Renormalizability of Gauge Theories*

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By generalization of methods developed by Kamefuchi, O'Raifeartaigh, and Salam, conditions for renormalizability of general gauge theories of massive vector mesons are derived. These conditions are stated explicitly in Eqs. (39) and (40) of the text. It is shown that all theories based on simple Lie groups (with the one exception of the neutral vector meson theory in interaction with a conserved current) are unrenormalizable.

1. INTRODUCTION

A NUMBER of recent investigations[1] have discussed and established the nonrenormalizability of massive vector meson theories of Yang-Mills type. The purpose of this paper is twofold: first, to clarify the essentially simple structure of these proofs and to extend it to all other gauge theories; secondly, to suggest that one can indeed secure renormalizability of vector meson theories if no bare meson mass terms occur in the Lagrangian. Nonzero physical masses can, however, be computed in a self-consistent manner employing, for example, the techniques used previously by Nambu.[2]

2. NEUTRAL PSEUDOVECTOR THEORY; THE γ_5 GAUGE

All ingredients required for the investigation of renormalizability of gauge theories are present in the simple theory of a single neutral pseudovector meson U_μ interacting with a nucleon.[3] Write the Lagrangian,

$$L=L_1+L_2+L_{\text{meson}}, \tag{1}$$

$$\begin{aligned} L_1&=-\bar{\psi}\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ig\gamma_5 U_\mu\right)\psi,\\ L_2&=-m\bar{\psi}\psi,\\ L_{\text{meson}}&=-(1/4)F_{\mu\nu}F_{\mu\nu}-(1/2)\kappa^2 U_\mu U_\mu, \end{aligned} \tag{2}$$

where

$$F_{\mu\nu}=\frac{\partial}{\partial x_\mu}U_\nu-\frac{\partial}{\partial x_\nu}U_\mu. \tag{3}$$

For the (γ_5 gauge) transformations,

$$\psi=S\psi'=\exp[(-i\gamma_5 g)B/\kappa]\psi', \tag{4}$$

$$U_\mu=U_\mu'+\frac{1}{\kappa}\frac{\partial B}{\partial x_\mu}, \tag{5}$$

L_1 is form-invariant, but L_2 is not.

(1) The first step is to replace U_μ by the Stueckelberg combination,

$$U_\mu=A_\mu+(1/\kappa)(\partial B/\partial x_\mu). \tag{6}$$

We find

$$\begin{aligned} L_{\text{meson}}\equiv&-(1/2)[(\partial_\mu A_\nu)^2+\kappa^2 A_\nu^2]\\ &-(1/2)[(\partial_\mu B)^2+\kappa^2 B^2]+(1/2)(\partial_\mu A_\mu+\kappa B)^2. \end{aligned} \tag{7}$$

One imposes the subsidiary condition

$$(\partial_\mu A_\mu+\kappa B)^{\text{in}}|\Psi\rangle=0. \tag{8}$$

With

$$\begin{aligned} (A_\mu{}^{\text{in}}A_\nu{}^{\text{in}})_+&=\delta_{\mu\nu}\Delta_F,\\ (B^{\text{in}}B^{\text{in}})_+&=\Delta_F, \end{aligned} \tag{9}$$

the S matrix equals

$$T\exp\left[-ig\int\bar{\psi}^{\text{in}}\gamma_\mu\gamma_5\left(A_\mu{}^{\text{in}}+\frac{1}{\kappa}\frac{\partial B^{\text{in}}}{\partial x_\mu}\right)\psi^{\text{in}}\right]. \tag{10}$$

(2) The next step is make a change of variables ψ to ψ':

$$\psi=S\psi'=\exp(-i\gamma_5 gB/\kappa)\psi', \tag{11}$$

so that $L(\psi,A_\mu,B)$ changes to

$$\begin{aligned} L'=-\bigg[&\bar{\psi}'\gamma_\mu\left(\frac{\partial}{\partial x_\mu}-ig\gamma_5 A_\mu\right)\psi'\\ &+m\bar{\psi}'\exp\left(-2i\gamma_5 g\frac{B}{\kappa}\right)\psi'\bigg]+L_{\text{meson}}(A_\mu,B). \end{aligned} \tag{12}$$

