Therefore, in the quadrant $\mathrm{Re}\lambda>0$, $\mathrm{Im}\lambda<0$,

$$[S^\dagger S]_{jj}<e^{-2\pi\mathrm{Im}\lambda}=e^{2\pi|\mathrm{Im}\lambda|}. \qquad (41)$$

Thus, the positive diagonal elements of $S^\dagger S$ are bounded. Since the two diagonal elements are $|S_{11}|^2+|S_{12}|^2$ and $|S_{21}|^2+|S_{22}|^2$, each element of the S matrix is bounded in this quadrant, for physical K.

6. CONCLUSIONS

It has been shown that Froissart's work on single-channel scattering can be generalized to the many-channel case. The singularities of the Jost matrix were discussed in detail and appear in Eqs. (27) and (28). Since the S matrix is given by the quotient of the two Jost matrices, the singularities of S will include the origin, those in Eqs. (27) and (28) and poles which arise at points where the determinant of one Jost matrix vanishes. Thus, in particular, for a potential of type (2), $S(\lambda,K)$ is meromorphic in the entire λ plane except for an essential singularity at infinity.

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Regge Poles and High-Energy Limits in Field Theory

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It is shown that the Bethe-Salpeter scattering amplitude in the ladder approximation is meromorphic in the complex angular momentum half-plane, $\mathrm{Re}l>-3/2$. There is always at least one Regge pole in this region.

The T matrix element for complex l can be written in the form, $N_lD_l^{-1}$, where N_l and D_l have convergent perturbation expansions. D_l has only a right-hand cut in the squared energy variable, with a branch point at the elastic scattering threshold and at each production threshold. N_l has a left-hand cut, and in addition a right-hand cut beginning from the first three-particle threshold. The Regge poles are zeros of D_l. Much of the information about the trajectories of Regge poles is contained in the lowest-order expression for D_l. The general properties of the trajectories are the same as for the case of scattering from a Yukawa potential. For sufficiently small coupling constant a Regge trajectory $\alpha(s)$ may apparently be expanded in a perturbation series, valid except near thresholds in s.

The connection between the Regge poles of the ladder graphs and the high-energy behavior of the "strip" graphs is discussed. In the $\lambda\varphi^3$ theory it is shown that the second-order expression for the leading Regge trajectory, for the sum of the ladder graphs, determines the leading term in the high-energy limit of the nth order strip graph. This relationship has been checked in fourth-order perturbation theory, and is evidence for the consistency of a perturbation approach to the calculation of Regge trajectories.

1. INTRODUCTION

It has been suggested recently that the ideas of Regge,[1,2] concerning certain asymptotic properties of potential scattering amplitudes, may be applicable in elementary particle physics.[3–6] Their applicability depends on the nature of the behavior of elementary particle scattering amplitudes in the complex angular momentum plane. Though a certain domain of analyticity in the l plane follows from assuming the validity of the Mandelstam representation,[7,8] it is doubtful that

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such general considerations can establish the nature and the location of the singularities in the l plane. The investigation of Regge poles in field theory must therefore fall back on the examination of approximation schemes.

In the present work we have investigated the behavior in the complex angular momentum plane of the Bethe-Salpeter scattering amplitude in the ladder approximation. The resulting set of graphs includes, in addition to relativistic effects, some limited contributions from states with more than two particles. For simplicity, we consider the scattering of two bosons through the mechanism of boson exchange.

In Secs. 2 and 3 it is shown that the scattering amplitude has only poles in the half plane, $\mathrm{Re}\,l>-3/2$. The proof involves writing the T matrix in the form ND^{-1}, where N and D have convergent perturbation expansions and are explicitly analytic in the angular momentum variable. In Sec. 4 we show the relation of this ND^{-1} factorization to the conventional one in which N has no right-hand cut and D no left-hand cut.

In Sec. 5 we study the trajectory of the first Regge pole in the high energy limit and in the weak coupling limit. Section 6 deals with the relation between this leading Regge term and the high energy limit of the "strip" approximation graphs. In Sec. 7 the results are generalized to include exchange of systems with continuous mass distributions.

