Non-Abelian Gauge Fields. Relativistic Invariance

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A simple criterion for Lorentz invariance in quantum field theory is stated as a commutator condition relating the energy density to the momentum density. With its aid a relativistically invariant radiation-gauge formulation is devised for a non-Abelian vector-gauge field coupled to a spin-$\frac{1}{2}$ Fermi field.

INTRODUCTION

It has been the historical role of gauge-variant field systems to pose the greatest challenge to relativistic quantum-field theory. From the first beginnings of a general quantum electrodynamics in the hands of Heisenberg and Pauli, difficulties were encountered, owing to the absence in the Lagrange function of the time derivative of some of the field variables, which frustrated the application of the simplest canonical quantization scheme. Two general responses to this situation can be distinguished. In the first of these, the physical system is accepted for what it is; the gauge variance of the field is interpreted to mean that not all the field components at a given time are fundamental dynamical variables, and the latter are identified. We shall describe this view point as the method of the radiation gauge.[1] It can be characterized by the desire to clarify the quantum nature of the system, be it at the expense of manifest Lorentz invariance. The second approach reverses this order of priority. Although there are several versions, all share the feature that the physical system is modified in order to restrict the group of gauge transformations and thereby extend the status of fundamental dynamical variable to all field components. The states of physical interest must then be related to the states of this larger system. These devices will be described collectively as the method of the Lorentz gauge.[2]

It is usual to assert that the two viewpoints are equivalent, and that a Lorentz gauge method has the advantage of calculational simplicity. But, against the validity of the Lorentz gauge methods as the basis of a general theory, must be arrayed the body of experience which indicates that the nature of the quantum vector space for a system with an infinite number of degrees of freedom is intimately associated with the dynamics and that no operator transformation connecting the states of different dynamical systems can be guaranteed to exist.[3] For this reason, combined with the conviction that an intrinsic method is superior in economy of concept to the artifice of imbedding the physical system in another kinematical and dynamical framework, we reject all Lorentz gauge formulations as unsuited to the role of providing the fundamental operator foundation for a gauge variant field system.

The radiation gauge formulation is three dimensional in structure, and Lorentz invariance must be verified by explicit calculation. In the electromagnetic or Abelian gauge-field situation one can easily exhibit the operator gauge transformation that is induced by a Lorentz transformation. The covariance of all aspects of the theory can then be checked directly. Such a program is vastly more complicated for non-Abelian gauge fields, particularly since the Lagrange function is ambiguous, in a manner that influences Lorentz transformation properties. Thus, we are in grave need of a simple criterion for Lorentz invariance if we are to select a satisfactory theory from a class of theories that are acceptable in three dimensions. This is what we propose to supply in the present paper.

THE COMMUTATOR CONDITION

Consider a field system for which the fundamental dynamical variables obey equal time commutator or anticommutator relations of the form

$$x^0=x^{0\prime}:\quad [\chi(x),\chi(x')]_{\pm}=c(\mathbf{x}-\mathbf{x}'),$$

where $c(\mathbf{x}-\mathbf{x}')$ is a numerical matrix function. Let the system be characterized by Hermitian momentum density operators $T^0{}_k(x)$, $k=1, 2, 3$, such that the operators of linear momentum,

$$P_k=\int(d\mathbf{x})T^0{}_k(x),$$

and angular momentum,

$$J_{kl}=\int(d\mathbf{x})[x_kT^0{}_l-x_lT^0{}_k],$$

obey the commutation relations appropriate to the three-dimensional translation-rotation group,

$$[P_k,P_l]=0,$$
$$-i[P_k,J_{lm}]=\delta_{km}P_l-\delta_{kl}P_m,$$
$$-i[J_{kl},J_{mn}]=\delta_{km}J_{ln}-\delta_{lm}J_{kn}-\delta_{kn}J_{lm}+\delta_{ln}J_{km}.$$

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[1] For a recent discussion of the gravitational field, in this spirit, see R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **117**, 1595 (1960).

[2] See, for example, J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

[3] Some remarks of this nature are made by R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 12 (1955).

The significance of these operators as generators of translations and rotations should also be expressed by

$$[\chi(x),P_k]=(1/i)\partial_k\chi(x),$$
$$[\chi(x),J_{kl}]=[x_k(1/i)\partial_l-x_l(1/i)\partial_k+S_{kl}]\chi(x),$$

where the finite-dimensional Hermitian spin matrices S_{kl} obey the angular-momentum commutation relations.

We now assert, as a sufficient condition for invariance under the group of proper, orthochronous Lorentz transformations, that the Hermitian energy density operator $T^{00}(x)$ obeys the equal-time commutator condition[4]:

$$-i[T^{00}(x),T^{00}(x')]=-(T^0{}_k(x)+T^0{}_k(x'))\partial^k\delta(\mathbf{x}-\mathbf{x}'),$$

at least for the systems now under consideration, in which only spin values of $\frac{1}{2}$ and 1 occur. It is also required, of course, that $T^{00}(x)$ be a three-dimensional scalar function of the field operators at the given time, with no explicit coordinate dependence.