The new variables ψ' have been defined in such a way that after the transformation the B field disappears from the combination $\bar{\psi}\gamma_\mu(\partial/\partial x_\mu-igA_\mu-ig\partial B/\partial x_\mu)\psi$. This is always possible because of the basic gauge structure of the theory. Further the variable change (11) implies (as can be seen by an adiabatic switching off of g) that

$$\psi^{\text{in}}=\psi'^{\text{in}}. \tag{13}$$

Thus the work of Chisholm,[4] Kamefuchi, O'Raifeartaigh, and Salam,[1] or the more rigorous theorems proved by Borchers[5] guarantee the intuitive result that the S matrix (10) set up using the original interaction is

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[1] A. Komar and A. Salam, Nuclear Phys. **21**, 624 (1960); S. Kamefuchi and H. Umezawa *ibid.* **23**, 399 (1961); S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, *ibid.* (to be published).
[2] Y. Nambu, Phys. Rev. **122**, 345 (1961).
[3] A. Salam, Nuclear Phys. **18**, 681 (1960); S. Kamefuchi, *ibid.* **18**, 691 (1960).
[4] Y. Chisholm, Nuclear Phys. **26**, 469 (1961).
[5] H. J. Borchers, Nuovo cimento **15**, 784 (1960).

identical with the S matrix

$$T\exp\left(-i\int\{\bar{\psi}'^{\mathrm{in}}(g\gamma_\mu\gamma_5A_\mu{}^{\mathrm{in}})\psi'^{\mathrm{in}} + m\bar{\psi}'^{\mathrm{in}}[\exp(-2i\gamma_5gB^{\mathrm{in}}/\kappa)-1]\psi'^{\mathrm{in}}\}\right), \quad (14)$$

which is based on the transformed Lagrangian (12). The possible nonrenormalizability of the theory had its origin so far as the original S matrix is concerned in the derivative term containing $\partial B^{\mathrm{in}}/\partial x_\mu$. In the expression (14) this shows itself (more conclusively) in the exponential factor containing B^{in} which has made its appearance in the interaction Lagrangian. Further the expression (14) shows clearly that the nonrenormalizability is associated with the non-gauge-invariant part of the Lagrangian (i.e., the nucleon mass term). If $m=0$ the theory would be renormalizable.

3. GENERAL GAUGE THEORIES

In this section we briefly review the structure of gauge theories.

Consider a set of Hermitian fields χ with the linear kinetic energy terms,

$$\chi\beta_\mu\frac{\partial}{\partial x_\mu}\chi. \quad (15)$$

β_μ are matrices characteristic of the spin of the particles.[6]

We assume that (15) is invariant for the transformation

$$\chi=S\chi', \quad (16)$$

where

$$S=\exp ig(T^ib^i), \quad (17)$$

and b^i's are constants. Here T^i are a set of n Hermitian matrices (which of course commute with β_μ) and which satisfy[7]

$$[T^i,T^j]=C_k{}^{ij}T_k. \quad (18)$$

If now S depends on x_μ, to preserve invariance $\partial/\partial x_\mu$ must be replaced in the well-known manner by the combination $(\partial/\partial x_\mu-igU_\mu)$. Here $U_\mu=T^iU_\mu{}^i$ and $U_\mu{}^i$ are a set of n vector fields whose transformation character is given by[8]

$$U_\mu'=S^{-1}U_\mu S+(i/g)S^{-1}(\partial S/\partial x_\mu). \quad (19)$$

This can be inferred from the relation,

$$(\partial/\partial x_\mu-igU_\mu)\chi=S(\partial/\partial x_\mu-igU_\mu')\chi', \quad (20)$$

which is necessary in order that

$$\chi\beta_\mu(\partial/\partial x_\mu-igU_\mu)\chi=\chi'\beta_\mu(\partial/\partial x_\mu-igU_\mu')\chi'. \quad (21)$$

