=\tilde{k}$.

The character of the roots of the above equation depends on the numerical value of $\tilde{k}$. For those values of $\tilde{k}$ for which α is real, the waves are stationary and concurrent. For those values of $\tilde{k}$ for which α is complex the waves corresponding to $\mathrm{Im}(\alpha)<0$ are "amplified and concurrent."

Using the formulations (105) and (125), one can determine the stability condition for any particular value of ω' and k' satisfying the relationship,

$$\omega'=ck'\beta\cos\theta. \quad (126)$$

Thus, an instability occurs if for $\omega'=\tilde{\omega}$ the equation (105) yields complex roots for δ in which $\mathrm{Im}(\delta)>0$. This instability is convective if for $k'=\tilde{k}$ Eq. (125) gives complex roots for α in which $\mathrm{Im}(\alpha)<0$. The instability is, however, nonconvective if for $k'=k$ the roots of Eq. (125) are real.

APPENDIX A

The phenomenological macroscopic parameters of a structurally isotropic plasma were formulated independently by Gertzenshtein[14] and by Linhard.[13] Both investigators pointed out, apparently for the first time, the occurrence of space dispersion in a plasma. There are at present two formulations of the macroscopic parameters of a thermal plasma. In one formulation the plasma is assumed to be "nonmagnetic," i.e., $\mu=1$, whereas in the other formulation it is assumed to be "magnetic." In the "nonmagnetic formulation" the properties of plasma may be accounted for by means of two functions: $\epsilon_t(\omega,\mathbf{k})$ and $\epsilon_l(\omega,\mathbf{k})$. The expression $\epsilon_t(\omega,\mathbf{k})$ represents the "transverse capacitivity" of plasma and the expression $\epsilon_l(\omega,\mathbf{k})$ represents the "longitudinal capacitivity." In the "magnetic formulation" one uses also two functions: $\epsilon(\omega,\mathbf{k})$ and $\mu(\omega,\mathbf{k})$. The function $\epsilon(\omega,\mathbf{k})$ is identical with $\epsilon_l(\omega,\mathbf{k})$ and represents the capacitivity of plasma. The function $\mu(\omega,\mathbf{k})$ represents the magnetic permeability and can be expressed as

$$1/\mu(\omega,\mathbf{k})=1-(\omega^2/c^2k^2)[\epsilon_t(\omega,\mathbf{k})-\epsilon_l(\omega,\mathbf{k})]. \quad \text{(A1)}$$

Therefore, if the plasma is described in terms of an electron-velocity distribution $f(v)$, one obtains

$$\frac{1}{\mu(\omega,k)}=1-\frac{\omega\omega_0}{c^2k^2}\int\frac{(\mathbf{k}\cdot\mathbf{v})f(v)d\mathbf{v}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2}. \quad \text{(A2)}$$

The formulation of the "nonmagnetic plasma" was given both by Gertzenshtein and by Linhard, whereas the formulation of the "magnetic plasma" was introduced by Linhard. These two formulations have been investigated further by Rukhadze and Silin.[15]

Therefore, there are two alternative descriptions of a plasma in terms of its macroscopic parameters. In accordance with one description, the magnetic intensity **H** and the electric displacement **D** are expressed in terms of the microscopic field quantities **B** and **E** as follows:

$$\mathbf{H}=\mathbf{B}, \quad \text{(A3)}$$

$$\begin{vmatrix}D_t\\D_l\end{vmatrix}=\begin{pmatrix}\epsilon_t & 0\\0 & \epsilon_l\end{pmatrix}\begin{vmatrix}E_t\\E_l\end{vmatrix}. \quad \text{(A4)}$$

(The transverse components of **E** and **D** are designated as E_t and D_t and the corresponding longitudinal components are E_l and D_l.) In accordance with the other description, one can write

$$\mathbf{H}=[1/\mu(\omega,\mathbf{k})]\mathbf{B}, \quad \text{(A5)}$$

and

$$\mathbf{D}=\epsilon(\omega,\mathbf{k})\mathbf{E}. \quad \text{(A6)}$$

Linhard was concerned primarily with the fact that one can apply to plasma parameters two different formulations. The physical nature of plasma was not considered and the question as to whether plasma is "magnetic" or "nonmagnetic" was not discussed. On the other hand, Rukhadze and Silin assumed that a thermal plasma is magnetic ($\mu\neq 1$).