2. THE BETHE-SALPETER EQUATION

We consider the scattering of two distinguishable bosons, a and b, both of mass m. These interact through exchange of a third boson of mass μ with coupling constant g to both particles a and b. The two-body Green's function is defined as

$$G(x_1,x_2,x_3,x_4)$$
$$=\langle 0|T\varphi_a(x_1)\varphi_b(x_2)\varphi_a(x_3)\varphi_b(x_4)|0\rangle$$
$$=\int d^4p\,d^4q\,d^4W\,G(p,q,W)\exp[ip(x_1-x_2)-iq(x_3-x_4)+\tfrac{1}{2}iW(x_1+x_2-x_3-x_4)]. \quad (1)$$

In momentum space the Bethe-Salpeter equation in the ladder approximation is[9–12]

$$[(p+\tfrac{1}{2}W)^2+m^2-i\epsilon][(p-\tfrac{1}{2}W)^2+m^2-i\epsilon]G(p,q,W)$$
$$=\delta^4(p-q)+\frac{g^2}{i(2\pi)^4}\int\frac{d^4p'}{(p'-p)^2+\mu^2-i\epsilon}G(p',q,W). \quad (2)$$

In the center-of-mass system we have, $W=(\sqrt{s},0,0,0)$. The T matrix element is determined from the Green's function by the following limit,

$$T=\lim F(\mathbf{p},p_0,s)[G(p,q,\sqrt{s})-\delta^4(p-q)/F(\mathbf{p},p_0,s)]\times F(\mathbf{q},q_0,s), \quad (3)$$

where

$$F(\mathbf{p},p_0,s)=[\mathbf{p}^2+m^2-i\epsilon-(p_0-\tfrac{1}{2}\sqrt{s})^2]\times[\mathbf{p}^2+m^2-i\epsilon-(p_0+\tfrac{1}{2}\sqrt{s})^2] \quad (4)$$

and the limit in Eq. (3) is to be taken as

$$\mathbf{p}\to\mathbf{n}_f(\tfrac{1}{4}s-m^2)^{1/2},\quad \mathbf{q}\to\mathbf{n}_i(\tfrac{1}{4}s-m^2)^{1/2},$$
$$p_0\to 0,\qquad q_0\to 0.$$

Here, $\mathbf{n}_i$ and $\mathbf{n}_f$ are unit vectors in the directions of the initial and the final momenta.

From Eqs. (2) and (3), we may solve for the T matrix element in the form,

$$\langle f|T|i\rangle=\langle\mathbf{p},0|B(1-K)^{-1}|\mathbf{q},0\rangle, \quad (5)$$

where

$$\langle\mathbf{p},p_0|B|\mathbf{p}',p_0'\rangle=(2\pi)^{-4}g^2/[(p-p')^2+\mu^2-i\epsilon]$$

and

$$\langle\mathbf{p},p_0|K|\mathbf{p}',p_0'\rangle=\frac{(2\pi)^{-4}g^2}{iF(\mathbf{p},p_0,s)[(p-p')^2+\mu^2-i\epsilon]}. \quad (6)$$

B and K are here to be considered as integral operators in a four-dimensional space, in which operator products are defined by

$$\langle\mathbf{p},p_0|AB|\mathbf{p}',p_0'\rangle=\int d^4p''\,\langle\mathbf{p},p_0|A|\mathbf{p}'',p_0''\rangle\times\langle\mathbf{p}'',p_0''|B|\mathbf{p}',p_0'\rangle.$$

Taking the partial wave projection of Eq. (5) we obtain, for integral l,[13]

$$T_l=\langle|\mathbf{p}|,0|B_l(1-K_l)^{-1}||\mathbf{p}|,0\rangle, \quad (7)$$

where

$$\langle|\mathbf{p}|,p_0|B_l||\mathbf{p}'|,p_0'\rangle=\frac{g^2}{(2\pi)^3}Q_l\left(\frac{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mu^2-i\epsilon-(p_0-p_0')^2}{2|\mathbf{p}||\mathbf{p}'|}\right),$$
$$\langle|\mathbf{p}|,p_0|K_l(s)||\mathbf{p}'|,p_0'\rangle=\frac{g^2}{i(2\pi)^3F(|\mathbf{p}|,p_0,s)}Q_l\left(\frac{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mu^2-i\epsilon-(p_0-p_0')^2}{2|\mathbf{p}||\mathbf{p}'|}\right). \quad (8)$$

