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Formulation of the Scattering Functions in Terms of the Unitary Representations of the Inhomogeneous Lorentz Group*

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A group-theoretical basis of the axiomatic S-matrix theory is presented. The unitarity condition is taken as the scalar product of the scattering functions which then are shown to transform according to the unitary representations of the inhomogeneous Lorentz group. The general transformation property of the invariant scattering functions is given which allows one, in principle, to write down these functions for arbitrary processes.

I. INTRODUCTION

STAPP has given a postulational formulation of the S-matrix theory based essentially on relativistic invariance and the principle of maximal analyticity.[1] On this basis he proves the CPT theorem and the connection between spin and statistics, the latter with one additional assumption. He has further derived, by using the unitarity condition, the same singularities for the S matrix as are obtained from the perturbation theory.

In view of the importance of these results for a complete S-matrix theory independent of the quantum field theory, it is desirable to give a general mathematical formulation of the scattering functions, incorporating their transformation properties, their invariance, and the unitarity condition. Such a formulation can be obtained as an application of the theory of unitary representations of the inhomogeneous Lorentz group which, like the S-matrix theory, makes no assumptions about the existence of underlying fields for the particles.

Unlike the situation in quantum field theory, the Lorentz group is applied in this paper directly to scattering functions rather than to the state vectors of field theory. The S-matrix elements, which involve the variables of both the initial and the final states, are considered as the primary quantities satisfying a certain completeness relation. For this reason we may refer to them as scattering functions, or S functions, instead of the S matrix. The completeness relation expresses the conservation of the total probability (i.e., the unitarity condition of the quantum field theory) and is an invariant relation. It is then shown that the S functions transform under the unitary representations of the Lorentz group. This, together with the invariance of the S functions, fixes the transformation properties of these functions. A systematic group theoretical method is thus provided to construct the invariant amplitudes for arbitrary processes.

In the sense of the above-mentioned product relation, the scattering theory is a linear theory. There is no dynamics involved, however, in the present formulation. The philosophy of the S-matrix theory is that the dynamics will follow from the postulate of maximal analyticity.

II. THE SCATTERING FUNCTIONS

Quite generally, a complete scattering experiment is described by the scattering function S_{ij}, which represents the transition probability amplitude to state i when the system was in the state j with probability one. Hence, we must have the normalization of the probabilities (assuming discrete indices for the time being),

$$\begin{aligned} \textstyle\sum_i S_{ij}^* S_{ij} &= 1, \quad \text{for all } j, \\ \textstyle\sum_j S_{ij}^* S_{ij} &= 1, \quad \text{for all } i. \end{aligned} \tag{1}$$

If the initial state is taken to be a mixture, then the condition that the total probability is unity, together with Eqs. (1), implies

$$\begin{aligned} \textstyle\sum_i S_{ij}^* S_{ik} &= \delta_{jk}, \\ \textstyle\sum_i S_{ij}^* S_{kj} &= \delta_{ik}. \end{aligned} \tag{2}$$

The S matrix is "diagonal" with respect to selection rules, but Eqs. (2) are valid for the whole matrix. Even if, for certain attributes of the particles, superselection rules operate,[2] Eqs. (2) may be assumed to hold by definition, as the phases are not observable.

We now take the S_{ij} to be the elements of a linear space. In particular, the projections on definite values of momenta, spin, and particle type will be written as

$$\langle K, -\tilde{K}' | S \rangle = S(K, -\tilde{K}'). \tag{3}$$

One can also use, for example, projections on definite values of the total angular momenta and introduce scattering functions of the form $S(J, -\tilde{J}')$. In Eq. (3), according to Stapp's convention, K stands for an ordered set of variables (k_i, λ_i, t_i), one for each final particle where k_i is the momentum four-vector of the ith particle with $k_i^2 = m_i^2$, λ_i is the spin component with $2s_i+1$ values, and t_i is the type of the particle including possible internal quantum numbers. Similarly, K' is the set of variables of the initial particles written

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[1] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

[2] G. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

in S, by convention, in the reverse order and all with negative signs.

Let the S functions transform under Lorentz transformations according to the law

$$S'(K, -\tilde{K}') = U_{KK'} S(K, -\tilde{K}'). \quad (4)$$

We shall show in the next section that the operator $U_{KK'}$ is unitary.

