Unitarity and Production Amplitudes

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(Received January 18, 1962)

A development of the N/D formulation of the unitary S matrix is presented for multichannel reactions including production processes. The relevant amplitudes are expressed in terms of helicity amplitudes and the coupled, unitarity relations of Blankenbecler. The analytic continuation of the amplitudes in the presence of anomalous thresholds is considered in detail. An extension of the Levinson theorem to the multichannel production case is discussed. An application of the formalism developed here is given in another paper.

I. INTRODUCTION

Both theoretical and experimental studies on the higher resonances in pion-nucleon scattering have led us to hope that these can be eventually understood by including the effect of the competing production processes. The work of Peierls,[1] Goebel and Schnitzer,[2] Carruthers,[3] and Ball and Frazer[4] shows how the opening of a production channel excites the elastic channel by unitarity, thereby giving rise to a resonance in the scattering process, even though the mechanisms considered by them differ considerably in detail.

The construction of the unitary S matrix for the multichannel reaction has been studied intensively in the last few years.[5,6] In particular, Meetz[7] has considered a detailed application of the generalized unitarity relation, as formulated by Blankenbecler, to pion-nucleon scattering. He considers the unitarity relation for given angular momentum and parity states, and shows how the generalized N/D method enables us to construct scattering and production amplitudes which satisfy simultaneously the requirements of analyticity and unitarity imposed upon them. The original version of Blankenbecler's unitarity formulation[6] involves too many variables to be tractable in practical applications, and this difficulty is overcome by decomposing the amplitudes involved into partial waves.

While we concur fully with the philosophy behind Meetz's formulation, we consider Meetz's work incomplete in the following respects: (1) Meetz decomposes the production and three-particle scattering amplitudes in the total center-of-mass system, following the work of Ciulli and Fischer.[8] Whereas this is satisfactory in the nonrelativistic limit, the partial wave amplitudes so defined do not satisfy the simple unitarity relation Meetz attributes to them. (2) The production and three-particle amplitudes have anomalous singularities,[9] with the concommitant problem of the continuation of the unitarity relation beyond its "domain of definition." This analytic continuation of the N/D solution requires a careful discussion.

In the present article we shall address ourselves to the N/D formulation of the unitary S matrix for multichannel reactions including production processes. We shall concentrate on the decomposition of the production amplitudes in terms of the helicities of the particles involved, and the modification of the N/D solution necessary in the presence of anomalous singularities. The description of the angular momentum states of a many-particle system in terms of longitudinal spin components (helicites) is not only applicable to relativistic situations, but also enables the reduction of the S matrix and the unitarity relation thereof to simpler forms.

In this paper we shall proceed from a rather formal point of view, leaving the application to a physically interesting case—pion-nucleon scattering—to the following paper.[10]

In the next section, we discuss the choice of variables to describe two- and three-particle systems and which are convenient for later purposes. The invariant amplitudes for two-particle scattering, production, and three-particle scattering are defined and the unitarity relations between these amplitudes are noted. The unitarity formulation of Blankenbecler is contrasted with the

* Supported in part by the Air Force Office of Scientific Research, Air Research and Development Command.

† Research supported in part by the United States Air Force under Grant No. AF-AFOSR-61-19 monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

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[1] R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961); Phys. Rev. **118**, 325 (1960).

[2] C. Goebel and H. J. Schnitzer, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 298.

[3] P. Carruthers, Ann. Phys. (New York) **14**, 229 (1961).

[4] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).

[5] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960); M. Nauenberg, thesis (unpublished).

[6] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

[7] K. Meetz, Phys. Rev. **125**, 714 (1962).

[8] S. Ciulli and J. Fischer, Nuovo cimento **12**, 204 (1959).

[9] L. F. Cook, Jr., and J. Tarski, Phys. Rev. Letters **5**, 585 (1960); J. Math. Phys. **3**, 1 (1963). Additional references are given in these papers.

[10] L. F. Cook, Jr., and B. W. Lee (to be published).

usual statement of the unitarity relation, viz.,

$$T-T^{\dagger}=2\pi i T^{\dagger}\delta(E-H)T,$$

and a justification is given for Blankenbecler's formulation of the unitarity relation which expresses the *absorptive* part of an amplitude in terms of quadratic forms of amplitudes.

In Sec. III, the decomposition of the amplitudes concerned into helicity amplitudes[11] is demonstrated in detail, and the validity of the decomposition of the production amplitude is considered. The reduction of the unitarity relation to definite angular momentum and parity "sectors" is carried out.

In Sec. IV, we define new amplitudes which are free from kinematical singularities and which satisfy simple unitarity relations. The N/D method is formulated for these amplitudes. The inversion problem of the infinite-dimensional, continuous denominator matrix $(\mathbf{D})$ is solved with the aid of the Fredholm theory.

Section V deals with the analytic continuation of the N/D solution in the presence of anomalous singularities. An extension of the Levinson theorem[12] to the multichannel production case is discussed. The effect of anomalous singularities on the threshold behavior of the production amplitude is also noted there.

II. KINEMATICS AND UNITARITY

A. Definition of Variables and Amplitudes

While the formalism presented in this paper is applicable to any elastic scattering and production process, we shall confine our attention to pion-nucleon scattering and one-pion production for ease in description and to avoid confusion.

We shall neglect the trivial isotopic spin dependence but will take into full account the nucleon spin in the following. We label the momenta associated with various processes as follows:

$$\begin{aligned}
&(\mathrm{I}) \quad N(p)+\pi(k)\rightarrow N(p')+\pi(k'),\\
&(\mathrm{II}) \quad N(p)+\pi(k)\rightarrow N(q)+\pi(k_1)+\pi(k_2),\\
&(\mathrm{II}') \quad N(q)+\pi(k_1)+\pi(k_2)\rightarrow N(p)+\pi(k),\\
&(\mathrm{III}) \quad N(q)+\pi(k_1)+\pi(k_2)\rightarrow N(q')+\pi(k_1')+\pi(k_2').
\end{aligned} \tag{1}$$

For both two- and three-particle systems we introduce the square of the total center-of-mass energy as the invariant variable s:

$$s=-(p+k)^2,$$
or
$$s=-(q+k_1+k_2)^2, \tag{2}$$

where $p^2=\mathbf{p}^2-p_0^2=-m^2$, $q^2=-m^2$, and $k_1^2=k_2^2=-\mu^2$.

In a two-particle system, we introduce the polar and azimuthal angles $\Omega=(\theta,\phi)$ $[\Omega'=(\theta',\phi')]$ of k (k') relative to some arbitrarily chosen Cartesian coördinates in the center-of-mass system.

In a three-particle system, there are several ways to specify the configuration, and a preference of one description over the other can be determined only in the context of the underlying dynamics. Here, we shall discuss only one of the possibilities, which we shall utilize in the following paper, but the extension of our method to other descriptions should be transparent.

We define the direction of $\mathbf{K}\equiv\mathbf{k}_1+\mathbf{k}_2$ by two angles $\Phi=(\gamma,\delta)$ measured with respect to some coördinate axes in the *three-particle center-of-mass* system. In this total center-of-mass system the direction of q is opposite to that of $\mathbf{K}$, but the relative orientation of the two-pion system has yet to be specified. To this end we designate the direction of $\mathbf{k}_1$ by angles $\Xi=(\alpha,\beta)$ measured with respect to some Cartesian axes in the *two-pion rest frame.* In addition we define the invariant variable σ by

$$\sigma=-(k_1+k_2)^2, \tag{3}$$

which is the square of the "mass" of the two-pion system. It is clear that when the Lorentz transformation connecting the coördinate systems in the total center-of-mass and the two-pion rest frames are given, the variables s, σ, Φ, and Ξ will describe the configuration of the three particles completely. We postpone the specification of this Lorentz transformation until Sec. III.

Alternatively, we may define the direction of the sum of q and k_1, say, in the total center-of-mass system and the direction of k_1 in the rest frame of the nucleon and the first pion. Clearly, these alternatives correspond to two ways of combining the three angular momenta.

In terms of definite momenta of the particles involved we define the following Lorentz invariant amplitudes:

$$\begin{aligned}
\langle\mathbf{k}',\mathbf{p}'\lambda';\Omega'|M_{22}(s)|\mathbf{k},\mathbf{p}\lambda;\Omega\rangle&=(2k_0')^{1/2}\bar{u}_{\lambda'}(p')\langle k'|f|k,p\lambda^{(\mathrm{in})}\rangle(2k_0)^{1/2}(p_0/m)^{1/2}=M_{22}(s,\Omega',\Omega;\lambda',\lambda),\\
\langle\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi|M_{32}(s,\sigma)|\mathbf{k},\mathbf{p}\lambda;\Omega\rangle&=(4k_{10}k_{20})^{1/2}\bar{u}_{\nu}(q)\langle k_1,k_2^{(\mathrm{out})}|f|k,p\lambda^{(\mathrm{in})}\rangle(2k_0)^{1/2}(p_0/m)^{1/2}=M_{32}(s,\sigma,\Phi\Xi,\Omega;\nu\lambda),\\
\langle\mathbf{k},\mathbf{p}\lambda;\Omega|M_{23}(s,\sigma)|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle&=(2k_0)^{1/2}(p_0/m)^{1/2}\langle k,p\lambda^{(\mathrm{out})}|f^{\dagger}|k_1,k_2^{(\mathrm{in})}\rangle u_{\nu}(q)(4k_{10}k_{20})^{1/2}=M_{23}(s,\sigma,\Omega,\Phi\Xi;\lambda\nu),\\
\langle\mathbf{k}_1',\mathbf{k}_2',\mathbf{q}'\nu';\Phi'\Xi'|M_{33}(s,\sigma',\sigma)|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle&\\
=(4k_{10}'k_{20}')^{1/2}\bar{u}_{\nu'}(q')\langle k_1',k_2'^{(\mathrm{out})}|f|k_1,k_2,q\nu^{(\mathrm{in})}\rangle&(4k_{10}k_{20})^{1/2}(q_0/m)^{1/2}=M_{33}(s,\sigma',\sigma,\Phi'\Xi',\Phi\Xi;\nu',\nu),
\end{aligned} \tag{4}$$

where λ, λ', ν, and ν' are helicity indices of the nucleon, $(=1/2,-1/2)$, and $f=f(0)$ is the nucleon current operator,

$$(\gamma\cdot\partial+m)\Psi(x)=f(x).$$

The matrix elements exhibited above are understood to be the on-the-energy-shell ones. Two-pion states are

[11] M. Jacob and G. C. Wick, Ann. Phys. (New York) **7**, 404 (1959).

[12] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).

normalized as

$$|\mathbf{k}_1,\mathbf{k}_2\rangle=(2!)^{-1/2}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|\mathrm{vac}\rangle.$$

The unconventional contraction of the nucleon operator, rather than the pion operator, is deliberate; it serves to fix the mass σ and to explicate the correlation of the two-pion system.

It must be emphasized that some of the angular variables shown in Eq. (4) are redundant. In $M_{22}(s,\Omega',\Omega,\lambda',\lambda)$, for instance, we may choose the direction of $\mathbf{k}$ as the z direction and the plane of $\mathbf{k}$ and $\mathbf{k}'$ as the zx plane. Then M_{22} depends only on s, θ' (or alternatively, $\cos\theta'$) and the helicity indices λ, λ'. Nonetheless, it will be convenient to maintain this redundancy to discuss the unitarity relations.