Defining

$$F_{\mu\nu}=(\partial/\partial x_\mu-igU_\mu)U_\nu-(\partial/\partial x_\nu-igU_\nu)U_\mu,$$

it is easy to verify, from (19), that

$$F_{\mu\nu}'=S^{-1}F_{\mu\nu}S. \quad (22)$$

Thus the Lagrangian for the vector fields, if it must also be gauge invariant, should have[9] the form

$$-(1/4)\,\mathrm{Tr}F_{\mu\nu}F_{\mu\nu}. \quad (23)$$

Summarizing, a gauge-invariant theory is given by

$$\mathfrak{L}=[-\chi\beta_\mu(\partial/\partial x_\mu-igU_\mu)\chi-m\chi\chi-\tfrac{1}{4}\,\mathrm{Tr}F_{\mu\nu}F_{\mu\nu}] \quad (24)$$

and this theory is invariant for the transformations

$$\chi=S\chi',$$
$$U_\mu=SU_\mu'S^{-1}-(i/g)(\partial S/\partial x_\mu)S^{-1}. \quad (25)$$

A noninvariant meson mass term $\mathfrak{L}_{\mathrm{mass}}$ may be added to (24), where

$$\mathfrak{L}_{\mathrm{mass}}=-\mathrm{Tr}(1/2)\kappa^2U_\mu U_\mu. \quad (26)$$

4. RENORMALIZABILITY

To study renormalizability the first step (as in Sec. 2) is to introduce the Stueckelberg fields,

$$U_\mu=A_\mu+(1/\kappa)\partial B/\partial x_\mu. \quad (27)$$

Here also $A_\mu=A_\mu{}^i(x)T^i$ and $B=B^i(x)T^i$. The second step is to change the variables χ to χ' and U_μ to U_μ'

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[6] For stating the gauge principle it seems desirable to start with (Dirac type) linear (rather than the Klein-Gordon type quadratic) field equations for all fields, in the manner suggested by Schwinger in his action principle papers. This is because a restriction to linear (rather than quadratic) derivatives in free Lagrangians leaves less scope for ambiguity in the specification of the Lagrangians. This in turn avoids ambiguities in the interaction terms of gauge theories which are generated by the principle of replacing $\partial/\partial x_\mu$ by combinations like $(\partial/\partial x_\mu-igU_\mu)$.

[7] The structure constants are required further to satisfy the relations $C_k{}^{ij}=-C_k{}^{ji}$; $C_l{}^{im}C_m{}^{jk}+C_l{}^{jm}C_m{}^{ki}+C_l{}^{km}C_m{}^{ij}=0$.

[8] If $S=\exp(gX)$, then

$$S^{-1}\frac{\partial S}{\partial x_\mu}=g\frac{\partial X}{\partial x_\mu}-\frac{g^2}{2!}\left[X,\frac{\partial X}{\partial x_\mu}\right]+\frac{g^3}{3!}\left[X,\left[X,\frac{\partial X}{\partial x_\mu}\right]\right]-\cdots,$$

$$S^{-1}\frac{\partial X}{\partial x_\mu}S=\frac{\partial X}{\partial x_\mu}-g\left[X,\frac{\partial X}{\partial x_\mu}\right]+\frac{g^2}{2!}\left[X,\left[X,\frac{\partial X}{\partial x_\mu}\right]\right]-\cdots.$$

If $X=ib^iT^i$, from (18) it is clear that $S^{-1}(\partial S/\partial x_\mu)$ must have the form B^iT^i.