The latter property implies that the energy operator,

$$P^0=\int(d\mathbf{x})T^{00}(x),$$

obeys

$$[P^0,P_k]=[P^0,J_{kl}]=0.$$

Furthermore, the three infinitesimal generators of Lorentz transformations,

$$J^0{}_k=\int(d\mathbf{x})[x^0T^0{}_k-x_kT^{00}]=x^0P_k-\int(d\mathbf{x})x_kT^{00},$$

evidently constitute a three-dimensional vector and thus

$$-i[J^0{}_k,J_{lm}]=\delta_{km}J^0{}_l-\delta_{kl}J^0{}_m.$$

In addition,

$$-i[J^0{}_k,P_l]=\int(d\mathbf{x})x_k\partial_lT^{00}=-\delta_{kl}P^0.$$

Lacking from the list of commutation relations obeyed by the ten infinitesimal generators of the inhomogeneous Lorentz group,

$$[P_\mu,P_\nu]=0,$$
$$-i[P_\mu,J_{\nu\lambda}]=g_{\mu\lambda}P_\nu-g_{\mu\nu}P_\lambda,$$
$$-i[J_{\mu\nu},J_{\lambda\kappa}]=g_{\mu\lambda}J_{\nu\kappa}-g_{\nu\lambda}J_{\mu\kappa}-g_{\mu\kappa}J_{\nu\lambda}+g_{\nu\kappa}J_{\mu\lambda};$$

are the commutators $[P^0,J^0{}_k]$, $[J^0{}_k,J^0{}_l]$; and it is just these that are supplied by the commutator condition on the energy density.

On integrating over the three-dimensional domain of the variable x', the commutator condition becomes

$$-i[T^{00}(x),P^0]=-\partial_kT^{0k}(x),$$

which is the assertion of local energy conservation,

$$\partial_0T^{00}+\partial_kT^{0k}=0.$$

A subsequent x integration, including the factor x_k, gives

$$-i[x^0P_k-J^0{}_k,\ P^0]=-\int(d\mathbf{x})x_k\partial_lT^{0l},$$

or

$$-i[P^0,J^0{}_k]=P_k.$$

If the additional factor of x_l' is included in the x' integration, the result is

$$-i[T^{00}(x),x^0P_l-J^0{}_l]=-T^0{}_l(x)-\partial_k[x_lT^{0k}(x)],$$

or equivalently

$$[T^{00}(x),J^0{}_l]=[x^0(1/i)\partial_l-x_l(1/i)\partial^0]T^{00}(x)+2iT^0{}_l(x).$$

This is an infinitesimal transformation statement about the tensor character of $T^{\mu\nu}(x)$. An x integration, with the factor x_k, now yields

$$-i[J^0{}_k,J^0{}_l]=-\int(d\mathbf{x})(x_kT^0{}_l-x_lT^0{}_k)=-J_{kl},$$

which completes the set of commutation relations obeyed by the infinitesimal Lorentz generators.

These unitary group properties, combined with the invariance of the fundamental field commutation relations under unitary transformations, comprise the content of the requirement of Lorentz invariance. It will be noted that the energy density commutator equation could contain additional terms, which do not contribute to the various three-dimensional integrals. No such terms will appear, however, for the system we now examine,[5] a non-Abelian vector gauge field coupled with a spin-$\frac{1}{2}$ field.

THE ENERGY DENSITY OPERATOR

The fundamental dynamical variables of our system are a spin-$\frac{1}{2}$ Hermitian Fermi field $\psi(x)$, and a transverse vector Hermitian Bose field $\phi_k(x)$, $G^{0kT}(x)$, $k=1,\ 2,\ 3$. The former obeys the equal-time anticommutation relation,

$$x^0=x^{0\prime}:\quad\{\psi(x),\psi(x')\}=\delta(\mathbf{x}-\mathbf{x}'),$$

as a matrix equation in the four-component spinor indices. This field also has an additional internal multiplicity in order to realize the properties that are represented by n imaginary antisymmetrical matrices T_a, which obey the group commutation properties,

$$[T_b,T_c]=\sum_{a=1}^{n}T_at_{abc}.$$

[4] I am unaward of any similar statement in the literature. Although the work of P. A. M. Dirac, Phys. Rev. **73**, 1092 (1948), is certainly directly related, the possibility of its application to the tensor $T_{\mu\nu}$ is there specifically rejected.

[5] This discussion is a continuation of a previous paper, J. Schwinger, Phys. Rev. **125**, 1043 (1962).

The n-dimensional matrix,

$$t_b = (t_{abc}),$$

also obeys this group commutation law.