B. Unitarity

For the sake of brevity in this section, we shall suppress the angular and helicity variables of the amplitudes. The unitarity relations of Blankenbecler,[6] valid for $s\geqq(m+\mu)^2$, may be written, neglecting the contributions from four (or more) particle intermediate states, as:

$$\begin{aligned}
&M_{22}(s+i\epsilon)-M_{22}(s-i\epsilon)\\
&\quad=2\pi i\textstyle\sum_2 M_{22}(s+i\epsilon)M_{22}(s-i\epsilon)\\
&\qquad+2\pi i\textstyle\sum_3 M_{23}(s+i\epsilon,\sigma''+i\epsilon)M_{32}(s-i\epsilon,\sigma''-i\epsilon),\\
&M_{32}(s+i\epsilon,\sigma)-M_{32}(s-i\epsilon,\sigma)\\
&\quad=2\pi i\textstyle\sum_2 M_{32}(s+i\epsilon,\sigma)M_{22}(s-i\epsilon)\\
&\qquad+2\pi i\textstyle\sum_3 M_{33}(s+i\epsilon,\sigma,\sigma''+i\epsilon)M_{32}(s-i\epsilon,\sigma''-i\epsilon),\\
&M_{23}(s+i\epsilon,\sigma)-M_{23}(s-i\epsilon,\sigma)\\
&\quad=2\pi i\textstyle\sum_2 M_{22}(s+i\epsilon)M_{23}(s-i\epsilon,\sigma)\\
&\qquad+2\pi i\textstyle\sum_3 M_{23}(s+i\epsilon,\sigma''+i\epsilon)M_{33}(s-i\epsilon,\sigma''-i\epsilon,\sigma),\\
&M_{33}(s+i\epsilon,\sigma',\sigma)-M_{33}(s-i\epsilon,\sigma',\sigma)\\
&\quad=2\pi i\textstyle\sum_2 M_{32}(s+i\epsilon,\sigma')M_{23}(s-i\epsilon,\sigma)\\
&\qquad+2\pi i\textstyle\sum_3 M_{33}(s+i\epsilon,\sigma',\sigma''+i\epsilon)M_{33}(s-i\epsilon,\sigma''-i\epsilon,\sigma).
\end{aligned}\tag{5}$$

Here $\sum_2$ and $\sum_3$ are

$$\begin{aligned}
\textstyle\sum_2&=\frac{2m}{4(2\pi)^3}\frac{P(s)}{s^{1/2}}\theta[s-(m+\mu)^2]\int d\Omega''\textstyle\sum_{\lambda''},\\
\textstyle\sum_3&=\frac{2m}{32(2\pi)^6}\int d\sigma''\frac{Q(s,\sigma'')}{s^{1/2}}\left(\frac{\sigma''-4\mu^2}{\sigma''}\right)^{1/2}\theta(\sigma''-4\mu^2)\\
&\quad\times\theta[(s^{1/2}-m)^2-\sigma'']\int d\Phi''d\Xi''\textstyle\sum_{\nu''},
\end{aligned}\tag{6}$$

where

$$d\Omega=d\phi\,d\cos\theta,$$
$$d\Phi=d\delta\,d\cos\gamma,$$
$$d\Xi=d\beta\,d\cos\alpha,$$

P, Q are the magnitudes of 3-momenta of the two- and three-particle channels:

$$\begin{aligned}
2(s)^{1/2}P(s)&=[s-(m+\mu)^2]^{1/2}[s-(m-\mu)^2]^{1/2},\\
2(s)^{1/2}Q(s,\sigma)&=[s-(m+\sigma)^2]^{1/2}[s-(m-\sigma)^2]^{1/2},
\end{aligned}\tag{7}$$

and the double primes denote variables of intermediate states. Thus, written out in full, the first line of Eq. (5) reads

$$\begin{aligned}
&M_{22}(s+i\epsilon,\Omega',\Omega;\lambda',\lambda)-M_{22}(s-i\epsilon,\Omega',\Omega;\lambda',\lambda)\\
&=2\pi i\rho_2(s)\int d\Omega''\textstyle\sum_{\lambda''}M_{22}(s+i\epsilon,\Omega',\Omega'';\lambda',\lambda'')M_{22}(s-i\epsilon,\Omega'',\Omega;\lambda'',\lambda)\\
&\quad+2\pi i\int d\sigma''\,\rho_3(s,\sigma'')\int d\Phi''d\Xi''\textstyle\sum_{\nu''}M_{23}(s+i\epsilon,\Omega',\Phi''\Xi'';\lambda',\nu'')M_{32}(s-i\epsilon,\Phi''\Xi'',\Omega;\nu'',\lambda),
\end{aligned}\tag{5'}$$

where

$$\begin{aligned}
\rho_2(s)&=\frac{2m}{4(2\pi)^3}\frac{P(s)}{s^{1/2}}\theta[s-(m+\mu)^2],\\
\rho_3(s,\sigma)&=\frac{2m}{32(2\pi)^6}\frac{Q(s,\sigma)}{s^{1/2}}\theta(\sigma-4\mu^2)\\
&\quad\times\theta[(s^{1/2}-m)^2-\sigma]\left[\frac{\sigma-4\mu^2}{\sigma}\right]^{1/2}.
\end{aligned}\tag{8}$$

Note that, although $\sum_2$ and $\sum_3$ are invariant phase-space integrals, they are most conveniently expressed in particular Lorentz frames, and our choice of kinematical variables is motivated in this context.

A few remarks are in order about the unitarity relations in Eq. (5). These relations are quite distinct from Cutkosky's[13] generalized unitarity which connects the *imaginary part* of a graph to all possible partitions of it. Equation (5) states the relations between the *absorptive parts* of amplitudes and the full amplitudes. Let us examine the second line of Eq. (5). We have, from Eq. (4),

$$\begin{aligned}
&M_{32}(s+i\epsilon,\sigma\pm i\epsilon)\\
&\quad=i(8k_{10}k_{20}k_0)^{1/2}\int d^4x\,e^{-i(q+p)\cdot x/2}\bar{u}(q)\\
&\qquad\times\langle k_1,k_2^{(\mathrm{out,in})}|\theta(x)[f(x/2),f^\dagger(-x/2)]|k\rangle u(p).
\end{aligned}$$

We continue $M_{32}(s+i\epsilon,\sigma\pm i\epsilon)$ in σ to a small value, $\sigma<4\mu^2$, in which case $M_{32}(s,\sigma)$ can be continued in s

[13] See, also, R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

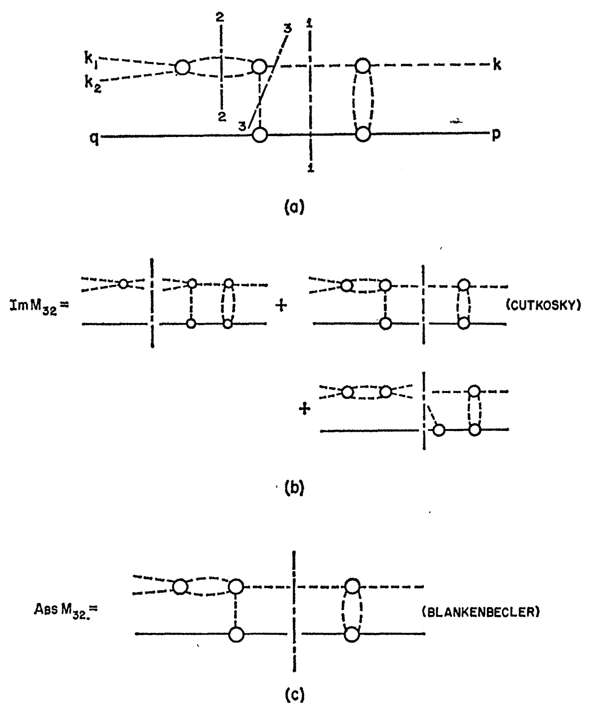

FIG. 1. Comparison of the generalized unitarity of Cutkosky and the unitarity relation of Blankenbecler. In computing the imaginary part of M_{32}, one must partition the diagram (a) along 2 and 3, but in computing the "absorptive" part, one may not partition along 2 or 3.

into the lower half plane (i.e., there will be a "gap"—singularity-free region—along the real s axis). Continuing σ back to the original value, $\mathrm{Re}\sigma > 4\mu^2$, we can define $M_{32}(s-i\epsilon,\, \sigma\pm i\epsilon)$, which we maintain is identical with the advanced amplitude:

$$M_{32}(s-i\epsilon,\, \sigma\pm i\epsilon) = -i(8k_{10}k_{20}k_0)^{1/2}\int d^4x\, e^{-i(q+p)\cdot x/2}\bar{u}(q) \times\langle k_1,\, k_2^{(\mathrm{out,in})}|\theta(-x)[f(x/2),\, f^\dagger(-x/2)]|k\rangle u(p).$$

By taking the difference of $M_{32}(s+i\epsilon,\,\sigma)$ and $M_{32}(s-i\epsilon,\,\sigma)$, the second line of Eq. (5) follows in the standard manner.[14] In this connection the prescription of $\sigma''\pm i\epsilon$ should be meticulously observed.

The amplitude $M_{33}(s,\sigma',\sigma)$ contains two disconnected graphs, in which one (or the other) of the pions does not interact with the remaining pion and nucleon, while the disconnected graph in which the nucleon is not interacting is missing. Note

$$\langle N2\pi|S|N2\pi\rangle = \delta_{\nu'\nu}\delta(\mathbf{q}'-\mathbf{q})\langle k_1'k_2'^{(\mathrm{out})}|k_1k_2^{(\mathrm{in})}\rangle - (2\pi)^4 i\delta(q'+k_1'+k_2'-q-k_1-k_2)M_{33}.$$

[14] We are well aware of the lack of rigor in our argument. When two pions in the final state are assumed to be noninteracting, however, this statement can be in fact proven. See Y. S. Kim, Phys. Rev. **124**, 1241 (1961) and Princeton University thesis (unpublished).

The significance of all these remarks will become clearer if we consider the particular diagram shown in Fig. 1.[13] In the usual unitarity, the imaginary part of the diagram Fig. 1(a), in the physical region of the production $[s\geq(m+\sigma^{1/2})^2,\, \sigma\geq 4\mu^2]$ is the sum of all partitions (1, 2, and 3). See Fig. 1(b). In Eq. (5), one is not allowed to partition the diagram along 2 and 3 [Fig. 1(c)], and therefore, in M_{33}, the disconnected graph in which the nucleon is noninteracting is excluded. An advantage of using the unitarity expressed by Eq. (5) is that, in this case, we need not be concerned with the details of the pion-pion interaction, but may substitute the result of a previous study, or a phenomenological description of the pion-pion correlation.

At this point we may formulate the linear unitarity relations in terms of Blankenbecler's generalized N and D functions for the amplitudes M_{ij}. Such relations would involve, however, multiple integrals over angles, which make these relations intractable from the practical point of view. We shall therefore postpone the N/D formulation until we have disposed of the complicating angular dependence of the amplitudes.

III. HELICITY AMPLITUDES

A. Decomposition of the Elastic Amplitude

The decomposition of an elastic scattering amplitude into helicity amplitudes has been studied extensively elsewhere,[11] but for the sake of completeness, we will summarize the results here. It follows from the rotational invariance of the S matrix that the invariant amplitude $\langle\lambda'\Omega'|M_{22}(s)|\lambda\Omega\rangle$ may be written

$$\langle \mathbf{k}',\mathbf{p}'\lambda';\Omega'|M_{22}(s)|\mathbf{k},\mathbf{p}\lambda;\Omega\rangle = (1/4\pi)\sum_{JM}(2J+1)\langle\lambda'|M_{22}^J(s)|\lambda\rangle \times d_{M,-\lambda'}^J(\theta')e^{i(M+\lambda')\phi'}d_{M,-\lambda'}^J(\theta)e^{-i(M+\lambda)\phi}, \quad (9)$$

where $\langle\lambda'|M_{22}^J(s)|\lambda\rangle$ is, apart from a multiplicative factor, the helicity amplitude of Jacob and Wick (JW).