[9] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 435 (1951).
[10] S. Okubo, D. Feldman, Phys. Rev. **117**, 279, 292 (1960).
[11] G. C. Wick, Phys. Rev. **96**, 1124 (1952).
[12] N. N. Khuri, Nuovo cimento **5**, 1023 (1961).
[13] The definition of our $T(\cos\theta)$ and T_l are such that $T(\cos\theta)=(1/4\pi p^2)\Sigma(2l+1)T_lP_l(\cos\theta)$, $T_l=(2/\pi)^2p(p^2+m^2)^{1/2}e^{i\delta_l}\sin\delta_l$.

B_l and K_l are now to be considered as integral operators in a two-dimensional space in which operator products are defined by

$$\langle|\mathbf{p}|,p_0|AB||\mathbf{p}'|,p_0'\rangle = \int_0^\infty d|\mathbf{p}''|\int_{-\infty}^{\infty} dp_0''\langle|\mathbf{p}|,p_0|A||\mathbf{p}|'',p_0''\rangle \times\langle|\mathbf{p}|'',p_0''|B||\mathbf{p}'|,p_0'\rangle$$

The appropriate continuation of T_l into the complex angular momentum plane will be the one provided by Eqs. (7) and (8). $Q_l(x)$, the Legendre function of the second kind, is analytic in l except for poles at the negative integers.

3. MEROMORPHY OF THE T MATRIX

The expression for T_l, Eq. (7), can be developed as the ratio of two series by substitution in (7) of the identity[14]

$$(1-K_l)^{-1} = -[(\delta/\delta K_l^T)\,\mathrm{Det}(1-K_l)]/\mathrm{Det}(1-K_l). \quad (9)$$

The series expansions of numerator and denominator may be read off from the expansion, in powers of K_l, of

$$\mathrm{Det}(1-K_l) = \exp \mathrm{Tr} \log(1-K_l).$$

We write T_l in the form,

$$T_l(s) = N_l(s)D_l(s)^{-1},$$

where

$$N_l(s) = -\langle|\mathbf{p}|,0|B_l[(\delta/\delta K_lT)D_l]||\mathbf{p}|,0\rangle, \qquad D_l(s) = \mathrm{Det}(1-K_l). \quad (10)$$

This will be a useful representation if the series for D_l converges. The terms in the expansion of D_l are of the form

$$[\mathrm{Tr}(K_l)^{m_1}][\mathrm{Tr}(K_l)^{m_2}]\cdots.$$

It is possible to see from Eq. (8) and from the properties of $Q_l(x)$ that the integrals involved in $\mathrm{Tr}(K_l)^m$ converge in the half-plane $\mathrm{Re}l > -3/2$.[15] It can be further shown that the series expansions of D_l and N_l converge for $\mathrm{Re}l > -3/2$. Here we outline the argument; the details are in the appendix.

We consider first the region, $|\mathrm{Re}\sqrt{s}| < 2m$. In this region the contours for all the p_0 integrations in $\mathrm{Tr}K_l^m$ may be rotated counterclockwise to the imaginary axis, following Wick.[11] Beginning with the new kernel resulting from this distortion, we now make changes of variable which reduce the problem to one with a finite region of integration and a bounded kernel $\bar{K}_l$. Convergence of the series for both $N_l(s)$ and $D_l(s)$ now follows from standard results. Convergence for all s not on the real axis between $s=4m^2$ and $s=\infty$ follows from suitably deforming the p_0 integration contours.

Since K_l is explicitly analytic in l, the only singularity in the half l plane, $\mathrm{Re}l > -3/2$, of either N_l or D_l, is the fixed pole of $Q_l(x)$ at $l=-1$. It is shown below that both $N_l(s)$ and $D_l(s)$ have simple poles at $l=-1$. The singularities of $T_l(s)$ for $\mathrm{Re}l > -3/2$ are, therefore, only poles located at the zeros of $D_l(s)$.