At a given time, the transverse vector fields commute with ψ, and obey the commutation relations

$$[\phi_k(x),\phi_l(x')] = [G^{0kT}(x),G^{0lT}(x')] = 0,$$
$$i[\phi_k(x),G^{0lT}(x')] = [\delta^l{}_k\delta(\mathbf{x}-\mathbf{x}')]^T.$$

These are matrix equations in the n-dimensional internal space to which the matrices t_b refer. We also employ the fields defined by

$$f^2G_{kl} = \partial_k\phi_l - \partial_l\phi_k + i(\phi_k t\phi_l)$$

and

$$\partial_k G^{0k} - i'\,t\phi_k{}'.\,G^{0k} = k^0 = \tfrac{1}{2}\psi T\psi$$

where

$$G^{0k} = G^{0kT} - \partial^k\Psi.$$

The explicit form of G^{0k} is given symbolically by

$$\mathbf{G} = [1 + \nabla\mathfrak{D}_\phi(\nabla - i'\,t\boldsymbol{\phi}')]:\mathbf{G}^T - \nabla\mathfrak{D}_\phi k^0,$$

in which

$$(-\nabla^2 + i'\,t\boldsymbol{\phi}(x)'\cdot\nabla)\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}').$$

We propose the following Hermitian operators as candidates for the momentum and energy densities of this system:

$$T^0{}_k(x) = \tfrac{1}{2}\psi(x).[(1/i)\partial_k - 'T\phi_k(x)']\psi(x) + \tfrac{1}{2}\partial^l[\tfrac{1}{4}\psi(x)\sigma_{kl}\psi(x)] + f^2G^{0m}(x).G_{km}(x) = T^0{}_k(x)^F + T^0{}_k(x)^B,$$

and

$$T^{00}(x) = \tfrac{1}{2}\psi(x).\alpha^k[(1/i)\partial_k - 'T\phi_k(x)']\psi(x) - \tfrac{1}{2}im\psi(x)\beta\psi(x) + \tfrac{1}{2}f^2[(G^{0k}(x))^2 + \tfrac{1}{2}(G^{kl}(x))^2] + t_\phi(\mathbf{x}) = T^{00}(x)^F + T^{00}(x)^B.$$

Note that symmetrized and antisymmetrized multiplications are called for in Bose and Fermi terms, respectively. The scalar function $t_\phi(\mathbf{x})$ will not be specified here. Its determination is the essential contribution of this paper. The Dirac matrices α^k and β are real, and are respectively symmetrical and antisymmetrical. The spin matrices,

$$\sigma_{kl} = (1/2i)[\alpha_k,\alpha_l],$$

are imaginary and antisymmetrical.

In order to verify that $T^0{}_k$ produces correct three-dimensional transformation properties we remark that

$$f^2G^{0m}(x).G_{km}(x) = G^{0m}(x).\partial_k\phi_m(x) + k^0(x)\phi_k(x) - \partial_m[G^{0m}(x).\phi_k(x)],$$

in which a reordering of symmetrized products is involved. Accordingly,

$$T^0{}_k(x) = \tfrac{1}{2}\psi(x).(1/i)\partial_k\psi(x) + \tfrac{1}{2}\partial^l[\tfrac{1}{4}\psi(x)\sigma_{kl}\psi(x)] + G^{0m}(x).\partial_k\phi_m(x) - \partial_m[G^{0m}(x).\phi_k(x)],$$

and therefore

$$P_k = \int(d\mathbf{x})T^0{}_k = \int(d\mathbf{x})\left[\frac{1}{2}\psi.\frac{1}{i}\partial_k\psi + G^{0mT}.\partial_k\phi_m\right],$$

which exhibits the momentum operator in terms of the canonical variables. The immediate result is

$$[\chi(x),P_k] = (1/i)\partial_k\chi(x),$$

in which χ may be ψ, ϕ_l, or G^{0lT}. Similarly,

$$J_{kl} = \int(d\mathbf{x})\{\tfrac{1}{2}\psi.[x_k(1/i)\partial_l - x_l(1/i)\partial_k + \tfrac{1}{2}\sigma_{kl}]\psi + G^{0mT}.(x_k\partial_l - x_l\partial_k)\phi_m + G^0{}_k{}^T.\phi_l - G^0{}_l{}^T.\phi_k\}$$

from which we derive

$$[\psi(x),J_{kl}] = [x_k(1/i)\partial_l - x_l(1/i)\partial_k + \tfrac{1}{2}\sigma_{kl}]\psi(x),$$
$$[\phi^m(x),J_{kl}] = [x_k(1/i)\partial_l - x_l(1/i)\partial_k]\phi^m(x) - i\delta_k{}^m\phi_l(x) + i\delta_l{}^m\phi_k(x),$$

and a similar equation for $G^{0mT}(x)$. These properties ensure the appropriate transformation behavior for P_k and J_{kl}, and thereby the validity of the commutation relations for the infinitesimal generators of the three-dimensional translation-rotation group.