It is worth remarking that irrespective of any particular representation for the generating matrices T^i, the transformation properties of the vector fields $U_\mu{}^i$ depend only on the structure constants of the group. Thus infinitesimally (19) is equivalent to

$$U_\mu'=U_\mu+[U_\mu,gX]+(i/\kappa)\partial X/\partial x_\mu,$$

i.e.,

$$U_\mu'^i=U_\mu{}^i+igC_i{}^{jk}U_\mu{}^jb^k-(1/\kappa)\partial b^i/\partial x_\mu.$$

[9] According to our earlier prescription, one should write a linearized version of (23). This will have the form

$$\mathrm{Tr}\{-(1/2)F_{\mu\nu}(\partial_\mu U_\nu-\partial_\nu U_\mu-igU_\mu U_\nu-igU_\nu U_\mu)-(1/2)F_{\mu\nu}F_{\mu\nu}.\}$$

For convenience of exposition, however, we continue to work with (23).

where,[10] as in Sec. 2, the transformation matrix S has the form

$$S=\exp(igB/\kappa), \tag{28}$$

and χ' and U_μ' are defined as in (25). Clearly, from gauge invariance,

$$\mathfrak{L}(\chi,U_\mu)=\mathfrak{L}(\chi',U_\mu'), \tag{29}$$

while

$$\begin{aligned}\mathfrak{L}_{\text{mass}}&=-\tfrac{1}{2}\operatorname{Tr}\kappa^2U_\mu U_\mu\\&=-\tfrac{1}{2}\operatorname{Tr}\left[\kappa^2U_\mu'U_\mu'+\frac{\kappa^2}{g^2}\frac{\partial S}{\partial x_\mu}\frac{\partial S^{-1}}{\partial x_\mu}-\frac{2i\kappa}{g}U_\mu'S^{-1}\frac{\partial S}{\partial x_\mu}\right]\\&=-\tfrac{1}{2}\operatorname{Tr}\left\{\kappa^2\left(U_\mu'+\frac{1}{\kappa}\frac{\partial B}{\partial x_\mu}\right)^2+\left[\frac{\kappa^2}{g^2}\frac{\partial S}{\partial x_\mu}\frac{\partial S^{-1}}{\partial x_\mu}-\left(\frac{\partial B}{\partial x_\mu}\right)^2\right]-\frac{2i\kappa}{g}U_\mu'\left(S^{-1}\frac{\partial S}{\partial x_\mu}-ig\frac{\partial B}{\partial x_\mu}\right)\right\}.\end{aligned} \tag{30}$$

One may now set up the S matrix for the theory based on $\mathfrak{L}(\chi',U_\mu')+\mathfrak{L}_{\text{mass}}(U_\mu',B)$ and use χ' and U_μ' variables.[11] Now

$$\frac{i}{g}S^{-1}\frac{\partial S}{\partial x_\mu}=-\frac{1}{\kappa}\frac{\partial B}{\partial x_\mu}+O(g), \tag{31}$$

so that

$$U_\mu'=A_\mu+O(g).$$

Therefore

$$U'^{\text{in}}=A_\mu{}^{\text{in}}; \tag{33}$$

also

$$\chi'^{\text{in}}=\chi^{\text{in}}. \tag{34}$$

The new S matrix equals

$$T\exp i\int\mathfrak{L}'(A_\mu{}^{\text{in}},\chi^{\text{in}})+\mathfrak{L}''(A_\mu{}^{\text{in}},B^{\text{in}}), \tag{35}$$

where

$$\mathfrak{L}'(A^{\text{in}},\chi^{\text{in}})=\mathfrak{L}(A^{\text{in}},\chi^{\text{in}})$$

minus the free Lagrangians

$$\left(-\chi^{\text{in}}\beta_\mu\frac{\partial}{\partial x_\mu}\chi^{\text{in}}-m\chi^{\text{in}}\chi^{\text{in}}-\frac{\operatorname{Tr}}{4}\left(\frac{\partial A_\mu{}^{\text{in}}}{\partial x_\nu}-\frac{\partial A_\nu{}^{\text{in}}}{\partial x_\mu}\right)^2\right),$$

and

$$\begin{aligned}\mathfrak{L}''(A^{\text{in}},B^{\text{in}})=&-\tfrac{1}{2}\operatorname{Tr}\left[\frac{\kappa^2}{g^2}\frac{\partial S}{\partial x_\mu}\frac{\partial S^{-1}}{\partial x_\mu}-\left(\frac{\partial B^{\text{in}}}{\partial x_\mu}\right)^2\right]\\&-\tfrac{1}{2}\operatorname{Tr}\left[\frac{-2i\kappa}{g}A_\mu{}^{\text{in}}\left(S^{-1}\frac{\partial S}{\partial x_\mu}-ig\frac{\partial B^{\text{in}}}{\partial x_\mu}\right)\right]\end{aligned} \tag{36}$$