When $\mathrm{Re}l < -3/2$, the traces of $(K_l)^m$ diverge. A continuation procedure, such as that devised by Froissart for potential scattering,[16] is required for the investigation of the analytic properties of T_l in this region. Term by term continuation of the series for D_l is not sufficient for a proof, since the continued series may not converge. Nevertheless, it seems likely that the singularities of D_l in this region will be the singularities of the term by term continuation, a result which is true for the Yukawa potential case.[17] The first term in the expansion of D_l shows poles in the l plane at each negative integer, as in the Yukawa case.

4. SINGULARITIES OF N_l AND D_l IN THE s PLANE

We now show how our factorization $T_l = N_l D_l^{-1}$ is related to the conventional one in which N_l has only a left-hand cut and D_l has only a right-hand cut. We begin by considering the singularities of D_l. A typical trace occurring in the series development of D_l is

$$\mathrm{Tr}K_l^n(s) = [g^2/i(2\pi)^3]^n \int_0^\infty dq_1\cdots dq_n \int_{-\infty}^{\infty} d\omega_1\cdots d\omega_n \times \frac{Q_l\left(\dfrac{q_1^2+q_2^2+\mu^2-i\epsilon-(\omega_1-\omega_2)^2}{2q_1q_2}\right)\cdots Q_l\left(\dfrac{q_n^2+q_1^2+\mu^2-i\epsilon-(\omega_n-\omega_1)^2}{2q_nq_1}\right)}{F(q_1,\omega_1,s)\cdots F(q_n,\omega_n,s)}. \quad (11)$$

A standard Landau type of analysis may be followed to locate the singularities of (11) in the variable s.[18]

(a) Two zeros of $F(q_i,\omega_i,s)$ may pinch the ω_i integration contour. This happens when $s=4m^2$ and provides the beginning of the two-particle cut in $D_l(s)$.

(b) Singularities arise from pinches of the ω integration contours between branch points of the Q_l's and

[14] M. Baker, Ann. Phys. (New York) 4, 271 (1958). See also, J. Schwinger, Phys. Rev. **93**, 615 (1954); **94**, 1362 (1954).

[15] The only properties of $Q_l(x)$ which are needed here are the behavior at infinity, $Q_l(x) \to \mathrm{const}x^{-l-1}$, and the fact that the singularities of $Q_l(x)$ at $x=\pm1$ are logarithmic.

[16] M. Froissart (to be published).

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[18] L. D. Landau, Nuclear Phys. **13**, 181 (1959); R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

zeros of the denominators. These give branch points at $s=(2m+\mu)^2$, $(2m+2\mu)^2$, etc., that is, at the thresholds for the production of different numbers of the exchanged particles.

(c) There will be additional singularities following from the pinch analysis. It is shown in the Appendix that $D_l(s)$ has in fact only the right-hand cut, that is, that the additional singularities lie on unphysical sheets.

In the region $4m^2<s<(2m+\mu)^2$ we may evaluate the discontinuity of $D_l(s)$ across the cut by means of Cutkosky's method, which involves the following replacement,[19]

$$[F(q,\omega,s)]^{-1}\rightarrow(2\pi i)^2\delta[q^2+m^2-(\omega-\tfrac{1}{2}\sqrt{s})^2]\times\delta[q^2+m^2-(\omega+\tfrac{1}{2}\sqrt{s})^2]. \quad (12)$$

The replacement is to be made in all possible ways in integrals such as that of Eq. (11), including simultaneous replacement for several factors F^{-1}. The reculting sum of terms will equal the discontinuity of $D_l(s)$.

What is to be shown is that $N_l(s)$ as defined in Eq. (10) has no cut in the region, $4m^2<s<(2m+\mu)^2$. One way to show this is to write out the nth order terms in the series for $N_l(s)$ defined by Eq. (10), take the discontinuity using Cutkosky's method and see that all terms cancel. Writing out terms can be avoided by the following equivalent formal manipulation,

$$\begin{aligned}D_l(s-i\epsilon)/D_l(s+i\epsilon)&=\mathrm{Det}[1-K_l(s-i\epsilon)]/[1-K_l(s+i\epsilon)]\\&=\mathrm{Det}\{1+[K_l(s+i\epsilon)-K_l(s-i\epsilon)]\times[1-K_l(s+i\epsilon)]^{-1}\}. \quad (13)\end{aligned}$$