Specializing to the case of $\theta=0$, $\phi'=0$, we obtain

$$M_{22}(s,(\theta,0),0;\lambda'\lambda) = (1/4\pi)\sum_J(2J+1)\langle\lambda'|M_{22}^J(s)|\lambda\rangle d_{\lambda'\lambda}^J(\theta'), \quad (10)$$

where we have used the relation $d_{M\lambda}^J(0)=\delta_{M\lambda}$. By virtue of the orthogonality property of $d_{\lambda'\lambda}(\theta)$ (JW 22), the helicity amplitude $\langle\lambda'|M_{22}^J(s)|\lambda\rangle$ may be written

$$\langle\lambda'|M_{22}^J(s)|\lambda\rangle = 2\pi\int_{-1}^{1} d\cos\theta'\, d_{\lambda'\lambda}^J(\theta') \times M_{22}(s,\,(\theta',0),\,\Omega=0;\,\lambda',\,\lambda). \quad (11)$$

Note that the amplitude $M_{22}(s,(\theta',0),0;\lambda',\lambda)$ is a function of two variables s and θ', and redundant variables are absent.

B. Decomposition of the Production Amplitude

First we shall specify the orientation of the coordinate frame C', located in the two-pion center-of-mass system in which the angles $\Xi=(\alpha,\beta)$ are defined (see II A), with respect to the coördinate system C in the total center-of-mass system. The coördinate system C' is such that the z_c axis is in the direction of $-\mathbf{q}_c$ (the subscript c designates quantities measured in the two-pion system C'). The configuration of the two pions in the C frame is obtained from that of the C' frame by applying the pure Lorentz transformation Z along the z_c axis with the velocity $Q/(Q^2+\sigma)^{1/2}$, followed by the rotation $R_{\delta,\gamma,-\delta}$ where, as in JW,

$$R_{a,b,c}=e^{-iJ_z a}e^{-iJ_y b}e^{-iJ_z c}$$

(actually $R_{\delta,\gamma,\eta}$ with arbitrary η will do, but this choice turns out to be the simplest). Then

$$|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle\equiv|\mathbf{q}\nu\rangle R_{\delta,\gamma,-\delta}Z|\mathbf{k}_{1c},\mathbf{k}_{2c};\Xi\rangle$$
$$=|\mathbf{q}\nu\rangle R_{\delta,\gamma,-\delta}Z\sum_l\sum_{\Lambda=-l}^{l}|\mathbf{K}_c,\Lambda(l)\rangle\langle l,\Lambda|\Xi\rangle,$$

where $|\mathbf{q}\nu\rangle$ is the nucleon state of momentum $\mathbf{q}$ and helicity ν, $|\mathbf{K}_c,\Lambda(l)\rangle$ is the state of two pions with total momentum $\mathbf{K}_c=\mathbf{k}_{1c}+\mathbf{k}_{2c}=0$ and angular momentum l with its projection on the z_c axis Λ. Since the pure Lorentz transformation Z is along the z_c axis, Λ is in fact the Lorentz-invariant helicity of the two-pion system. We assume the bras and kets are normalized according to the prescription of JW. Using the formula (JW 6)

$$R_{\delta,\gamma,-\delta}Z|\mathbf{K}_c=0,\Lambda(l)\rangle=|\mathbf{K}\Lambda(l);\Phi\rangle,$$

we obtain

$$|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle=\sum_{l,\Lambda}|\mathbf{K}\Lambda(l),\mathbf{q}\nu;\Phi\rangle\langle l\Lambda|\Xi\rangle$$
$$=\sum_{l,\Lambda}|\mathbf{K}\Lambda(l),\mathbf{q}\nu;\Phi\rangle Y_{l\Lambda}^*(\alpha,\beta),\quad(12)$$

so that

$$\langle\mathbf{k},\mathbf{p}\lambda;\Omega|M_{23}(s,\sigma)|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle$$
$$=\sum_{l,\Lambda}\langle\mathbf{k},\mathbf{p}\lambda;\Omega|M_{23}(s,\sigma)|\mathbf{K}\Lambda(l),\mathbf{q}\nu;\Phi\rangle Y_{l\Lambda}^*(\alpha,\beta).\quad(13)$$

Now the matrix element $\langle\mathbf{k},\mathbf{p}\lambda;\Omega|M_{23}(s,\sigma)|\mathbf{K}\Lambda(l),\mathbf{q}\nu;\Phi\rangle$ has the same transformation properties under rotation as the two-particle scattering amplitude. Therefore, we define the helicity amplitudes $\langle\lambda|M_{23}{}^{Jl}(s,\sigma)|\nu,\Lambda\rangle$ by

$$\langle\mathbf{k},\mathbf{p}\lambda;\Omega|M_{23}(s,\sigma)|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle$$
$$=\sum_{l,\Lambda}\sum_{J,M}\frac{2J+1}{4\pi}\langle\lambda|M_{23}{}^{Jl}(s,\sigma)|\nu,\Lambda\rangle Y_{l\Lambda}^*(\alpha,\beta)$$
$$\times d_{M,-\lambda}{}^J(\theta)e^{i(M+\lambda)\phi}d_{M,\Lambda-\nu}{}^J(\gamma)e^{-i(M-\Lambda+\nu)\delta}.\quad(14)$$

We now specialize to the case $\gamma=\phi=0$. Then

$$\langle\lambda|M_{23}{}^{Jl}(s,\sigma)|\nu,\Lambda\rangle$$
$$=2\pi\int d\Xi\,Y_{l\Lambda}(\alpha,\beta)\int_{-1}^{1}d\cos\theta\,d_{\Lambda-\nu,-\lambda}{}^J(\theta)$$
$$\times M_{23}(s,\sigma,(\theta,0),\Phi=0,\Xi;\lambda,\nu).\quad(15)$$

As expected, there are five independent variables, s, σ, θ, α, and β in the production amplitude.

Similarly

$$\langle\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi|M_{32}(s,\sigma)|\mathbf{k},\mathbf{p}\lambda,\Omega\rangle$$
$$=\sum_{l,\Lambda}\sum_{J,M}\frac{2J+1}{4\pi}\langle\nu,\Lambda|M_{32}{}^{Jl}(s,\sigma)|\lambda\rangle Y_{l\Lambda}(\alpha,\beta)$$
$$\times d_{M,-\lambda}{}^J(\theta)e^{-i(M+\lambda)\phi}d_{M,\Lambda-\nu}{}^J(\gamma)e^{i(M-\lambda+\nu)\delta}.\quad(16)$$

Finally, time-reversal invariance implies the connection

$$\langle\nu,\Lambda|M_{32}{}^{Jl}(s+i\epsilon,\sigma+i\epsilon)|\lambda\rangle$$
$$=\langle\lambda|M_{23}{}^{Jl}(s-i\epsilon,\sigma-i\epsilon)|\nu,\Lambda\rangle^*.\quad(17)$$

What we have done here is based on the physical observation that the two-pion state of angular momentum l and helicity Λ behaves exactly as a particle of spin l and helicity Λ insofar as the kinematics of the production process is concerned. Thus, the amplitude $\langle\nu,\Lambda|M_{32}{}^{Jl}(s,\sigma)|\lambda\rangle$ may be looked at as the inelastic, scattering amplitude: a state of angular momentum J consisting of a nucleon of helicity λ and a pion producing a nucleon of helicity ν and a particle of spin l, helicity Λ, and (variable) mass σ. Hence, when the amplitude $\langle\nu,\Lambda|M_{32}{}^{Jl}(s,\sigma)|\lambda\rangle$ has a sharp peak in σ, the picture of an unstable particle naturally arises.

It is obvious now that the three-particle scattering amplitude M_{33} can be decomposed into helicity amplitudes in a similar manner. Arguing as before, we obtain the equation defining the helicity amplitudes $\langle\nu',\Lambda'|M_{33}{}^{Jl'l}(s,\sigma',\sigma)|\nu,\Lambda\rangle$ as the following:

$$\langle\mathbf{k}_1',\mathbf{k}_2',\mathbf{q}'\nu';\Phi'\Xi'|M_{33}(s,\sigma',\sigma)|\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}\nu;\Phi\Xi\rangle$$
$$=\sum_{l'=0}^{\infty}\sum_{\Lambda'=-l'}^{l'}Y_{l'\Lambda'}(\Xi')\sum_{l=0}^{\infty}\sum_{\Lambda=-l}^{l}Y_{l\Lambda}^*(\Xi)$$
$$\times\frac{1}{4\pi}\sum_{J,M}(2J+1)\langle\nu',\Lambda'|M_{33}{}^{Jl'l}(s,\sigma',\sigma)|\nu,\Lambda\rangle$$
$$\times d_{M,\Lambda'-\nu'}{}^J(\gamma')e^{i(M+\nu'-\Lambda')\delta'}$$
$$\times d_{M,\Lambda'-\nu}{}^J(\gamma)e^{-i(M+\nu-\Lambda)\delta}.\quad(18)$$

At this juncture it should be noted that the helicity amplitude expansion for the production amplitudes cannot be continued into the complex plane of the cosine of certain angles.[15] This follows because production amplitudes have singularities for physical values of cosines of angles. Note, for instance, that the amplitude M_{23} has a branch cut in the variable $-(q+k_1)^2$ for $(m+\mu)^2\leq-(q+k_1)^2<\infty$, and this condition is satisfied

[15] S. B. Treiman (private communication).

for physical values of s, σ, $\cos\theta$, $\cos\alpha$, and $\cos\beta$. However, as emphasized by Wick,[16] the expansion is likely to converge[17] for physical values of $\cos\theta$ and $\cos\alpha$, i.e., $-1\leq\cos\theta$, $\cos\alpha\leq 1$, even though it may not be continued to complex values. We shall have no occasion to continue the production amplitude either in $\cos\theta$ or $\cos\alpha$. As to the rapidity of convergence, we remark that our experience with strong interactions indicates, in many cases, that only a few angular momentum states need be considered, at least at reasonable energies.