with

$$S=\exp(igB^{\text{in}}/\kappa). \tag{37}$$

It is easy to satisfy oneself that no nonrenormalizable infinities arise from the terms contained in $\mathfrak{L}'(\chi^{\text{in}},A_\mu{}^{\text{in}})$. However, $\mathfrak{L}''(A_\mu{}^{\text{in}},B^{\text{in}})$ has exponential terms from the explicit occurrence of $S(x)$ and these will produce horrible infinities unless either

$$\kappa=0, \tag{38}$$

or

$$\operatorname{Tr}\left[\frac{\kappa^2}{g^2}\frac{\partial S}{\partial x_\mu}\frac{\partial S^{-1}}{\partial x_\mu}-\left(\frac{\partial B^{\text{in}}}{\partial x_\mu}\right)^2\right]=0, \tag{39}$$

and

$$\operatorname{Tr}\left[A_\mu{}^{\text{in}}\left(S^{-1}\frac{\partial S}{\partial x_\mu}-ig\frac{\partial B^{\text{in}}}{\partial x_\mu}\right)\right]=0. \tag{40}$$

Thus, the problem of finding renormalizable massive vector meson gauge theories is the problem of finding those gauge transformations (and the corresponding matrices T^i) which satisfy (39) and (40). These conditions are the major results of this paper.

(1) For a massive neutral vector meson interacting for example with a nucleon

$$S=\exp(igB/\kappa).$$

Thus, both (39) and (40) are satisfied, and the theory is renormalizable.

(2) In general, (39) and (40) can be satisfied, provided

$$\operatorname{Tr}T^iT^j\equiv 0. \tag{41}$$

Now as remarked earlier (footnote 8) the matrix representations of T^i appropriate to the fields $U_\mu{}^i$ are given by the structure constants $C_i{}^{jk}$ themselves. For simple Lie groups, Ionides and Gell-Mann and Glashow[12] have shown that for such matrices

$$\operatorname{Tr}T^iT^j=\lambda\delta^{ij}, \tag{42}$$

so that (39) and (40) can not be satisfied.

The only non-gauge-invariant term in the above work was the vector meson mass term. Clearly any other noninvariant interaction term $L(\chi)$ will get transformed into $L(S\chi')$ and the exponentials of the B field contained in S will inevitably produce nonrenormalizable infinities in the same way as those arising from the transformation of the nucleon term $(m\bar{\psi}\psi)$ to $m\bar{\psi}'\exp(-2i\gamma_5 gB/\kappa)\psi$ in Sec. 2.

[10] The change of variables U_μ to U_μ' is necessary in the general case treated above because here $U_\mu{}^i$ (as well as χ) are "source" fields. In the neutral pseudovector meson case of Sec. 2 the meson field was not a source field and did not itself undergo a transformation involving S.

[11] That Borcher's results can be extended to the case of vector mesons has been proved by S. Kamefuchi (private communication).

[12] P. Ionides, Nuclear Phys. (to be published); S. Glashow and Gell-Mann, Ann. Phys. (New York) **15**, 439 (1961).

5. RENORMALIZABLE VECTOR MESON THEORIES

The only way to make gauge theories renormalizable seems to be to take the bare mass $\kappa=0$. One may then hope to compute the physical mass in a self-consistent manner as follows: Replace

$$-\tfrac{1}{4}(\partial U_\mu/\partial x_\nu-\partial U_\nu/\partial x_\mu)^2$$

in the free-meson Lagrangian by the non-gauge-invariant term $-\frac{1}{2}(\partial U_\mu/\partial x_\nu)^2$. This is quite similar to the corresponding procedure in electrodynamics. The equations of motion now read ($\kappa=0$)

$$\Box^2 U_\mu=J_\mu, \tag{43}$$

instead of

$$\frac{\partial}{\partial x_\nu}\left[\frac{\partial}{\partial x_\nu}U_\mu-\frac{\partial}{\partial x_\mu}U_\nu\right]=J_\mu. \tag{44}$$