Cutkosky's method amounts to the replacement,

$$\langle q\omega|K_l(s+i\epsilon)-K_l(s-i\epsilon)|q'\omega'\rangle\rightarrow\frac{g^2}{i(2\pi)^3}(2\pi i)^2\times\delta[q^2+m^2-(\omega-\tfrac{1}{2}\sqrt{s})^2]\delta[q^2+m^2-(\omega+\tfrac{1}{2}\sqrt{s})^2]\times Q_l\left(\frac{q^2+q'^2+\mu^2-i\epsilon-(\omega-\omega')^2}{2qq'}\right). \quad (14)$$

This will be valid for the region $4m^2<s<(2m+\mu)^2$, in accord with the above remarks. Noting that

$$\delta[q^2+m^2-(\omega-\tfrac{1}{2}\sqrt{s})^2]\delta[q^2+m^2-(\omega+\tfrac{1}{2}\sqrt{s})^2]=\frac{\delta[q-(\tfrac{1}{4}s-m^2)^{1/2}]\delta(\omega)}{[s(\tfrac{1}{4}s-m^2)]^{1/2}} \quad (15)$$

the right-hand side of Eq. (13) may be evaluated as

$$\frac{D_l(s-i\epsilon)}{D_l(s+i\epsilon)}=1+\frac{i\pi^2}{[s(\tfrac{1}{4}s-m^2)]^{1/2}}\langle p,0|B_l(1-K_l)^{-1}|p,0\rangle. \quad (16)$$

The infinite determinant in Eq. (13) was evaluated by noting that aside from the ones along the diagonal, there is only one nonvanishing column of the matrix of which the determinant is being taken, as indicated by the δ functions in Eq. (15). From Eqs. (7) and (16) it now follows that, in the region, $4m^2<s<(2m+\mu)^2$,

$$D_l(s-i\epsilon)/D_l(s+i\epsilon)=e^{2i\delta(s,l)}. \quad (17)$$

This suffices to show the identity of our ND^{-1} factorization with the conventional one in the elastic region.[20] We note that in the elastic region our amplitude T_l satisfies a two-particle unitarity condition extended to complex angular momentum.[21]

According to the arguments of Bardakci,[7] for nonintegral l there is a kinematical cut in $N_l(s)$, running from $s=4m^2$ to $s=-\infty$, which can be factored out in the form

$$N_l(s)=(s-4m^2)^l n_l(s). \quad (18)$$

Here, $n_l(s)$ has a left-hand cut beginning at $s=4m^2-\mu^2$ and a right-hand cut beginning from $s=(2m+\mu)^2$.

Another property of $D_l(s)$ which will be required will be the behavior at $s=\infty$. In the Appendix it is shown that $D_l(s)-1$ approaches zero faster than $s^{-1/2}$ as s approaches infinity.

5. THE LEADING REGGE POLE

Before discussing the roots of $D_l(s)=0$, we need to establish the result, already mentioned, that the singularity of $D_l(s)$ at $l=-1$ is a simple pole. To prove this we separate $Q_l(x)$ into a singular and a regular part,

$$Q_l(x)=1/(l+1)+R(x,l). \quad (19)$$

The operator $K_l(s)$ (Eq. (8)) is thus written as

$$\langle q\omega|K_l(s)|q'\omega'\rangle=1/[(l+1)F(q,\omega,s)]+R_l(q\omega,q'\omega') \quad (20)$$

where the operator R_l is regular at $l=-1$. D_l may be factored in the form,

$$D_l(s)=\mathrm{Det}(1-K_l+R_l)\times\mathrm{Det}[1-R_l(1-K_l+R_l)^{-1}]. \quad (21)$$