In a relativistically invariant theory the transition amplitudes are invariant. By this we mean

$$S'(K, -\tilde{K}') = S(K, -\tilde{K}'). \quad (5)$$

That is, two different observers operating with the same values of the initial and final variables measure the same transition amplitudes. Also, the same experiment viewed from two different coordinate frames gives the same transition amplitudes. Now $U_{KK'}$ in Eq. (4) involves the transformations of both the spin and the momentum variables. If we denote by ΛK the set of variables $(\Lambda k_i, \lambda_i, t_i)$—i.e., with only momenta transformed—and use (5), we can write the transformation property of the *invariant functions* in the form

$$S(K, -\tilde{K}') = U'_{KK'} S(\Lambda^{-1}K, -\Lambda^{-1}\tilde{K}'). \quad (6)$$

The U' is such that first transforming the momentum arguments of S alone, then applying U', we must get back the original S function.

III. NORMALIZATION

In terms of the sets of variables $(K, -\tilde{K}')$ Eqs. (2) can be written as

$$\begin{aligned} &\sum_K \int S^*(K, -\tilde{K}') S(K, -\tilde{K}'') = \delta(K', -K''), \\ &\sum_K \int S^*(K', -\tilde{K}) S(K'', -\tilde{K}) = \delta(K', -K''), \end{aligned} \quad (7)$$

where, corresponding to continuous momentum variables k_i, the symbol $\sum\int$ means integrals over the invariant volume elements in the momentum space and sums over the spin indices.

Equations (3) have to be interpreted in the sense of distributions. The momenta are never measured with absolute accuracy so that we can multiply both sides of (7) with some smooth normalized test function $f(K', K'')$ and integrate. This procedure corresponds physically to the introduction of wave packets in the initial or final states. We can then write, if necessary, a proper norm

$$\langle S|S\rangle \equiv \sum_{K,K',K''} \int S^*(K, -\tilde{K}') S(K, -\tilde{K}'') f(K', K'') = \sum_K \int f(K,K) = 1. \quad (8)$$

We can take now both Eqs. (7) [or Eq. (8)] as the definition of a "scalar" product or, better, of a completeness relation in which, however—unlike the usual scalar product—the summation is over part of the indices. With respect to the set of indices K it is an ordinary scalar product, and we shall use it in this sense.

We split U in Eq. (4) into two parts,

$$U = VW, \quad (9)$$

where V acts on the first set of variables of $S(K, -\tilde{K}')$ and W on the second set of variables. This separation can always be made, since V and W operate on different spin spaces and on independent momenta. Equations (7) remain invariant under either transformation V or W alone. For example, in the first of Eqs. (7) we can replace the first set of variables by ΛK and transform the spins. We obtain then the equation

$$\sum_K \int V^\dagger S^*(K, -\tilde{K}') V S(K, -\tilde{K}'') = \delta(K', -\tilde{K}''), \quad (10)$$

which is equal to the first part of (7), since we are summing over the transformed indices with an invariant volume. Hence, $V^\dagger V = 1$. If we now transform the same equation with W it follows from the invariance of the $\delta(K', -\tilde{K}'')$ that $W^\dagger W = 1$.

By exactly the same argument, we see from the second part of Eq. (7) that V and W satisfy the equations $VV^\dagger = 1$ and $WW^\dagger = 1$. Thus, V and W, and consequently $U = VW$, are unitary.

IV. TRANSFORMATION PROPERTIES OF THE SCATTERING FUNCTIONS

Having established that the scattering functions transform according to the unitary representations of the inhomogeneous Lorentz group, we now discuss the form of these representations suitable for the variables occuring in $S(K, -\tilde{K}')$. The unitary representations have been discovered by Wigner.[3,4] All we need is to take the appropriate direct products of the representations given by Wigner and Wightman.[5]

Let us denote by $U(a,A)$ these unitary representations. Here, the four-vector a stands for an element of the translation group and A represents an element of the 2×2 unimodular group, which is isomorphic to the proper Lorentz group. For particles of nonzero mass, the transformation property of the S functions is given by

$$\begin{aligned} (U(a,A)S)(K, -\tilde{K}') &= \exp(i \textstyle\sum_j k_j a) \prod_i D^{(j_i)}(A_{k_i \leftarrow p_i}{}^{-1} A A_{q_i \leftarrow p_i}) \\ &\quad \times S(\Lambda(A^{-1})K, -\Lambda(A^{-1})\tilde{K}'), \end{aligned} \quad (11)$$

[3] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[4] See also A. S. Wightman, in *Dispersion Relations and Elementary Particles*, edited by C. deWitt (John Wiley & Sons, Inc., New York, 1960).