C. Unitarity of Helicity Amplitudes

When Eqs. (9), (14), (16), and (18) are substituted in Eq. (5), the angular integrals can be performed easily. In terms of the helicity amplitudes, the unitarity relations, Eq. (5), become very simple:

$$
\begin{aligned}
&(1/2\pi i)\langle\lambda'|M_{22}{}^J(s+i\epsilon)-M_{22}{}^J(s-i\epsilon)|\lambda\rangle\\
&\quad=\rho_2(s)\textstyle\sum_{\lambda''}\langle\lambda'|M_{22}{}^J(s+i\epsilon)|\lambda''\rangle\langle\lambda''|M_{22}{}^J(s-i\epsilon)|\lambda\rangle\\
&\qquad+\int d\sigma''\,\rho_3(s,\sigma'')\sum_{l'',\Lambda'',\nu''}\langle\lambda'|M_{23}{}^{Jl''}(s+i\epsilon,\sigma''+i\epsilon)|\nu'',\Lambda''\rangle\langle\nu'',\Lambda''|M_{32}{}^{Jl''}(s-i\epsilon,\sigma-i\epsilon)|\lambda\rangle,\\
&(1/2\pi i)\langle\nu,\Lambda|M_{32}{}^{Jl}(s+i\epsilon,\sigma)-M_{32}{}^{Jl}(s-i\epsilon,\sigma)|\lambda\rangle\\
&\quad=\rho_2(s)\textstyle\sum_{\lambda''}\langle\nu,\Lambda|M_{32}{}^{Jl}(s+i\epsilon,\sigma)|\lambda''\rangle\langle\lambda''|M_{22}{}^J(s-i\epsilon)|\lambda\rangle\\
&\qquad+\int d\sigma''\,\rho_3(s,\sigma'')\sum_{l'',\Lambda'',\nu''}\langle\nu,\Lambda|M_{33}{}^{Jll''}(s+i\epsilon,\sigma,\sigma''+i\epsilon)|\nu'',\Lambda''\rangle\langle\nu'',\Lambda''|M_{32}{}^{Jl''}(s-i\epsilon,\sigma''-i\epsilon)|\lambda\rangle,\\
&(1/2\pi i)\langle\lambda|M_{23}{}^{Jl}(s+i\epsilon,\sigma)-M_{23}{}^{Jl}(s-i\epsilon,\sigma)|\nu,\Lambda\rangle\\
&\quad=\rho_2(s)\textstyle\sum_{\lambda''}\langle\lambda|M_{22}{}^J(s+i\epsilon)|\lambda''\rangle\langle\lambda''|M_{23}{}^{Jl}(s-i\epsilon,\sigma)|\nu,\Lambda\rangle\\
&\qquad+\int d\sigma''\,\rho_3(s,\sigma'')\sum_{l'',\Lambda'',\nu''}\langle\lambda|M_{23}{}^{Jl''}(s+i\epsilon,\sigma''+i\epsilon)|\nu'',\Lambda''\rangle\langle\nu'',\Lambda''|M_{33}{}^{Jl''l}(s-i\epsilon,\sigma''-i\epsilon,\sigma)|\nu\Lambda\rangle,\\
&(1/2\pi i)\langle\nu',\Lambda'|M_{33}{}^{Jl'l}(s+i\epsilon,\sigma',\sigma)-M_{33}{}^{Jl'l}(s-i\epsilon,\sigma',\sigma)|\nu,\Lambda\rangle\\
&\quad=\rho_2(s)\textstyle\sum_{\lambda''}\langle\nu',\Lambda'|M_{32}{}^{Jl'}(s+i\epsilon,\sigma')|\lambda''\rangle\langle\lambda''|M_{23}{}^{Jl}(s-i\epsilon,\sigma)|\nu,\Lambda\rangle\\
&\qquad+\int d\sigma''\,\rho_3(s,\sigma'')\sum_{l'',\Lambda'',\nu''}\langle\nu',\Lambda'|M_{33}{}^{Jl'l''}(s+i\epsilon,\sigma',\sigma''+i\epsilon)|\nu'',\Lambda''\rangle\langle\nu'',\Lambda''|M_{33}{}^{Jl''l}(s-i\epsilon,\sigma''-i\epsilon,\sigma)|\nu,\Lambda\rangle.
\end{aligned}
\tag{19}
$$

The decomposition into helicity amplitudes has the following advantages over that of Ciulli and Fischer[8] who decompose the production amplitude in the total center-of-mass system: (1) In the present scheme, the angular momentum l of the two-pion system refers to that in the rest frame of the two pions. Hence, the final interaction in the state (J,l) is due to the pion-pion interaction in the state of angular momentum l. In the Ciulli-Fischer scheme, however, the angular momentum of the two-pion system is measured in the total center-of-mass system, and therefore, the resulting amplitudes do not correspond to a definite angular momentum state of the two-pion system in its rest frame (this has been pointed out by Peierls[1] and Carruthers[3]). (2) The unitarity relations take a particularly simple form for the helicity amplitudes. This is because the angular variables Φ, Ξ, in the three-particle, phase-space intergral in Eq. (6) coincide with those in terms of which the helicity decomposition, Eq. (15), is carried out. It is clear that, in terms of the Ciulli-Fischer amplitudes, the unitarity relations cannot be expressed as simply as Eq. (19), contrary to Meetz's[7] assertion.

Rather than using Eq. (19), it is more convenient to define eigenamplitudes of parity as in (JW 57). To this end we form eigenstates of parity:

$$|JM\Pi\rangle=|JM;\lambda=1/2\rangle\pm(-1)^{J+1/2}|JM;\lambda=-1/2\rangle,\quad(\Pi=\pm),\tag{20}$$

where Π is the "total parity" of the pion-nucleon system. We define the eigenamplitudes of parity $M_{22}{}^{J\Pi}$ by

$$
\begin{aligned}
M_{22}{}^{J\Pi}(s)&\equiv(1/2)\langle JM\Pi|M_{22}(s)|JM\Pi\rangle\\
&=\langle(1/2)|M_{22}{}^J(s)|(1/2)\rangle\pm(-1)^{J+1/2}\\
&\quad\times\langle-(1/2)|M_{22}{}^J(s)|1/2\rangle,\quad(\Pi=\pm).
\end{aligned}\tag{21}
$$

The amplitude $M_{22}{}^{J\Pi}(s)$ is, of course, independent of M.

Similarly, for the three-particle system we define

$$
\begin{aligned}
|JM\Pi,l,\xi\rangle&=|JM,l;\nu=1/2,\Lambda=\xi\rangle\\
&\quad\pm(-1)^{J-1/2}|JM,l;\nu=-1/2,\Lambda=-\xi\rangle,\\
&\xi=0,\cdots l,\quad\Pi=\pm1.
\end{aligned}\tag{22}
$$

[16] G. C. Wick (private communication).

[17] E. T. Whittaker and G. N. Watson, *Course of Modern Analysis* (Cambridge University Press, New York, 1952), p. 323.

For each parity state, there are $2l+1$ states of angular momentum J. We define the eigenamplitudes of parity $M_{32}{}^{J\Pi,l\xi}$ and $M_{33}{}^{J\Pi,l'\xi';l\xi}$ by:

$$M_{32}{}^{J\Pi,l\xi}\equiv(1/2)\langle JM\Pi,l,\xi|M_{32}|JM\Pi\rangle$$
$$=\langle(1/2),\xi|M_{32}{}^{Jl}(s,\sigma)|(1/2)\rangle\pm(-1)^{J-1/2}\langle-(1/2),-\xi|M_{32}{}^{Jl}(s,\sigma)|(1/2)\rangle,\quad(\Pi=\pm),\tag{23}$$

$$M_{33}{}^{J\Pi,l'\xi';l\xi}\equiv(1/2)\langle JM\Pi,l',\xi'|M_{33}|JM\Pi,l,\xi\rangle$$
$$=\langle(1/2),\xi'|M_{33}{}^{Jl',l}(s,\sigma',\sigma)|(1/2),\xi\rangle\pm(-1)^{J-1/2}$$
$$\times\langle-(1/2),-\xi'|M_{33}{}^{Jl',l}(s,\sigma',\sigma)|(1/2),\xi\rangle,\quad(\Pi=\pm).\tag{24}$$

The amplitude $M_{23}{}^{J\Pi,l\xi}$ can be defined so that

$$M_{23}{}^{J\Pi,l\xi}(s+i\epsilon,\sigma+i\epsilon)=[M_{32}{}^{J\Pi,l\xi}(s-i\epsilon,\sigma-i\epsilon)]^*.\tag{25}$$

Since the parity is conserved, amplitudes of the same parity are related by the unitarity relations. Substituting Eqs. (22), (24), (25), and (26) into Eq. (19), and suppressing the superscripts $J\Pi$, we obtain

$$\begin{aligned}
&(1/2\pi i)[M_{22}(s+i\epsilon)-M_{22}(s-i\epsilon)]\\
&\quad=\rho_2(s)M_{22}(s+i\epsilon)M_{22}(s-i\epsilon)+\sum_{l,\xi}\int d\sigma\,\rho_3(s,\sigma)M_{23}{}^{l\xi}(s+i\epsilon,\sigma+i\epsilon)M_{32}{}^{l\xi}(s-i\epsilon,\sigma-i\epsilon),\\
&(1/2\pi i)[M_{23}{}^{l\xi}(s+i\epsilon,\sigma)-M_{23}{}^{l\xi}(s-i\epsilon,\sigma)]\\
&\quad=\rho_2(s)M_{22}(s+i\epsilon)M_{23}{}^{l\xi}(s-i\epsilon,\sigma)+\sum_{l'\xi'}\int d\sigma'\,\rho_3(s,\sigma')M_{23}{}^{l'\xi'}(s+i\epsilon,\sigma'+i\epsilon)M_{33}{}^{l'\xi',l\xi}(s-i\epsilon,\sigma'-i\epsilon,\sigma),\\
&(1/2\pi i)[M_{32}{}^{l\xi}(s+i\epsilon,\sigma)-M_{32}{}^{l\xi}(s-i\epsilon,\sigma)]\\
&\quad=\rho_2(s)M_{32}{}^{l\xi}(s+i\epsilon,\sigma)M_{22}(s-i\epsilon)+\sum_{\xi'}\int d\sigma'\,\rho_3(s,\sigma')M_{33}{}^{l\xi,l'\xi'}(s+i\epsilon,\sigma,\sigma'+i\epsilon)M_{32}{}^{l'\xi'}(s-i\epsilon,\sigma'-i\epsilon),\\
&(1/2\pi i)[M_{33}{}^{l'\xi',l\xi}(s+i\epsilon,\sigma',\sigma)-M_{33}{}^{l'\xi',l\xi}(s-i\epsilon,\sigma',\sigma)]\\
&\quad=\rho_2(s)M_{32}{}^{l'\xi'}(s+i\epsilon,\sigma')M_{23}{}^{l\xi}(s-i\epsilon,\sigma)+\sum_{\xi''}\int d\sigma''\,\rho_3(s,\sigma'')M_{33}{}^{l'\xi',l''\xi''}(s+i\epsilon,\sigma',\sigma''+i\epsilon)M_{33}{}^{l''\xi'',l\xi}(s-i\epsilon,\sigma''-i\epsilon,\sigma).
\end{aligned}\tag{26}$$

IV. *N/D* FORMULATION

The elastic amplitude $M_{22}{}^{J\Pi}(s)$ can be expressed in terms of the phase shift $\delta_{J\Pi}$ as:

$$M_{22}{}^{J\Pi}(s)=\left[\frac{1}{\pi\rho_2(s)}\right]e^{i\delta_{J\Pi}}\sin\delta_{J\Pi}$$
$$=\frac{16\pi^2}{m}(s)^{1/2}\frac{e^{i\delta_{J\Pi}}\sin\delta_{J\Pi}}{P}.\tag{27}$$

In the notation of CGLN,[18] we find

$$M_{22}{}^{J\Pi}(s)=(\pi/m)\{(E+m)[A_L(s)+[(s)^{1/2}-m]B_L(s)]$$
$$+(E-m)[-A_{L\pm1}(s)+[(s)^{1/2}+m]B_{L\pm1}(s)]\},$$
$$\text{for}\quad J=L\pm1/2,$$

where L is the "orbital" angular momentum of the state $J\Pi$;

$$L=J\mp1/2,\quad\text{if}\quad\Pi=(-1)^{J\pm1/2},$$

and E is the c.m. nucleon energy,

$$E=(s+m^2-\mu^2)/2(s)^{1/2}.$$

The amplitude $M_{22}(s)$, considered as a function of s, contains kinematical cuts. To remove these, we define a new function $F_{22}{}^{J\Pi}(w)$, $w^2=s$, by

$$F_{22}{}^{J\Pi}(w)=g_2{}^L(w)M_{22}{}^{J\Pi}(w^2),\tag{28}$$

$$g_2{}^L(w)=\frac{2m}{E+m}\left(\frac{E+m}{P}\right)^{2L}.\tag{29}$$

The new amplitude $F_{22}{}^{J\Pi}(w)$ has dynamical branch cuts in the manner described by Frazer and Fulco.[19] The $2L$ power of P in the denominator of $g_2{}^L$ does not introduce additional poles in $F_{22}{}^{J\Pi}(w)$, since $M_{22}{}^{J\Pi}$ goes to zero as P^{2L} as P approaches zero.