It is worth noting here that a momentum density must obey equal-time commutation relations of the form

$$-i[T^0{}_k(x),T^0{}_l(x')] = -T^0{}_k(x')\partial_l\delta(\mathbf{x}-\mathbf{x}') - T^0{}_l(x)\partial_k\delta(\mathbf{x}-\mathbf{x}') + \tau_{kl}(x,x'),$$

where

$$\tau_{kl}(x,x') = -\tau_{lk}(x',x)$$

and

$$\int(d\mathbf{x})\tau_{kl}(x,x') = \int(d\mathbf{x})[x_k\tau_{lm}(x,x') - x_l\tau_{km}(x,x')] = 0.$$

The latter are the conditions demanded by the infinitesimal transformation equations,

$$[T^0{}_k(x),P_l] = (1/i)\partial_l T^0{}_k(x),$$
$$[T^0{}_m(x),J_{kl}] = [x_k(1/i)\partial_l - x_l(1/i)\partial_k]T^0{}_m(x) - i\delta_{km}T^0{}_l(x) + i\delta_{lm}T^0{}_k(x),$$

which also imply the commutation properties of P_k and J_{kl}. The Hermitian operators $\tau_{kl}(x,x')$ should vanish at finite $|\mathbf{x}-\mathbf{x}'|$ but generally are not identically zero for field systems with spin. Concerning the system now under discussion we shall only remark that $\tau_{kl}(x,x')$ has terms containing no higher than second derivatives of $\delta(\mathbf{x}-\mathbf{x}')$.

The Fermi field contribution to the energy density obeys the commutator condition,

$$-i[T^{00}(x)^F,T^{00}(x')^F] = -(T^0{}_k(x)^F + T^0{}_k(x')^F)\partial^k\delta(\mathbf{x}-\mathbf{x}'),$$

as one can easily confirm. In order to evaluate $[T^{00}(x)^F,T^{00}(x')^B]$, for example, we first compute $[T^{00}(x)^F,G^{0l}(x')]$ which differs from zero in virtue of the commutator

$$i[\phi_k(x),G^{0l}(x')]=\delta^l{}_k\delta(\mathbf{x}-\mathbf{x}')-[\partial_k-i`t\phi_k(x)']\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l,$$

and of the density operators $k^0{}_a$ contained in $G^{0l}(x')$. The result obtained by combining the two contributions is simply

$$-i[T^{00}(x)^F,G^{0l}(x')]=\delta(\mathbf{x}-\mathbf{x}')k^l(x),$$

where

$$k_a{}^l(x)=\tfrac{1}{2}\psi(x)\alpha^l T_a\psi(x).$$

It follows that

$$-i[T^{00}(x)^F,\tfrac{1}{2}(G^{0l}(x'))^2]=\delta(\mathbf{x}-\mathbf{x}')k_l(x).G^{0l}(x),$$

which is a symmetrical function of x and x', and accordingly

$$[T^{00}(x)^F,T^{00}(x')^B]+[T^{00}(x)^B,T^{00}(x')^F]=0.$$

Turning to the evaluation of $[T^{00}(x)^B,T^{00}(x')^B]$, we observe that

$$-i[f^2G_{kl}(x),G^{0m}(x')]=-[\partial_k-i`t\phi_k(x)']\delta_l{}^m\delta(\mathbf{x}-\mathbf{x}')+[\partial_l-i`t\phi_l(x)']\delta_k{}^m\delta(\mathbf{x}-\mathbf{x}')-i`tG_{kl}(x)'\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^m,$$

and therefore

$$-i[\tfrac{1}{4}f^2(G_{kl}(x))^2,G^{0m}(x')]=-G^{km}(x)[\partial_k-i`t\phi_k(x)']\delta(\mathbf{x}-\mathbf{x}')$$

since

$$(G_{kl}(x)tG_{kl}(x))=0.$$

As a result,

$$(1/i)[\tfrac{1}{4}f^2(G_{kl}(x))^2,\tfrac{1}{2}f^2(G^{0m}(x'))^2]+(1/i)[\tfrac{1}{2}f^2(G^{0m}(x))^2,\tfrac{1}{4}f^2(G_{kl}(x'))^2]=-[T^0{}_k(x)^B+T^0{}_k(x')^B]\partial^k\delta(\mathbf{x}-\mathbf{x}'),$$

and the verification of the commutator condition would be completed, were it known that

$$[\tfrac{1}{2}f^2(G^{0k}(x))^2+t_\phi(\mathbf{x}),\tfrac{1}{2}f^2(G^{0l}(x'))^2+t_\phi(\mathbf{x}')]=0.$$

Let us recall the commutator

$$i[G^{0k}(x),G^{0l}(x')]=\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').i`tG^{0l}(x')'+i`tG^{0k}(x)'.\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l,$$

of which one important consequence is that

$$(G^{0k}(x)tG^{0k}(x)),$$

unlike the similar construction with G_{kl}, does not vanish. Indeed,

$$(G^{0k}(x)tG^{0k}(x))=-\textstyle\sum_a t_a\partial_k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})t_a.G^{0k}(x),$$

where only the first variable of $\mathfrak{D}_\phi$ is to be differentiated. We shall now simply record the outcome of computing $[(G^{0k}(x))^2,(G^{0l}(x'))^2]$ and invite the reader to duplicate this calculation:

$$-i[\tfrac{1}{2}(G^{0k}(x))^2,\tfrac{1}{2}(G^{0l}(x'))^2]=-\gamma_\phi(\mathbf{x},\mathbf{x}')_k.G^{0k}(x')+G^{0k}(x).\gamma_\phi(\mathbf{x}',\mathbf{x})_k,$$

in which

$$f^2\gamma_\phi(\mathbf{x},\mathbf{x}')^k=(1/i)[t_\phi(\mathbf{x}),G^{0k}(x')]$$

and

$$t_\phi(\mathbf{x})=\tfrac{1}{8}f^2\textstyle\sum_a\mathrm{Tr}[t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})t_a\partial_k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})].$$