Notice that $\partial J_\mu/\partial x_\mu=0$ (from the gauge invariance of the theory) as before, so that

$$\Box^2\partial U_\mu/\partial x_\mu=0. \tag{45}$$

One may now impose on one space-like surface

$$(\partial U_\mu/\partial x_\mu)|\Psi\rangle=(\partial/\partial t)(\partial U_\mu/\partial x_\mu)|\Psi\rangle=0$$

so that from (45) it follows that

$$\partial U_\mu/\partial x_\mu|\Psi\rangle=0 \tag{46}$$

for all physical states at all times. We are now ready to set up a self-consistent scheme for computation of the physical mass. We follow identically the lines of Nambu's[2] calculation of electron self-mass in a γ_5-invariant theory, where one rewrites the Lagrangian of the theory $\bar\psi\gamma_\mu(\partial/\partial x_\mu)\psi+ie\bar\psi\gamma_\mu A_\mu\psi$ in the form $\bar\psi(\gamma_\mu(\partial/\partial x_\mu)+m)\psi+[ie\bar\psi\gamma_\mu\psi A_\mu-m\bar\psi\psi]$. The first part $\bar\psi(\gamma_\mu(\partial/\partial x_\mu)+m)\psi$ is treated as the free Lagrangian in an interaction representation while the rest is the interaction term containing, as usual, the self-mass part $m\bar\psi\psi$. The self-mass δm ($=m$) is computed as the sum of the usual proper self-energy graphs whose contributions are calculated using the electron propagator $1/(p+m)$. Thus, to the second order in e, this self-consistent method gives $m=\frac{3}{4}\alpha m\ln(\Lambda^2/m^2)$ as the equation determining m (Λ is a cutoff mass). Besides the trivial solution $m=0$, there is the second solution for m given by $1=\frac{3}{4}\alpha\ln(\Lambda^2/m^2)$. More generally, m is the solution of the relation

$$\int^{\Lambda 2/m2} x\rho_1(x^2)dx^2=\int^{\Lambda 2/m2}\rho_2(x^2)dx^2,$$

where ρ_1 and ρ_2 are Lehmann's spectral function.

For the vector meson case, the procedure is similar; one adds $-(1/2)\mu^2U_\mu U_\mu$ to the meson Lagrangian and subtracts the same term from the interaction terms. Equation (46) is unaltered so that the unwanted negative energy components of the vector field do not appear in the physical states as a result of the subsidiary condition. This, of course, is much weaker than the usual operator equation $\partial U_\mu^{\text{in}}/\partial x_\mu=0$ which normally holds for massive vector mesons and eliminates the zero-spin negative-energy particles from the theory. The meson propagator in the S matrix comes from the free-meson Lagrangian

$$-(1/2)(\partial U_\mu^{\text{in}}/\partial x_\nu)^2-(1/2)\mu^2U_\mu^{\text{in}}U_\mu^{\text{in}},$$

and therefore has the form $\delta_{\mu\nu}/(p^2-\mu^2)$. This ensures that all calculations in such a theory are similar to those in electrodynamics so far as the degree of divergence of S-matrix elements is concerned. Thus, the theory is renormalizable.

A proof of nonrenormalizability of general massive vector meson theories was first given by Ionides,[12] following the lines of the original proof of Kamefuchi and Umezawa, by constructing an explicit unitary transformation to eliminate the B field. A proof similar to that of Ionides[12] has also been given recently.[13] In our work, we have found the construction of such unitary transformations unnecessary.

Note added in proof. It is likely that the theory described in this section, like all self-consistent theories, contains zero mass particles; see J. Goldston, Abdus Salam, and S. Weinberg, Phys. Rev. (to be published).

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[13] H. Umezawa, M. Konuma, and M. Wada (to be published).