We know very little about the analyticity of $M_{33}{}^{J\Pi l'l}(s,\sigma',\sigma)$, but we conjecture that kinematical cuts can be removed in much the same manner. Define, suppressing ξ' and ξ,

$$F_{33}{}^{J\Pi l'l}(w,\sigma',\sigma)=[g_3{}^{l'I'}(w,\sigma')]^{1/2}M_{33}{}^{J\Pi l'l}$$
$$\times(s,\sigma',\sigma)[g_3{}^{lI}(w,\sigma)]^{1/2},\tag{30}$$

with

$$g_3{}^{lI}(w,\sigma)=\frac{2m}{Q_0(s,\sigma)+m}\left[\frac{Q_0(s,\sigma)+m}{Q(s,\sigma)}\right]^{2I},\tag{31}$$

where I is the lowest orbital momentum (in c.m.) of the

[18] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957). Hereafter referred to as CGLN.

[19] W. R. Frazer and J. R. Fulco, Phys. Rev. 119, 1420 (1960).

$[(\pi\pi)_l N]_{J\Pi}$ system compatible with the parity considerations, such that

$$I=\min|\mathfrak{J}|,\quad \mathbf{J}=\mathfrak{J}+\mathbf{l}+\boldsymbol{\sigma};\quad |\boldsymbol{\sigma}|=1/2,$$

and

$$Q_0(s,\sigma)=[Q^2(s,\sigma)+m^2]^{1/2}.$$

The factor $(Q_0+m)^{-1/2}$ is necessary to cancel the kinematical singularities arising from the spinor normalization factor. In nonrelativistic theory, one expects that the factor $[Q(s,\sigma)]^{-I}[Q(s,\sigma')]^{-I'}$ removes kinematical singularities arising from

$$Q(s,\sigma)=[(w+m)^2-\sigma^2]^{1/2}[(w-m)^2-\sigma^2]^{1/2}/2w$$

and $Q(s,\sigma')$. Note that in nonrelativistic theory

$$M_{33}{}^{J l' l}(s,\sigma',\sigma)=\mathcal{O}[Q^I(s,\sigma),Q^{I'}(s,\sigma')],\quad \text{as } Q \text{ or } Q'\to 0.$$

Finally, we define

$$F_{23}{}^{J\Pi l}(w,\sigma)=[g_2{}^L(w)]^{1/2}M_{23}{}^{J\Pi l}(s,\sigma')[g_3{}^{lI}(w)]^{1/2}. \quad (32)$$

We claim the new amplitude $F_{23}{}^{J\Pi l}$ is devoid of kinematical singularities. This conjecture can in fact be proved for certain class of diagrams one of which will be dealt with in the following paper.

In terms of these new amplitudes the unitarity relations, Eq. (26), may be expressed as, suppressing the superscripts $J\Pi$ and denoting henceforth l and ξ collectively by l,

$$\begin{aligned}
&F_{22}(w+i\epsilon)-F_{22}(w-i\epsilon)\\
&\quad=2\pi i\rho_2{}^L(w)F_{22}(w+i\epsilon)F_{22}(w-i\epsilon)+2\pi i\int d\sigma\sum_l \rho_3{}^{lI}(w,\sigma)F_{23}{}^l(w+i\epsilon,\sigma+i\epsilon)F_{32}{}^l(w-i\epsilon,\sigma-i\epsilon),\\
&F_{23}{}^l(w+i\epsilon,\sigma)-F_{23}{}^l(w-i\epsilon,\sigma)\\
&\quad=2\pi i\rho_2{}^L(w)F_{22}(w+i\epsilon)F_{23}{}^l(w-i\epsilon,\sigma)+2\pi i\int d\sigma'\sum_{l'} \rho_3{}^{l'I}(w,\sigma')F_{23}{}^{l'}(w+i\epsilon,\sigma'+i\epsilon)F_{33}{}^{l',l}(w-i\epsilon,\sigma'-i\epsilon,\sigma),\\
&F_{32}{}^l(w+i\epsilon,\sigma)-F_{32}{}^l(w-i\epsilon,\sigma)\\
&\quad=2\pi i\rho_2{}^L(w)F_{32}{}^l(w+i\epsilon,\sigma)F_{22}(w-i\epsilon)+2\pi i\int d\sigma'\sum_{l'} \rho_3{}^{l'I}(w,\sigma')F_{33}{}^{l,l'}(w+i\epsilon,\sigma,\sigma'+i\epsilon)F_{32}{}^{l'}(w-i\epsilon,\sigma'-i\epsilon),\\
&F_{33}{}^{l',l}(w+i\epsilon,\sigma',\sigma)-F_{33}{}^{l',l}(w-i\epsilon,\sigma',\sigma)\\
&\quad=2\pi i\rho_2{}^L(w)F_{32}{}^{l'}(w+i\epsilon,\sigma')F_{23}{}^l(w-i\epsilon,\sigma)\\
&\qquad+2\pi i\int d\sigma''\sum_{l''} \rho_3{}^{l''I}(w,\sigma')F_{33}{}^{l',l''}(w+i\epsilon,\sigma',\sigma''+i\epsilon)F_{33}{}^{l'',l}(w-i\epsilon,\sigma''-i\epsilon,\sigma),
\end{aligned} \quad (34)$$

where

$$\begin{aligned}
\rho_2{}^L(w)&=\rho_2(w^2)/g_2{}^L(w).\\
\rho_3{}^{lI}(w,\sigma)&=\rho_3(w^2,\sigma)/g_3{}^{lI}(w,\sigma).
\end{aligned} \quad (35)$$

As noted by Blankenbecler, the constraint on the amplitudes imposed by Eq. (34) will be satisfied by the solutions of the following set of linear equations:

$$\begin{aligned}
N_{22}(w)&=F_{22}(w)D_{22}(w)+\int_{4\mu^2}^{\infty}d\sigma\sum_l F_{23}{}^l(w,\sigma+i\epsilon)D_{32}{}^l(w,\sigma-i\epsilon),\\
N_{23}{}^l(w,\sigma)&=F_{22}(w)D_{23}{}^l(w,\sigma)+\int_{4\mu^2}^{\infty}d\sigma'\sum_{l'} F_{23}{}^{l'}(w,\sigma'+i\epsilon)D_{33}{}^{l',l}(w,\sigma'-i\epsilon,\sigma),\\
N_{32}{}^l(w,\sigma)&=F_{32}{}^l(w,\sigma)D_{22}(w)+\int_{4\mu^2}^{\infty}d\sigma'\sum_{l'} F_{33}{}^{l,l'}(w,\sigma,\sigma'+i\epsilon)D_{32}{}^{l'}(w,\sigma'-i\epsilon),\\
N_{33}{}^{l',l}(w,\sigma',\sigma)&=F_{32}{}^{l'}(w,\sigma')D_{23}{}^l(w,\sigma)+\int_{4\mu^2}^{\infty}d\sigma''\sum_{l''} F_{33}{}^{l',l''}(w,\sigma',\sigma''+i\epsilon)D_{33}{}^{l'',l}(w,\sigma''-i\epsilon,\sigma),
\end{aligned} \quad (36)$$

where

$$
\begin{aligned}
D_{22}(w)&=1-\int\frac{dw'}{w'-w}\rho_2{}^L(w')N_{22}(w'),\\
D_{32}{}^l(w,\sigma)&=-\int\frac{dw'}{w'-w}\rho_3{}^{lI}(w',\sigma)N_{32}(w',\sigma),\\
D_{23}{}^l(w,\sigma)&=-\int\frac{dw'}{w'-w}\rho_2{}^L(w')N_{23}(w,\sigma),\\
D_{33}{}^{l',l}(w,\sigma',\sigma)&=\delta_{l',l}\delta(\sigma'-\sigma)-\int\frac{dw'}{w'-w}\rho_3{}^{l'I}(w',\sigma')N_{33}{}^{l',l}(w',\sigma',\sigma)\equiv\delta_{l',l}\delta(\sigma'-\sigma)-K_{l',l}(w,\sigma',\sigma).
\end{aligned}
\tag{37}
$$

Formally, let us denote the discontinuities across the left-hand cuts of the amplitudes[20] F_{ij} by A_{ij}:

$$
\begin{aligned}
F_{22}(w+i\epsilon)-F_{22}(w-i\epsilon)&=2\pi iA_{22}(w),\\
F_{23}{}^l(w+i\epsilon,\,\sigma)-F_{23}{}^l(w-i\epsilon,\,\sigma)&=2\pi iA_{23}{}^l(w,\sigma),\\
F_{32}{}^l(w+i\epsilon,\,\sigma)-F_{32}{}^l(w-i\epsilon,\,\sigma)&=2\pi iA_{32}{}^l(w,\sigma),\\
F_{33}{}^{l',l}(w+i\epsilon,\,\sigma',\,\sigma)-F_{33}{}^{l',l}(w-i\epsilon,\,\sigma',\,\sigma)&=2\pi iA_{33}{}^{l',l}(w,\sigma',\sigma).
\end{aligned}
\tag{38}
$$

Then

$$
\begin{aligned}
N_{22}(w)&=\int_{-\infty}\frac{dw'}{w'-w}\left[A_{22}(w')D_{22}(w')+\int_{4\mu^2}^{\infty}d\sigma\sum_l A_{23}{}^l(w',\sigma)D_{32}{}^l(w',\sigma^*)\right],\\
N_{23}{}^l(w,\sigma)&=\int_{-\infty}\frac{dw'}{w'-w}\left[A_{22}(w',\sigma)D_{22}(w')+\int_{4\mu^2}^{\infty}d\sigma'\sum_{l'} A_{23}{}^{l'}(w',\sigma')D_{33}{}^{l',l}(w',\sigma'^*,\sigma)\right],\\
N_{32}{}^l(w,\sigma)&=\int_{-\infty}\frac{dw'}{w'-w}\left[A_{32}{}^l(w',\sigma)D_{22}(w')+\int_{4\mu^2}^{\infty}d\sigma'\sum_{l'} A_{33}{}^{l,l'}(w',\sigma,\sigma')D_{32}{}^{l'}(w',\sigma'^*)\right],\\
N_{33}{}^{l',l}(w,\sigma',\sigma)&=\int_{-\infty}\frac{dw'}{w'-w}\left[A_{32}{}^{l'}(w',\sigma')D_{23}{}^l(w',\sigma)+\int_{4\mu^2}^{\infty}d\sigma''\sum_{l''} A_{33}{}^{l',l''}(w',\sigma',\sigma'')D_{33}{}^{l'',l}(w',\sigma''^*,\sigma)\right].
\end{aligned}
\tag{39}
$$