In this result we recognize the statement that a function $t_\phi(\mathbf{x})$ exists such that the set of operators, $\frac{1}{2}f^2(G^{0k}(x))^2+t_\phi(\mathbf{x})$ for all $\mathbf{x}$, are commutative.

The intimate relationship that must exist between the two terms $\frac{1}{2}f^2(G^{0k}(x))^2$ and $t_\phi(\mathbf{x})$ is emphasized by writing the sum as the positive Hermitian operator,

$$\tfrac{1}{2}f^2\textstyle\sum_a[G_a{}^{0k}(x)-\tfrac{1}{2}\mathrm{Tr}t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})][G_a{}^{0k}(x)+\tfrac{1}{2}\mathrm{Tr}t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]=\tfrac{1}{2}f^2(G^{0k}(x))^2+t_\phi(\mathbf{x}),$$

where, it will be remembered, the elements of the matrices t_a are imaginary numbers. The possibility of this rearrangement is equivalent to a statement of the identity

$$\textstyle\sum_a[\mathrm{Tr}t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}),G_a{}^{0k}(x)]=-\tfrac{1}{2}\sum_a\mathrm{Tr}[t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]-\tfrac{1}{2}\sum_a[\mathrm{Tr}t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]^2.$$

Its verification will require the following theorem:

$$\textstyle\sum_a\mathrm{Tr}[t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})t_a(\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})-\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})\partial^k)]=\sum_a[\mathrm{Tr}t_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]^2.$$

FIELD TRANSFORMATION PROPERTIES

Now that we are in possession of the explicit operators[6] for P^0 and $J^0{}_k$, the equations of motion and the Lorentz transformation behavior of the various field quantities can be derived. Let us begin with the Fermi field $\psi(x)$ and remark that ($x^0=x^{0\prime}$)

$$[\psi(x),k^0(x')]=\delta(\mathbf{x}-\mathbf{x}')T\psi(x),$$

which has the consequence

$$[\psi(x),G^{0l}(x')]=-\partial'^l\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})T\psi(x).$$

Then we get

$$i\partial_0\psi(x)=[\psi(x),P^0]=`T\phi^0(x)'.\psi(x)+\alpha^k[(1/i)\partial_k-`T\phi_k(x)']\psi(x)-im\beta\psi(x),$$

where

$$\phi^0(x)=f^2\int(d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').\partial_l'G^{0l}(x').$$

[6] Despite the criticism of Lorentz gauge methods, it seems reasonable to suppose that no difficulty of a formal nature will appear in a Lorentz gauge treatment analogous to that given by Fermi for the electromagnetic field, provided one avoids all reference to state vector norms, for these cannot be finite. Accordingly, it behooves one to show that the elimination of longitudinal modes from the Lorentz gauge formulation reproduces the energy operator of the radiation gauge method. This I have succeeded in doing.

This equation of motion is also expressed by

$$\{\alpha^\mu[\partial_\mu - i\text{`}T\phi_\mu(x)\text{'}] + \beta m\}.\psi(x) = 0,$$

with

$$\alpha^0 = 1.$$

In evaluating the commutator of any field operator $F(x)$ with J_{0l}, it is convenient to write the latter as

$$J_{0l} = x_0 P_l - x_l P_0 + \int (d\mathbf{x}')(x_l' - x_l) T^{00}(x'),$$

so that

$$[F(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0]F(x) + \int (d\mathbf{x}')(x_l' - x_l)[F(x), T^{00}(x')].$$

The result of applying this procedure to $\psi(x)$ is

$$[\psi(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0 + \tfrac{1}{2}\sigma_{0l}]\psi(x) + \text{`}T\Lambda_l(x)\text{'}.\psi(x),$$

in which

$$\Lambda_l(x) = f^2 \int (d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').\partial_k'[(x_l' - x_l)G^{0k}(x')]$$

appears as an operator gauge function. We have also defined

$$\sigma_{0l} = -i\alpha_l.$$

A similar calculation can be performed for the densities $k^0(x)$ with the aid of the equal-time commutation relation

$$[k_b^0(x), k_c^0(x')] = \delta(\mathbf{x} - \mathbf{x}')\textstyle\sum_a k_a^0(x) t_{abc}.$$