[5] See also an extended version of the present paper, University of California Radiation Laboratory Report UCRL-9963 (unpublished).

where the left-hand side represents the transformed function at the arguments $(K, -\tilde{K}')$, and $D^{(j)}(A)$ are the well-known unitary representations of dimensions $2j+1$ (j integral or half-integral positive numbers) of the three-dimensional orthogonal group which, in this case, is the little group of the inhomogeneous Lorentz group. Finally, $A_{k\leftarrow p}$ is such that the corresponding Lorentz transformation carries the vector p into the vector k, i.e., $\Lambda(A_{k\leftarrow p})p=k$.

Since the S functions are invariant separately under translations and homogeneous Lorentz transformations, we obtain from this equation the conservation of the total momentum

$$\textstyle\sum_i k_i=0,$$

and, using (5), the transformation property of the S functions

$$S(K, -\tilde{K}')=\textstyle\prod_i \mathfrak{D}^{(j_i)}(A_{k_i\leftarrow p_i}{}^{-1} AA_{q_i\leftarrow p_i}) \times S(\Lambda(A^{-1})K, -\Lambda(A^{-1})\tilde{K}'), \quad (12)$$

where p_i are the momenta in the rest frames and $q_i=\Lambda(A^{-1})k_i$.

The definition and the conservation of the angular momenta will be obtained if $\mathfrak{D}^{(j)}$, corresponding to pure rotations, is brought to an exponential form.

Let us consider the special case of one spin-$\frac{1}{2}$ particle. In this case, $\mathfrak{D}^{(j)}(A)$ reduces to A itself and

$$A_{k\leftarrow p}=(k^\mu\sigma_\mu/m)^{\frac{1}{2}},$$

choosing $p=(m,0,0,0)$; σ_μ are the Pauli matrices $\sigma_\mu=(\mathbf{1},\boldsymbol{\sigma})$. Hence, we have

$$S(K, -\tilde{K}')=[(k^\mu\sigma_\mu/m)^{\frac{1}{2}}]^{-1}A(q^\mu\sigma_\mu/m)^{\frac{1}{2}} \times S(\Lambda(A^{-1})K, -\Lambda(A^{-1})\tilde{K}')$$

or

$$(k\cdot\sigma/m)^{\frac{1}{2}}S(K, -\tilde{K}')=A^{-1}((\Lambda k)\cdot\sigma/m)^{\frac{1}{2}}S(\Lambda K, -\Lambda\tilde{K}').$$

The same transformation property holds for the R functions,

$$R=S-1.$$

Stapp has introduced the M functions,

$$M(K, -\tilde{K}')=(k\cdot\sigma/m)^{\frac{1}{2}}R(K, -\tilde{K}'), \quad (13)$$

which then have the simple transformation property:

$$M(K, -\tilde{K}')=A^{-1}M(\Lambda K, -\Lambda\tilde{K}'). \quad (14)$$

These M functions are expected to be analytic functions of the invariant products of the momenta, not the R functions, since the transformation property of the R functions involves square roots and the components of the momenta separately.

In the general case, $\mathfrak{D}^{(j)}(A)$ themselves are direct products of the A's. One can therefore always split the factors $(k\cdot\sigma/m)^{\frac{1}{2}}$ for each spinor index, and the remaining M functions will transform simply with $\mathfrak{D}^{(j)}(A)$ instead of $\mathfrak{D}^{(j)}(A_{k\leftarrow p}{}^{-1}AA_{q\leftarrow p})$. Equations (13) and (14) agree then with Stapp's results. The application of the group representations has allowed us to give also the transformation property of the S or the R functions. Furthermore, in terms of the known forms of $\mathfrak{D}^{(j)}(A)$, one can write the general form of the M functions and consequently the invariant amplitudes for arbitrary many-body processes involving arbitrary spins.

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