Once the A_{ij} are given, the coupled integral equations (37) and (39) can be solved for N_{ij} and D_{ij}. After substituting N_{ij} and D_{ij} into Eq. (36), we must solve Eq. (36) for F_{ij}. In order to do this we must invert $D_{33}{}^{l,l'}(w,\sigma,\sigma')$. Consider the integral equation

$$
f^l(w,\sigma)=\sum_{l'}\int_{4\mu^2}^{\infty}D_{33}{}^{l,l'}(w,\,\sigma,\,\sigma'+i\epsilon)\phi^{l'}(w,\,\sigma'-i\epsilon)d\sigma'
$$

or

$$
\phi^l(w,\sigma)=f^l(w,\sigma)+\sum_{l'}\int_{4\mu^2}^{\infty}d\sigma'\,K_{l',l}(w,\,\sigma,\,\sigma'+i\epsilon)\phi^{l'}(w,\,\sigma'-i\epsilon).
\tag{40}
$$

This is a Fredholm equation[21] in σ with kernel $K_{l',l}(w,\sigma',\sigma)$. Now let $H_{l',l}(w,\sigma,\sigma')$ be the resolvent kernel. Then evidently

$$
\sum_{l''}\int_{4\mu^2}^{\infty}d\sigma''\,D_{33}{}^{l,l''}(w,\sigma,\sigma'')[\delta_{l'',l'}\delta(\sigma''-\sigma')+H_{l'',l'}(w,\sigma''^*,\sigma')]=\delta_{l,l'}\delta(\sigma-\sigma')
\tag{41}
$$

and

$$
H_{l',l}(w,\sigma',\sigma)=D_{l',l}(w,\sigma',\sigma)/D(w),
\tag{42}
$$

where $D_{l',l}(w,\sigma',\sigma)$ is the Fredholm minor[21]:

$$
D_{l',l}(w,\sigma',\sigma)=K_{l',l}(w,\sigma',\sigma)-\sum_{l''}\int_{4\mu^2}^{\infty}d\sigma''\begin{vmatrix}K_{l',l}(w,\sigma',\sigma) & K_{l',l''}(w,\sigma',\sigma'')\\ K_{l'',l}(w,\sigma''^*,\sigma) & K_{l'',l''}(w,\sigma''^*,\sigma'')\end{vmatrix}+\cdots
\tag{43}
$$

[20] Actually, great care must be exercised here since anomalous thresholds are present (see Sec. V).
[21] Reference 17, Chap. XI.

and $D(w)$ is the Fredholm determinant[21]:

$$D(w)=1-\sum_l\int_{4\mu^2}^{\infty}K_{l,l}(w,\sigma,\sigma^*)d\sigma+\frac{1}{2!}\int_{4\mu^2}^{\infty}d\sigma\int_{4\mu^2}^{\infty}d\sigma'\sum_{l,l'}\begin{vmatrix}K_{l,l}(w,\sigma,\sigma^*) & K_{l,l'}(w,\sigma,\sigma')\\ K_{l',l}(w,\sigma'^*,\sigma^*) & K_{l',l'}(w,\sigma'^*,\sigma')\end{vmatrix}+\cdots$$
$$\equiv \det D_{33}{}^{l',l}(w,\sigma,\sigma'). \tag{44}$$

Let us denote the inverse of $D_{33}{}^{l',l}(w,\sigma',\sigma)$ by $G_{l',l}(w,\sigma',\sigma)$:

$$G_{l',l}(w,\sigma',\sigma)\equiv\delta_{l',l}\delta(\sigma'-\sigma)+D_{l',l}(w,\sigma',\sigma)/D(w). \tag{45}$$

Now Eq. (36) can be solved for F_{ij}. Formally we write

$$F_{ij}(w)=\textstyle\sum_k N_{ik}(w)[D^{-1}(w)]_{kj}$$

or

$$\mathbf{F}(w)=\mathbf{N}(w)\cdot\mathbf{D}^{-1}(w),$$

where i, j, k stand for the channel index 2 or the channel index 3, l, and continuous σ. Further

$$\mathbf{N}(w)=(N_{ij}(w))=\begin{pmatrix}N_{22}(w), & N_{2m}(w)\\ N_{n2}(w), & N_{nm}(w)\end{pmatrix},$$

$$\mathbf{D}(w)=(D_{ij}(w))=\begin{pmatrix}D_{22}(w), & D_{2m}(w)\\ D_{n2}(w), & D_{nm}(w)\end{pmatrix}.$$

Then

$$\mathbf{D}^{-1}(w)=\frac{1}{\Delta(w)}\begin{pmatrix}1, & -\sum_r D_{2r}(w)G_{rm}(w)\\ -\sum_r G_{nr}(w)D_{r2}(w), & \Delta(w)G_{nm}+\sum_r G_{nr}(w)D_{r2}(w)\sum_s D_{2s}(w)G_{sm}(w)\end{pmatrix}.$$

Here the subscripts m, n, r, s stand for the channel 3, l, and continuous σ, and

$$\Delta(w)=D_{22}(w)-\textstyle\sum_{r,s}D_{2r}(w)G_{rs}(w)D_{s2}(w)$$
$$=D_{22}(w)-\sum_{l,l'}\int_{4\mu^2}^{\infty}d\sigma d\sigma'\,D_{23}{}^{l}(w,\sigma+i\epsilon)G_{l,l'}(w,\sigma-i\epsilon,\sigma'+i\epsilon)D_{32}{}^{l'}(w,\sigma'-i\epsilon). \tag{46}$$

Thus, for $i=j=2$,

$$F_{22}(w)=\frac{1}{\Delta(w)}\left[N_{22}(w)-\sum_{l,l'}\int d\sigma\int d\sigma'\,N_{23}{}^{l}(w,\sigma+i\epsilon)G_{l,l'}(w,\sigma-i\epsilon,\sigma'+i\epsilon)D_{32}{}^{l'}(w,\sigma'-i\epsilon)\right]. \tag{47}$$

Finally, it should be noted that

$$\det D_{ij}(w)=D(w)\Delta(w)=\det\mathbf{D}(w),$$
$$\det D_{33}{}^{l',l}(w,\sigma',\sigma)=D(w). \tag{48}$$

V. ANALYTIC CONTINUATION

Production amplitudes, as a rule, have anomolous singularities of the type discussed previously by a number of authors.[9] The formal solution obtained in the last section has, therefore, to be continued in the manner discussed by Mandelstam,[22] and Blankenbecler and Nambu,[23] in an external mass—in our case, σ—from a normal to an anomalous case.

To illustrate the analytic continuation of the N/D solution, let us assume that the two-particle scattering amplitude $F_{22}(w)$, and the three-particle amplitude $F_{33}(w,\sigma',\sigma)$ have only the right-hand cuts demanded by unitarity. The inclusion of the left-hand cut of F_{22} does not present any difficulties, but simply complicates the algebra. The left-hand cut of M_{33} arising from the disconnected graphs in which one of the pions is noninteracting would give rise to the isobar formation in the production process. This mechanism—the (3,3) isobar formation following the pion production—has been much discussed in the literature.[1–3,24] Another interesting diagram is the one in which the (3,3) isobar and the pion scatter with a nucleon exchange, a process whose effect on the pion production has been considered by Peierls[1] (see Fig. 2). While these diagrams will produce further complexities, we believe the simple example we shall discuss below is adequate to demonstrate the general procedure of continuation.

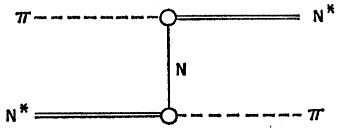

FIG. 2. A contribution to M_{33} considered by Peierls.[1] N^* denotes the excited state (3,3) of the nucleon.

If we had a representation of the production amplitude in five variables comparable to the Mandelstam representation for the scattering amplitude, we would have a complete knowledge of the left-hand cut. In the absence of such a representation, the best thing we can do is to choose a certain set of diagrams and to evaluate the left-hand cut. Let us consider the one-pion-exchange diagram (see Fig. 3). The singularities of the partial wave amplitudes $F_{23}{}^{l}(w,\sigma)$ associated with the one-pion exchange are well known. For $\sigma<2\mu^2(1+\mu/2m)$, the

[22] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).
[23] R. Blankenbecler and Y. Nambu, Nuovo cimento 18, 595 (1960).
[24] R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. 123, 333 (1961). Additional references are given in this paper.

branch point $w_0(\sigma)$ lying farthest to the right is

$$w_0(\sigma)=\{2m^2+\sigma +[\sigma(4m^2-\mu^2)(-\sigma+4\mu^2)/\mu^2]^{1/2}\}^{1/2}/\sqrt{2}. \quad (49)$$

Let us increase σ. When σ reaches $2\mu^2(1+\mu/2m)$, the branch point w_0 is at the elastic threshold $w_2=m+\mu$. If we give a small positive imaginary part to σ and increase σ further, the branch point moves as indicated by the line in Fig. 4. When $\sigma=4\mu^2$, the branch point is at $w_0(4\mu^2)=(m^2+2\mu^2)^{1/2}$, and as σ is increased further, the branch point moves into the complex plane as indicated in Fig. 4:

$$w_0(\sigma+i\epsilon)=\{2m^2+\sigma -i[\sigma(4m^2-\mu^2)(\sigma-4\mu^2)/\mu^2]^{1/2}\}^{1/2}/\sqrt{2}, \quad (50)$$
$$-(\pi/2)<\arg w_0(\sigma+i\epsilon)<0, \quad \sigma>4\mu^2.$$

When a small negative imaginary part, $-i\epsilon$, is attached to σ, the curve in Fig. 4 is reflected about the real w axis.

The particular configuration of the singularity in Fig. 4 is of importance because it crosses the real w axis above $w=m+\mu$. It follows that any quantity which involves an integral in w over $A_{23}(w,\sigma)$ from $w=m+\mu$ to $w=\infty$ is no longer defined. We must thus discuss the proper continuation of **N** and **D**. Since $D_{32}{}^l(w,\sigma)$ and $D_{33}{}^{ll'}(w,\sigma,\sigma')$ are defined by integrals extending from $w=m+2\mu$ to $w=\infty$, the only consideration involved for them is the proper continuation of $N_{32}{}^l(w,\sigma)$ and $N_{33}{}^{ll'}(w,\sigma,\sigma')$. However, $D_{22}(w)$ and $D_{23}{}^l(w,\sigma)$ must be properly continued, as well as $N_{22}(w)$ and $N_{23}{}^l(w,\sigma)$.

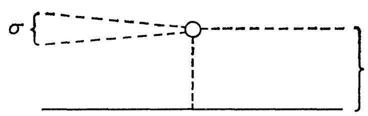

FIG. 3. One-pion exchange diagram for M_{32}. The dashed lines are pions, the solid line nucleons.

We first consider the continuation of $N_{22}(w)$, $N_{23}{}^l(w,\sigma)$, $D_{22}(w)$, and $D_{23}{}^l(w,\sigma)$. Recalling Eq. (36), we would expect the discontinuity of $N_{22}(w)$ and $N_{23}{}^l(w,\sigma)$ to be nonzero along a cut similar to that shown in Fig. 4. Further, $D_{22}(w)$ and $D_{23}{}^l(w,\sigma)$ will now exhibit anomalous cuts since the contours in Eq. (37) must be appropriately deformed. Let us make these remarks more precise.