We find

$$i\partial_0 k^0(x) = \text{`}t\phi^0(x)\text{'}.k^0(x) + [(1/i)\partial_l - \text{`}t\phi_l(x)\text{'}]k^l(x),$$

or

$$[\partial_\mu - i\text{`}t\phi_\mu(x)\text{'}].k^\mu(x) = 0,$$

and the Lorentz transformation law

$$[k^0(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0]k^0(x) - ik_l(x) + \text{`}t\Lambda_l(x)\text{'}.k^0(x).$$

A related result is

$$[k_m(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0]k_m(x) - i\delta_{ml}k^0(x) + \text{`}t\Lambda_l(x)\text{'}.k_m(x).$$

Turning to the transverse vector field $\phi_k(x)$, we first note that

$$i[\phi_k(x), \tfrac{1}{2}(G^{0m}(x'))^2] = \{\delta_{km}\delta(\mathbf{x} - \mathbf{x}') - [\partial_k - i\text{`}t\phi_k(x)\text{'}]\mathfrak{D}_\phi(\mathbf{x}\,\mathbf{x}')\partial_m'\}.G^{0m}(x')$$

in which ∂_m' is still understood to act on the function to its left. The immediate consequences are

$$\partial_0\phi_k(x) = f^2 G_{0k}(x) + [\partial_k - i\text{`}t\phi_k(x)\text{'}].\phi_0(x)$$

or equivalently,

$$f^2 G_{0k} = \partial_0\phi_k - \partial_k\phi_0 + (\phi_0 i t\phi_k),$$

and

$$[\phi_k(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0]\phi_k(x) - i\delta_{kl}\phi^0(x) + [(1/i)\partial_k - \text{`}t\phi_k(x)\text{'}].\Lambda_l(x).$$

Since both sides of the latter equation must be divergenceless we learn, incidentally, that

$$0 = \partial_l\phi^0 - \partial_0\phi_l + (\nabla^2 - i\text{`}t\boldsymbol{\phi}\text{'}\cdot\nabla)\Lambda_l,$$

which supplies an alternative construction for the operator gauge function,

$$\Lambda_l(x) = \int (d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').[\partial_l'\phi^0(x') - \partial_0\phi_l(x')],$$

as one can verify directly. This version of the gauge function can be used to rewrite the Lorentz transformation property of ϕ_k in the symbolic form

$$[\phi_k, J_{0l}] = [1 + (\nabla - i\text{`}t\boldsymbol{\phi}\text{'})\mathfrak{D}_\phi\nabla]_{km}.\{[x_0(1/i)\partial_l - x_l(1/i)\partial_0]\phi^m - i\delta_l{}^m\phi^0\}$$

which makes explicit the origin of the operator gauge transformation in the radiation gauge requirement of transversality. A closely related transformation law is that of $G_{km}(x)$, since

$$f^2\delta G_{km} = (\partial_k - i\text{`}t\phi_k\text{'})\delta\phi_m - (\partial_m - i\text{`}t\phi_m\text{'})\delta\phi_k$$

and the result of this calculation is

$$[G_{km}(x), J_{0l}] = [x_0(1/i)\partial_l - x_l(1/i)\partial_0]G_{km}(x) + i\delta_{kl}G_{0m}(x) - i\delta_{ml}G_{0k}(x) + \text{`}t\Lambda_l(x)\text{'}.G_{km}(x).$$

Thus far the Lorentz transformation properties present a comparatively simple picture. In addition to the anticipated geometrical transformations the various fields are subjected to an operator gauge transformation. Indeed, this would be a completely valid assertion for an Abelian gauge field, but it is not true for the time components of a non-Abelian gauge field. If we consider the field $G^{0k}(x)$, commutation properties which have already been employed show that

$$-i[G^{0k}(x), T^{00}(x')] = G^{mk}(x')[\partial_m' - i\text{`}t\phi_m(x')\text{'}]\delta(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}')k^k(x) - f^2 i\text{`}tG^{0k}(x)\text{'}.[\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l.G^{0l}(x')] + r_\phi{}^k(\mathbf{x},\mathbf{x}'),$$

where the last two terms appear as an evaluation of

$$-i[G^{0k}(x), \tfrac{1}{2}f^2(G^{0l}(x'))^2 + t_\phi(\mathbf{x}')].$$

The notation anticipates the structure of the function $r_\phi{}^k(\mathbf{x},\mathbf{x}')$. It is again understood that ∂'^l acts on the function to its left. The result of an x' integration is the equation of motion

$$\partial_0 G^{0k}(x) = i\text{`}t\phi_0(x)\text{'}.G^{0k}(x) - [\partial_l - i\text{`}t\phi_l(x)\text{'}]G^{lk}(x) - k^k(x) + r_\phi{}^k(\mathbf{x}),$$

while the additional factor $x_l' - x_l$ produces the Lorentz transformation formula

$$[G^{0k}(x),J_{0l}]=[x_0(1/i)\partial_l-x_l(1/i)\partial_0]G^{0k}(x)+iG^k{}_l(x)+{}^{\backprime}t\Lambda_l(x)^{\prime}.G^{0k}(x)+ir_\phi{}^k{}_l(\mathbf{x}).$$

Here we have defined

$$r_\phi{}^k(\mathbf{x})=\int(d\mathbf{x}')r_\phi{}^k(\mathbf{x},\mathbf{x}')$$

and

$$r_\phi{}^k{}_l(\mathbf{x})=\int(d\mathbf{x}')(x_l'-x_l)r_\phi{}^k(\mathbf{x},\mathbf{x}').$$

The novel transformation aspects of the operators $G^{0k}(x)$ are thus associated with the appearance of the function $r_\phi{}^k(\mathbf{x},\mathbf{x}')$.