Under certain conditions both formulations give similar results. Thus, one can describe the propagation of a transverse wave by assuming that plasma is either magnetic or nonmagnetic. However, there are instances when the knowledge of the magnetic behavior of plasma is essential. One faces then a difficult choice as to which one of the two formulations represents the physical reality. The appropriate decision does not depend on the mathematical formulation but on the fundamental physical assumptions. A plasma is considered as "magnetically polarizable" if an electromagnetic field induces a magnetic polarization and conversely, a plasma is not "magnetically polarizable" if there is no magnetic polarization in presence of an electromagnetic field. This problem was discussed in paper II. It was found that when a plasma is not in thermal equilibrium there is a magnetic polarization induced by an electromagnetic field. On the other hand, when a thermal equilibrium is established, there is no magnetic polarization.

APPENDIX B

Equations describing the behavior of a charge equilibrated electron plasma having temperature T may be represented in hydrodynamic form as follows[20]:

$$(d/dt)\mathbf{v}'+(1/mN)\,\mathrm{grad}p'+(e/m)\mathbf{E}'=0, \quad \text{(B1)}$$

$$\mathrm{curl}\mathbf{E}'=-(1/c)\partial\mathbf{B}'/\partial t, \quad \text{(B2)}$$

$$\mathrm{curl}\mathbf{B}'=(1/c)\partial\mathbf{E}'/\partial t+(4\pi/c)\mathbf{j}', \quad \text{(B3)}$$

$$\mathrm{div}\mathbf{B}'=0, \quad \text{(B4)}$$

$$\mathrm{div}\mathbf{E}'=-4\pi n'e, \quad \text{(B5)}$$

$$\mathbf{j}'=-Ne\mathbf{v}', \quad \text{(B6)}$$

$$(\partial n'/\partial t)+\mathrm{div}N\mathbf{v}'=0. \quad \text{(B7)}$$

The density of plasma is $N+n'$ where N is the unperturbed term and n' represents the perturbation due to the electromagnetic field $\mathbf{E}'$, $\mathbf{B}'$. Other small terms representing the disturbance are as follows: $\mathbf{v}'$ is the velocity; p' represents the pressure, and $\mathbf{j}'$ is the electric current. The quantity p' may be expressed as

$$p'=n'K_BT, \quad \text{(B8)}$$

where K_B is the Boltzmann constant. The electromagnetic field has the form of a plane wave

$$\mathbf{E}'=\mathbf{E}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t);\quad \mathbf{B}'=\mathbf{B}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t).$$

The terms representing the perturbation are:

$$n'=n\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t),\quad \mathbf{v}'=\mathbf{v}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t),$$

and

$$p'=p\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t).$$

Equation (B1) can be expressed as

$$-i\omega\mathbf{v}+(p/mN)\mathbf{k}+(e/m)\mathbf{E}=0. \quad \text{(B9)}$$

One obtains from (B7)

$$n=(N/\omega)(\mathbf{k}\cdot\mathbf{v}). \quad \text{(B10)}$$

Taking into account (B8) and (B10), Eq. (B1) can be expressed as

$$-i\omega\mathbf{v}+[iK_BT(\mathbf{k}\cdot\mathbf{v})\mathbf{k}/m\omega]+(e/m)\mathbf{E}=0. \quad \text{(B11)}$$

[20] See, for instance, J. F. Denisse and J. L. Delcroix, *Theorie des Ondes dans les Plasmas* (Dunod, Paris, 1961).