From Eq. (36) we may calculate the discontinuities of $N_{22}(w)$ *along the anomalous cut* (the heavy line in Fig. 4; real and complex).

$$[\mathrm{disc}N_{22}(w)]_a =(1/2i)[N_{22}(w_+)-N_{22}(w_-)]$$
$$=F_{22}[\mathrm{disc}D_{22}(w)]_a+\pi\int_{4\mu^2}^{\infty} d\sigma[A_{23}{}^{\mathrm{II}}(w,\sigma+i\epsilon) +\alpha_{23}(w,\sigma+i\epsilon)]D_{32}(w,\sigma-i\epsilon), \quad (51\mathrm{a})$$

where $w_\pm$ are as shown in Fig. 4, and the superscript l, and the accompanying summation, are suppressed. Several points must be emphasized: (1) We anticipate that $D_{22}(w)$ develops anomalous singularities where

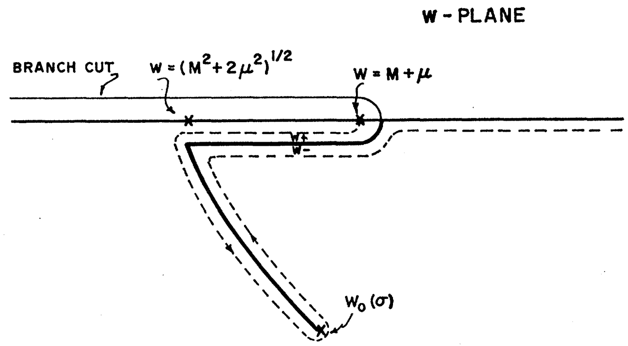

FIG. 4. The contour of integration of $D_{22}(w)$ and $D_{23}(w,\sigma)$. The heavy line indicates the intruding singularities of $N_{22}(w)$ and $N_{23}(w,\sigma)$; the dashed lines are the deformed contour for $D_{22}(w)$ and $D_{23}(w,\sigma)$.

$N_{22}(w)$ does, thus the first term on the right; (2) the discontinuity of $F_{23}(w)$ along the anomalous cut is written as

$$[\mathrm{disc}F_{23}(w,\sigma)]_a=\pi[A_{23}{}^{\mathrm{II}}(w,\sigma)+\alpha_{23}(w,\sigma)], \quad (51\mathrm{b})$$

where $-A_{23}{}^{\mathrm{II}}(w,\sigma)$ is the continuation of $A_{23}(w,\sigma)$, defined for $\sigma<2\mu^2(1+\mu/2m)$, to $\sigma>2\mu^2(1+\mu/2m)$, when w is on the anomalous cut, and $\alpha_{23}(w,\sigma)$ is in fact defined by Eq. (51b). We shall denote the continuation of $A_{23}(w,\sigma)$ for $\sigma>2\mu^2(1+\mu/2m)$ when w is on the normal cut by $A_{23}{}^{\mathrm{I}}(w,\sigma)$.

Equation (51a) is quite unsatisfactory since the expression for N_{22} is expressed in terms of α_{23}, which can be found only after $F_{23}(w,\sigma)$ is known. It will be assumed, and verified *a posteriori* to be consistent with unitarity and given analyticity on the left-hand cut, that

$$F_{22}(w)[\mathrm{disc}D_{22}(w)]_a +\pi\int_{4\mu^2}^{\infty} d\sigma\, \alpha_{23}(w,\sigma+i\epsilon)D_{32}(w,\sigma-i\epsilon)=0. \quad (51\mathrm{c})$$

The same remarks hold also for $N_{23}(w,\sigma)$.

Thus, we write proper analytic continuation of $N_{22}(w)$ and $N_{23}(w,\sigma)$ as

$$N_{22}(w)=\int_{4\mu^2}^{\infty} d\sigma\left[\int^{w_2}\frac{dw'}{w'-w}A_{23}{}^{\mathrm{I}}(w',\sigma+i\epsilon) \times D_{32}(w',\sigma-i\epsilon)+\int_{w_0(\sigma+i\epsilon)}^{w_2}\frac{dw'}{w'-w} \times A_{23}{}^{\mathrm{II}}(w',\sigma+i\epsilon)D_{32}(w',\sigma-i\epsilon)\right],$$

$$N_{23}(w,\sigma)=\int_{4\mu^2}^{\infty} d\sigma'\left[\int^{w_2}\frac{dw'}{w'-w}A_{23}{}^{\mathrm{I}}(w',\sigma'+i\epsilon) \times D_{33}(w',\sigma'-i\epsilon,\sigma)+\int_{w_0(\sigma'+i\epsilon)}^{w_2}\frac{dw'}{w'-w} \times A_{23}{}^{\mathrm{II}}(w',\sigma'+i\epsilon)D_{33}(w',\sigma'-i\epsilon,\sigma)\right], \quad (52)$$

where the first integrals in the square brackets are along the normal cut and the second along the anomalous (the heavy line in Fig. 4).

The paths of integrations from $w_2=m+\mu$ to ∞ in the definitions of $D_{22}(w)$ and $D_{23}(w,\sigma)$ must be deformed so as to avoid the intruding singularities of N_{22} and N_{23} in the integrands. Therefore, the paths of integrations must be deformed as indicated by the dashed line in Fig. 4. Thus, we obtain

$$D_{22}(w)=1-\int_{w_2}^{\infty}\frac{dw'}{w'-w}\rho_2{}^L(w')N_{22}(w')+2\pi\int_{4\mu^2}^{\infty}d\sigma'\int_{w_0(\sigma'+i\epsilon)}^{w_2}\frac{dw'}{w'-w}i[\rho_2{}^L(w')]^c\times A_{23}{}^{\rm II}(w',\sigma'+i\epsilon)D_{32}(w',\sigma'-i\epsilon), \tag{53}$$

$$D_{23}(w,\sigma)=-\int_{w_2}^{\infty}\frac{dw'}{w'-w}\rho_2{}^L(w')N_{23}(w',\sigma)+2\pi\int_{4\mu^2}^{\infty}d\sigma'\int_{w_0(\sigma'+i\epsilon)}^{w_2}\frac{dw'}{w'-w}i[\rho_2{}^L(w')]^c\times A_{23}{}^{\rm II}(w',\sigma'+i\epsilon)D_{33}(w',\sigma'-i\epsilon,\sigma),$$

where $[\rho_2{}^L(w)]^c$ is the analytic continuation of the $\rho_2{}^L(w)$ defined in Eq. (35) without the θ function.

Equation (53) shows clearly that both $D_{22}(w)$ and $D_{23}(w,\sigma)$ contain anomalous (real and complex) cuts. The complex cuts extend to infinity in the lower half w plane. The discontinuities across the anomalous cuts are given by

$$[{\rm disc}D_{22}(w)]_a=2\pi^2i[\rho_2{}^L(w)]^c\int_{4\mu^2}^{\infty}d\sigma\,A_{23}{}^{\rm II}(w,\sigma+i\epsilon)D_{32}(w,\sigma-i\epsilon),$$

$$[{\rm disc}D_{23}(w,\sigma)]_a=2\pi^2i[\rho_2{}^L(w)]^c\times\int_{4\mu^2}^{\infty}d\sigma'\,A_{23}{}^{\rm II}(w,\sigma'+i\epsilon)D_{33}(w,\sigma'-i\epsilon,\sigma).$$

Therefore, we see, from Eq. (51.c) and a similar equation for $[{\rm disc}D_{23}(w,\sigma)]_a$, that if

$$\alpha_{23}(w,\sigma)=-2\pi i[\rho_2{}^L(w)]^cF_{22}(w)A_{23}{}^{\rm II}(w,\sigma), \tag{51.d}$$

then Eq. (51.c) is satisfied.

The continuation of $N_{32}(w,\sigma)$ and $N_{33}(w,\sigma',\sigma)$ requires considerable care. Formally we write

$$N_{32}(w,\sigma)=\int^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm I}(w',\sigma)D_{22}(w')+\int_{w_0(\sigma)}^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm II}(w',\sigma)[D_{22}(w')]^c, \tag{54}$$

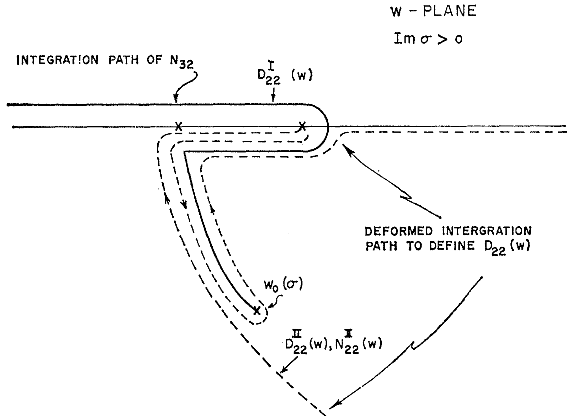

FIG. 5. The path of integration of N_{32} and N_{33} is indicated by the solid line. The path of integration of D_{22} and D_{23} in the integrands must be deformed as indicated by the dashed line to secure the proper continuation.

where $[D_{22}(w)]^c$ is the continuation of $D_{22}(w)$ along the path of integration from w_2 to $w_0(\sigma)$. The continuation of $D_{22}(w)$ is then effected by deforming the path of integration in Eq. (53) as indicated by the dashed line in Fig. 5. Then

$$N_{32}(w,\sigma)=\int_{-\infty}^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm I}(w',\sigma)D_{22}{}^{\rm I}(w')+\int_{w_0(\sigma)}^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm II}(w',\sigma)\times[D_{22}{}^{\rm II}(w')-2\pi i[\rho_2(w')]^cN_{22}{}^{\rm II}(w')], \tag{55}$$

where $D_{22}{}^{\rm I}(w)$, $D_{22}{}^{\rm II}(w)$, and $N_{22}{}^{\rm II}(w)$ are boundary values of $D_{22}(w)$ and $N_{22}(w)$ as w approaches the boundaries as indicated by arrows in Fig. 5. $N_{33}(w,\sigma',\sigma)$ can be continued likewise:

$$N_{33}(w,\sigma',\sigma)=\int^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm I}(w',\sigma')D_{23}{}^{\rm I}(w',\sigma)+\int_{w_0(\sigma')}^{w_2}\frac{dw'}{w'-w}A_{32}{}^{\rm II}(w',\sigma')\times[D_{23}{}^{\rm II}(w',\sigma)-2\pi i[\rho_2(w')]^cN_{23}{}^{\rm II}(w',\sigma)], \tag{56}$$

$D_{32}(w,\sigma)$ and $D_{33}(w,\sigma',\sigma)$ are as written down in Eq. (37).

There is a very interesting property of the N/D solution to be noted: While $\mathbf{D}$ develops anomalous singularities, the determinant of $\mathbf{D}$ is analytic except on the cut from $w_2=m+\mu$ to ∞.[25] We recall that

$$\det\mathbf{D}(w)=D(w)\Delta(w). \tag{48}$$

[25] This theorem and Eq. (61) were first noted in a conversation between R. Blankenbecler and the authors. For previous attempts to generalize the Levinson theorem, see, for example, M. T. Vaughn, R. Aaron, R. D. Amado, Phys. Rev. **124**, 1258 (1961).