This function can be evaluated by remarking that

$$-f^2G^{0k}(x).[i{}^{\backprime}tG^{0k}(x)^{\prime}.(\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l.G^{0l}(x'))]+G^{0k}(x).r_\phi{}^k(\mathbf{x},\mathbf{x}')=-i[\tfrac{1}{2}(G^{0k}(x))^2,\tfrac{1}{2}f^2(G^{0l}(x'))^2+t_\phi(\mathbf{x}')]=i[t_\phi(\mathbf{x}),G^{0l}(x')].G^{0l}(x'),$$

so that it suffices to identify the coefficients of $G^{0k}(x)$, after a reduction of the first term. A fairly explicit statement of the result is given by

$$r_\phi{}^k(\mathbf{x},\mathbf{x}')_a=-\tfrac{1}{8}f^2\sum_{bc}[it_a\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')]_{bc}\times\mathrm{Tr}[t_b\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^lt_c\partial'^l\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})]+\tfrac{1}{4}f^2\sum_b\mathrm{Tr}\{t_at_b\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^li[G_b{}^{0k}(x),\partial'^l\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})]\}.$$

In this and in previous manipulations the following relation has been useful:

$$i[\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')_{ab},G_c{}^{0l}(x'')]-i[\partial''^l\mathfrak{D}_\phi(\mathbf{x}'',\mathbf{x}')_{cb},G_a{}^{0k}(x)]=-[\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}'')it_c\partial''^l\mathfrak{D}_\phi(\mathbf{x}'',\mathbf{x}')]_{ab}-[\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')it_b\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x}'')\partial''^l]_{ac}-[\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})\partial^kit_a\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}'')\partial''^l]_{bc}.$$

Another aspect of the function $r_\phi{}^k(\mathbf{x},\mathbf{x}')$ should be noted. It is a consequence of the definition that

$$[\partial_k-i{}^{\backprime}t\phi_k(x)^{\prime}]r_\phi{}^k(\mathbf{x},\mathbf{x}')=\sum_{ab}t_bt_a\tfrac{1}{4}[\phi_a{}^k(x),[G_b{}^{0k}(x),f^2\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l.G^{0l}(x')]]-\sum_{ab}t_at_b\tfrac{1}{4}[G_b{}^{0k}(x),[\phi_a{}^k(x),f^2\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial'^l.G^{0l}(x')]].$$

Beyond remarking that an evaluation of these double commutators would indeed yield a function of the field $\boldsymbol{\phi}$, we shall not record the explicit result, for the important thing is the commutator structure, which is such that the constraint equation,

$$[\partial_k-i{}^{\backprime}t\phi_k(x)^{\prime}].G^{0k}(x)=k^0(x),$$

is maintained in time and under Lorentz transformations. Thus, a detailed term-by-term evaluation of

$$[(\partial_k-i{}^{\backprime}t\phi_k(x)^{\prime}).G^{0k}(x)-k^0(x),\ J_{0l}]$$

will finally yield zero only in virtue of the implied property,

$$[\partial_k-i{}^{\backprime}t\phi_k(x)^{\prime}]r_\phi{}^k{}_l(\mathbf{x})+r_{\phi l}(\mathbf{x})=-\sum_{ab}t_bt_a\tfrac{1}{4}[\phi_{ka}(x),[G_b{}^{0k}(x),\Lambda_l(x)]]+\sum_{ab}t_at_b\tfrac{1}{4}[G_b{}^{0l}(x),[\phi_{ka}(x),\Lambda_l(x)]],$$

which expression, incidentally, also equals

$$\sum_{ab}t_bt_a\tfrac{1}{4}[\phi_{ka}(x),[\Lambda_{lb}(x),G^{0k}(x)]]-\sum_{ab}t_at_b\tfrac{1}{4}[\Lambda_{lb}(x),[\phi_{ka}(x),G^{0k}(x)]].$$

The complete density of the internal property represented by the matrices T_a and t_a is given by

$$j_a{}^0(x)=\partial_kG_a{}^{0k}(x)=k_a{}^0(x)-i(\phi_k(x).t_aG^{0k}(x)),$$

and the total content of this property is described by the constant Hermitian operator,

$$\mathbf{T}_a=\int(d\mathbf{x})j_a{}^0(x).$$

Whenever the contributions to this integral are effectively confined to a spatially bounded region the scalar function $\Psi(x)$, which specifies the longitudinal component of $G^{0k}(x)$,

$$G^{0k}=G^{0kT}-\partial^k\Psi,$$

must have the asymptotic behavior

$$|\mathbf{x}|\rightarrow\infty:\quad\Psi_a(x)\sim\frac{1}{4\pi|\mathbf{x}|}\mathbf{T}_a.$$

No such slowly decreasing term will appear in the time derivative of G^{0k}, however, and we can identify the current j^k, which obeys the conservation law

$$\partial_0j^0+\partial_kj^k=0,$$

as

$$j^k(x)=-\partial_0G^{0k}(x)-\partial_lG^{lk}(x)=k^k(x)+i(\phi_0(x).tG^{0k}(x))+i(\phi_l(x)tG^{lk}(x))-r_\phi{}^k(\mathbf{x}).$$

We see that the function $r_\phi{}^k(\mathbf{x})$ also intervenes here.