One has the relationship[21]

$$\mathbf{j}' = -Ne\mathbf{v}' = c\,\mathrm{curl}\mathbf{M}' + (\partial\mathbf{P}'/\partial t), \tag{B12}$$

where $\mathbf{P}'$ and $\mathbf{M}'$ represent the electric and magnetic polarization. Since the plasma is nonmagnetic, one has $M'=0$. Using the relationship $\mathbf{P}'=\mathbf{P}\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, one can express the equality (B12) as follows:

$$\mathbf{v} = (i\omega/Ne)\mathbf{P}. \tag{B13}$$

[21] See, for instance, W. K. H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

Combining (B11) and (B13), one obtains

$$(\omega^2/Ne)\mathbf{P} - [K_BT(\mathbf{k}\cdot\mathbf{P})\mathbf{k}/Nem] + (e/m)\mathbf{E} = 0. \tag{B14}$$

Using the relationship (74), Eq. (B14) can be expressed as

$$(\omega^2/Ne)\mathbf{P} - [s^2(\mathbf{k}\cdot\mathbf{P})\mathbf{k}/3Ne] + (e/m)\mathbf{E} = 0, \tag{B15}$$

or

$$\omega^2\mathbf{P} - (s^2/3)(\mathbf{k}\cdot\mathbf{P})\mathbf{k} + (\omega_1^2/4\pi)\mathbf{E} = 0, \tag{B16}$$

where $\omega_1^2 = 4\pi Ne^2/m$.

Equation (B16) leads directly to the relationships (79), (83), and (84) given in the text.

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Phase Transition in Elastic Disks*

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The study of a two-dimensional system consisting of 870 hard-disk particles in the phase-transition region has shown that the isotherm has a van der Waals-like loop. The density change across the transition is about 4% and the corresponding entropy change is small.

A STUDY has been made of a two-dimensional system consisting of 870 hard-disk particles. Simultaneous motions of the particles have been calculated by means of an electronic computer as described previously.[1] The disks were again placed in a periodically repeated rectangular array. The computer program has been improved such that about 200 000 collisions per hour can be calculated by the LARC computer regardless of the number of particles in the system. This speed made it possible to follow large systems for several million collisions.

It became necessary to study larger systems in the phase transition region when for smaller ones in three dimensions, it did not seem to be possible for the two phases to exist together in equilibrium.[2,3] Even in the largest three-dimensional system investigated with the improved program (500 hard spheres), the particles were either all in the fluid phase or all in the crystalline phase. The system would typically remain in one phase for many collisions. The occasional shift from one phase to the other would be accompanied by a change of pressure. The equation of state was represented by two disconnected branches overlapping in the density range of the transition, since with the limited number of phase interchanges it was not possible to average the two branches.

Two-dimensional systems were then studied, since the number of particles required to form clusters of particles of one phase of any given diameter is less than in three dimensions. Thus, an 870 hard-disk system is effectively much larger than a 500 hard-sphere system. First, however, it was necessary to establish that small two-dimensional systems behave analogously to the three-dimensional systems. This is illustrated in Fig. 1 by the two disconnected branches drawn lightly through the triangular points for a 72-particle system. In that figure, the reduced pressure pA_0/NkT is plotted against the reduced area A/A_0, where A_0 is the area of the system at close packing. In the region of A/A_0 from 1.33 to 1.35 the system fluctuated infrequently between a high-pressure fluid branch and a low-pressure crystalline branch, while at A/A_0 of 1.31 and higher densities the solid phase was always stable.

For the larger 870-particle system, however, the two phases exist side by side. One piece of evidence for this coexistence is the cathode-ray tube pictures described earlier (see Fig. 2).[1] The trajectories of the particles plotted on the oscilloscope show regions where the particles are localized (crystallites) in between regions of mobile particles (fluid). Further evidence is the characteristically large pressure fluctuations in the phase transition region where two states can exist with almost equal probability. The extent of the fluctuations in a typical run of about 10 million collisions is obtained

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