$D(w)$ is seen to be analytic in the w-plane cut from $m+2\mu$ to ∞, by construction, Eq. (44). Let us take the discontinuity of $\Delta(w)$ across the anomalous cut [see Fig. 6]:

$$\begin{aligned}[\mathrm{disc}\Delta(w)]_a &= [\Delta(w_+)-\Delta(w_-)]/2i \\ &= i\pi[\rho_2(w)]^c\Bigg\{\int_{4\mu^2}^{\infty} d\sigma\, A_{23}^{\mathrm{II}}(w,\sigma+i\epsilon)D_{32}(w,\sigma-i\epsilon) \\ &\quad -\int_{4\mu^2}^{\infty} d\sigma\int_{4\mu^2}^{\infty} d\sigma'\int_{4\mu^2}^{\infty} d\sigma''\, A_{23}^{\mathrm{II}}(w,\sigma+i\epsilon) \\ &\quad \times D_{33}(w,\sigma-i\epsilon,\sigma'+i\epsilon)G(w,\sigma'-i\epsilon,\sigma''+i\epsilon) \\ &\quad \times D_{32}(w,\sigma''-i\epsilon)\Bigg\}=0. \end{aligned} \tag{57}$$

This has an important consequence. Adopting the matrix notation, write the S matrix as

$$\mathbf{S}=\mathbf{1}+2\pi i\boldsymbol{\varrho}^{1/2}\mathbf{F}\boldsymbol{\varrho}^{1/2}, \tag{58}$$

where the rows and columns are labeled by the channel $i=2,3$, the angular momentum quantum numbers l, and ξ, and the continuous variable σ, and $\boldsymbol{\varrho}^{1/2}$ is the diagonal matrix whose elements are $[\rho_2{}^L(w)]^{1/2}$ and $[\rho_3{}^{lI}(w,\sigma)]^{1/2}$. Since

$$\mathbf{FD}=\mathbf{N},$$

Eq. (58) may be rewritten as

$$\begin{aligned}\boldsymbol{\varrho}^{1/2}\mathbf{S}(w) &= [\mathbf{D}(w)+2\pi i\boldsymbol{\varrho}\mathbf{N}(w)]\mathbf{D}(w)^{-1}\boldsymbol{\varrho}^{1/2} \\ &= \mathbf{D}(w-i\epsilon)\mathbf{D}(w+i\epsilon)^{-1}\boldsymbol{\varrho}^{1/2}. \end{aligned} \tag{59}$$

Hence, taking the determinants of both sides, we obtain

$$\det\mathbf{S}(w)=\frac{\Delta(w-i\epsilon)D(w-i\epsilon)}{\Delta(w+i\epsilon)D(w+i\epsilon)}. \tag{60}$$

Taking the logarithmic derivatives of both sides of Eq. (60) and integrating from w_2 to ∞, we obtain in the usual manner

$$\ln\det\mathbf{S}(w_2)-\ln\det\mathbf{S}(\infty)=(n-n_{\mathrm{C.D.D.}})\pi, \tag{61}$$

where n is the number of bound states having the quantum number J, Π, (number of zeros of $\det\mathbf{D}$) and $n_{\mathrm{C.D.D.}}$ is the number of Castillejo-Dalitz-Dyson poles of $\det\mathbf{D}$. Equation (61) is a multichannel extension of the Levinson theorem,[12] since $\ln\det\mathbf{S}(w)$ is the sum of the eigenphase-shifts in the (J,Π) sector. It should be emphasized that this sum includes the phase shifts, not only from two-particle channels, but from three-particle channels as well.[25] While this theorem is proved under a restrictive condition ($A_{22}=A_{33}=0$), we believe this generally valid.

The reader can ascertain for himself that, while the expression for $F_{22}(w)$, Eq. (47), apparently has left-hand singularities, the cut of $N_{22}(w)$ is exactly canceled by the left-hand cut of the term $\sum_{m,n} N_{2m}(w)G_{mn}(w)D_{n2}(w)$. The same thing is true for $F_{33}(w)$ also.

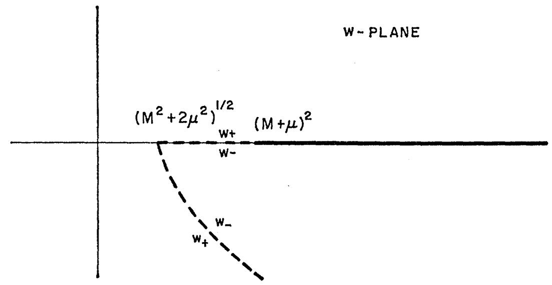

FIG. 6. The definition of $w_\pm$ across the anomalous cuts.

While it has been shown[6,26] that the N/D solution satisfies the symmetry properties of the T matrix, $F_{2m}(w)=F_{m2}(w)$ which follows from the time reversal invariance, it has yet to be shown that this property of the T matrix is preserved in the course of analytic continuation. In the matrix notation,

$$F_{2m}(w)=\frac{1}{\Delta}[\Delta\sum_n N_{2n}G_{nm}+(\sum_{n,r} N_{2n}G_{nr}D_{r2}) \times(\sum_s D_{2s}G_{sm})-N_{22}\sum_r D_{2r}G_{rm}]. \tag{62}$$

Let us compute the discontinuity of this function across the anomalous cut (w_+, w_- as indicated in Fig. 6):

$$\begin{aligned}[\mathrm{disc}F_{2m}(w)]_a &= [F_{23}(w_+,\sigma)-F_{23}(w_-,\sigma)]/2i \\ &= \pi A_{23}^{\mathrm{II}}(w,\sigma)[1-2\pi i[\rho_2{}^L(w)]^cF_{22}(w)]. \end{aligned} \tag{63}$$

On the other hand

$$F_{m2}(w)=\frac{1}{\Delta}[N_{m2}-\sum_{n,r} N_{mn}G_{nr}D_{r2}]. \tag{64}$$

Since N_{m2} and N_{mn} are discontinuous across the branch cut, we see that

$$\begin{aligned}[\mathrm{disc}F_{m2}(w)]_a &= [F_{32}(w_+,\sigma)-F_{32}(w_-,\sigma)]/2i \\ &= \pi A_{32}^{\mathrm{II}}(w,\sigma)\frac{1}{\Delta}\{[D_{22}^{\mathrm{II}}-\sum_{r,s} D_{2r}^{\mathrm{II}}G_{rs}D_{s2}] \\ &\quad -2i[\rho_2{}^L]^c[N_{22}^{\mathrm{II}}-\sum_{r,s} N_{2r}^{\mathrm{II}}G_{rs}D_{s2}]\}. \end{aligned} \tag{65}$$

Since both $\Delta(w)$ and $F_{22}(w)$ are analytic in this region,

[26] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

we can write

$$D_{22}^{\mathrm{II}}(w)-\sum_{r,s} D_{2r}^{\mathrm{II}}(w)G_{rs}(w)D_{s2}(w)=\Delta(w),$$

$$N_{22}^{\mathrm{II}}(w)-\sum_{r,s} D_{2r}^{\mathrm{II}}(w)G_{rs}(w)D_{s2}(w)=\Delta(w)F_{22}(w),$$

so that[27]

$$[\mathrm{disc}F_{m2}(w)]_a =\pi A_{32}^{\mathrm{II}}(w,\sigma)[1-2\pi i[\rho_2^L(w)]^c F_{22}(w)]. \quad (66)$$

Therefore, insofar as $A_{32}^{\mathrm{II}}(w,\sigma)=A_{23}^{\mathrm{II}}(w,\sigma)$, the discontinuities of $F_{2m}(w)$ and $F_{m2}(w)$ are the same across the anomalous cut. This, combined with the result of Bjorken and Nauenberg,[26] establishes the symmetry of the T matrix.

Furthermore, comparison of Eq. (63) and Eq. (51.b) gives

$$\alpha_{23}(w,\sigma)=-2\pi i[\rho_2^L(w)]^c F_{22}(w)A_{32}^{\mathrm{II}}(w,\sigma),$$

which is identically Eq. (51.d). Hence the consistency of the assumption, Eq. (51.c), is now established.

In concluding this section, we remark on the threshold behavior of the production amplitude when there is an anomalous singularity of the type discussed in this section. In nonrelativistic theory[28] one can deduce the threshold behavior as

$$F_{23}^{J\Pi l}(w,\sigma)\rightarrow \text{const}, \quad \text{as} \quad w\rightarrow m+\mu \quad \text{or} \quad m+(\sigma)^{1/2}. \quad (67)$$

In relativistic field theory, the threshold behavior of the production amplitude, Eq. (67), does not in general hold at $w=m+\mu$, but, as will be shown in the following paper, the amplitude $F_{23}^{J\Pi l}(w,\sigma)$ behaves as

$$\lim_{w\rightarrow m+\mu} F_{23}^{J\Pi l}(w,\sigma)=\text{const}[P(s)]^{-2L-2}, \quad (68)$$

whenever there is an anomalous singularity arising from the one-pion-exchange diagram [Fig. 3]. Let us examine how this discrepancy arises. Equation (67) is derived under the assumption that the potential, or the interaction is of short range, so that outside a certain "interaction radius" particles propagate essentially as free particles. The anomalous branch cut is connected with the fact that the exchanged pion in Fig. 3 is on the mass shell when $w=m+\mu$. Since the exchanged pion is on the mass shell, it can propagate an infinite distance without damping, and the "potential" associated with the one-pion exchange in the production process can no longer be considered as short range.

[27] These relations, Eqs. (63) and (66), can be verified for the one-channel case; see R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. 123, 692 (1961) or references 22, 23.

[28] See, e.g., L. Fonda and R. G. Newton, Phys. Rev. 119, 1394 (1960).

VI. CONCLUDING REMARKS

We have developed a formalism in which one can discuss the relation between elastic scattering and production processes. When we discuss higher resonances in pion-nucleon scattering, for instance, a number of models or approximations can be considered within this framework. If, for example, A_{33} is set equal to zero, we obtain what is essentially a generalization of the model of Ball and Frazer,[4] but, whereas they had to limit the size of the inelastic contribution to elastic scattering to satisfy the unitarity condition, the feedback implied by unitarity is automatically contained in the present formalism, so that the resulting amplitudes are guaranteed to satisfy unitarity.

In addition, if A_{33}, computed from the disconnected graphs in which one of the pions is not interacting, is included, we obtain the relativistic version of the Carruthers[3]-Goebel-Schnitzer[2] model, including the initial state rescattering effect. In this case, it might be more convenient to use, simultaneously, two schemes of coupling three angular momenta, i.e., the $[(\pi\pi)_l N]_{J\Pi}$ scheme and the $[(\pi N)_j \pi]_{J\Pi}$ scheme in the conventional terminology. The relativistic recoupling algebra for this case has been worked out in the helicity representation by Wick.[29]

We have obtained the N/D solution for the T matrix in the presence of anomalous singularities by analytic continuation in the external mass σ. The correctness of our procedure can be checked quite independently by noting that the T matrix so deduced has correct analytic properties, satisfies unitarity, and has the correct discontinuities across anomalous singularities.

During the completion of this paper, we became aware of a related study by Nauenberg and Pais[30] who concentrate on the cusp behavior due to production channel rather than the more formal aspects treated here.

ACKNOWLEDGMENTS

The authors have been benefited by conversations with Professor R. Blankenbecler, Professor C. Goebel, Professor M. L. Goldberger, Professor A. Klein, Professor M. Nauenberg, Professor R. Sawyer, Professor S. Treiman, and Professor G. C. Wick. We are particularly indebted to R. Blankenbecler who has actually participated in many stages of the studies reported in Sec. V, and to G. C. Wick for advice on the material presented in Sec. III. Part of this work was done during the authors' stay at the Brookhaven National Laboratory in the summer, 1961, and both authors wish to thank Professor M. Goldhaber and Professor G. C. Wick for the hospitality extended to them there. One of the authors (B. W. L.) wishes to thank Professor R. Oppenheimer for the hospitality extended to him at The Institute for Advanced Study.

[29] G. C. Wick (private communication); Ann. Phys. (to be published).

[30] M. Nauenberg and A. Pais (to be published).