Although the current $j^\mu(x)$ obeys a local conservation law, in contrast with $k^\mu(x)$, the density $j^0(x)$ does not have a localized character with respect to equal-time commutation relations, as does $k^0(x)$. Thus,

$$[j^0(x),j^0(x')]=-{}^{\backprime}tj^0(x)^{\prime}\delta(\mathbf{x}-\mathbf{x}')+\partial_k\partial_l'i[{}^{\backprime}t\phi^k(x)^{\prime}\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').{}^{\backprime}tG^{0l}(x')^{\prime}-{}^{\backprime}tG^{0k}(x)^{\prime}.\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}'){}^{\backprime}t\phi^l(x')^{\prime}].$$

The additional divergence term does not contribute to integrated quantities, and we reaffirm the commutation properties of the **T** operators,

$$[\mathbf{T}_b,\mathbf{T}_c]=\sum_a \mathbf{T}_a t_{abc},$$

while obtaining

$$[j^0(x),\mathbf{T}_a]=t_a j^0(x).$$

The Lorentz transformation properties of j^μ are more complicated than those of k^μ. For example,

$$[j^0(x),J_{0l}]=[x_0(1/i)\partial_l-x_l(1/i)\partial_0]j^0(x)-ij_l(x)+{}^{\backprime}t\Lambda_l(x)'.j^0(x)+{}^{\backprime}t\partial_k\Lambda_l(x)'.G^{0k}(x)+i\partial_k r_\phi{}^k{}_l(\mathbf{x}),$$

and by writing the latter as

$$[j^0(x),J_{0l}]=(1/i)\partial_k[\delta_l{}^k x_0 j^0(x)+x_l j^k(x)+i{}^{\backprime}t\Lambda_l(x)'.G^{0k}(x)-r_\phi{}^k{}_l(\mathbf{x})]$$

one can verify the Lorentz invariance of the conserved **T** operators,

$$[\mathbf{T}_a,J_{0l}]=0.$$

We consider, finally, the Lorentz transformation behavior of the field $\phi^0(x)$. One might begin with the explicit construction,

$$\phi^0(x)=f^2\int(d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').\partial_k'G^{0k}(x'),$$

which, incidentally, has a counterpart in

$$\phi^k(x)=f^2\int(d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\partial_l'G^{kl}(x'),$$

but it is simpler to return to the differential equation

$$f^2G_{0k}=\partial_0\phi_k-(\partial_k-i{}^{\backprime}t\phi_k').\phi_0.$$

From the known characteristics of ϕ_k and G_{0k} we now find that

$$[\phi^0(x),J_{0l}]=[x_0(1/i)\partial_l-x_l(1/i)\partial_0]\phi^0(x)-i\phi_l(x)+i[\partial_0-i{}^{\backprime}t\phi_0(x)'].\Lambda_l(x)+i\rho_{\phi l}(\mathbf{x}),$$

where

$$[\partial_k-i{}^{\backprime}t\phi_k(x)']\rho_{\phi l}(\mathbf{x})=-f^2r_{\phi kl}(\mathbf{x})+\sum_{ab}t_bt_a\tfrac{1}{4}[\phi_{ka}(x),[\Lambda_{lb}(x),\phi^0(x)]]-\sum_{ab}t_at_b\tfrac{1}{4}[\Lambda_{lb}(x),[\phi_{ka}(x),\phi^0(x)]],$$

and the divergence of this equation will supply the additional function of the field $\boldsymbol{\phi}$ that transcends the elementary geometrical and gauge transformations. The asymptotic behavior of $\phi^0(x)$ is given by

$$|\mathbf{x}|\to\infty:\quad \phi^0(x)\sim\frac{1}{4\pi|\mathbf{x}|}f^2\mathbf{T},$$

according to the differential equation

$$-\nabla^2\phi^0=f^2j^0-\nabla\cdot(i^{\backprime t}\boldsymbol{\phi}'\phi^0).$$

Note added in proof. The energy density commutator equation can be given a general dynamical basis by examining the modification in the $T^{\mu\nu}$ conservation equation produced by an external gravitational field. Also involved is a minimum assumption of time locality. For an analogous derivation of the null charge density equal-time commutator from the current conservation equation, one must consider systems such that the current is not an explicit function of the time derivative of an external vector potential.