Now we note two relations which follow from the separability of the singular part K_l-R_l of the kernel K_l:

$$\mathrm{Det}(1-K_l+R_l)=1-[1/(l+1)]g^2/i(2\pi)^3\times\int dqd\omega/F(q,\omega,s), \quad (22)$$

$$(1-K_l+R_l)^{-1}=1+(K_l-R_l)/\mathrm{Det}(1-K_l+R_l). \quad (23)$$

[19] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[20] The following statements can readily be verified: The quantity, $t(s,l)\equiv\pi^2T_l(s)/[s(s-4m^2)]^{1/2}$, is a real analytic function of s and l such that $t^*(s,l)=t(s^*,l^*)$ [see also reference 21]; the unitarity relation extended to complex values of l implies that $t(s,l)$ has the representation, $t(s+i\epsilon,l)=e^{i\delta(s,l)}\sin\delta(s,l)$, $t(s-i\epsilon,l)=t^*(s+i\epsilon,l^*)$, and $\delta^*(s,l)=\delta(s,l^*)$ for $4m^2\leq s\leq(2m+\mu)^2$. See also reference 8.

[21] D. Fivel, Phys. Rev. **125**, 1085 (1962).

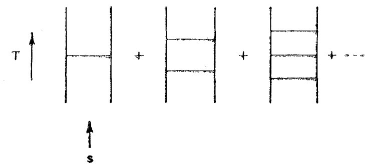

FIG. 1. The ladder graphs.

From Eq. (22) it is seen that the first factor in Eq. (21) has a simple pole at $l=-1$. From Eq. (23) it follows that the second factor in Eq. (21) is regular at $l=-1$.

Therefore, we may write

$$D_l(s)=1-f(s)/(l+1)-g(l,s) \quad (24)$$

where $g(l,s)$ is regular at $l=-1$. The condition for a Regge pole is

$$l=-1+f(s)+(l+1)g(l,s). \quad (25)$$

From the condition proved in the Appendix that both $f(s)$ and $g(l,s)$ approach zero as s approaches infinity, we see that in this limit a Regge pole moves to $l=-1$. Since $g(l,s)$ is regular in the l plane for $\text{Re}l>-3/2$, this is the only Regge pole which can lie in the half-plane, $\text{Re}l>-3/2$, in the limit of large s.

A completely analogous discussion could be made for the poles which move to the other negative integers as s approaches infinity. We shall henceforth ignore these other poles, which lie outside our proven domain of meromorphy, and concentrate on the leading Regge term.

If the coupling constant is very small it is clear that $D_l(s)$ can vanish only near one of its singularities, either in the s plane or in the l plane. Elsewhere all the terms in the series for $D_l(s)$ except unity go to zero as g^2 approaches zero. Since $D_l(s)$ is an entire function of g^2 we may hope to be able to study the trajectories of the Regge poles for small g^2 from the lowest-order expression for $D_l(s)$,

$$D_l(s)=1-\frac{g^2}{i(2\pi)^3}\int\frac{dqd\omega}{F(q,\omega,s)}Q_l\left(1+\frac{\mu^2}{2q^2}\right). \quad (26)$$

The ω integration in (26) may be performed to give

$$D_l(s)=1-\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty}\frac{ds'}{[s'(s'-4m^2)]^{1/2}}Q_l\left(1+\frac{2\mu^2}{s'-4m^2}\right)\times\frac{1}{s'-s-i\epsilon}. \quad (27)$$

Again we separate that part of D_l which is singular near $l=-1$,

$$D_l(s)=1-\frac{g^2}{8\pi^2(l+1)}\int_{4m^2}^{\infty}\frac{ds'}{[s'(s'-4m^2)]^{1/2}(s'-s-i\epsilon)}+\rho(l,s). \quad (28)$$

To order g^2 the solution to $D_l(s)=0$ is[22]

$$l=-1+\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty}\frac{ds'}{(s'-s)[s'(s'-4m^2)]^{1/2}}. \quad (29)$$

The term $\rho(l,s)$ from Eq. (28) enters first in the fourth order, by virtue of its regularity at $l=-1$. The higher-order terms in the expansion of $D_l(s)$ likewise enter the expansion, Eq. (29), for the root, only in higher order.

From Eq. (29) we see that at $s=-\infty$ the Regge pole begins at $l=-1$, moving to the right along the real axis as s is increased. As s approaches $4m^2$ the integral in (29) is not bounded. Our perturbation theory for the trajectory fails in this region, even for arbitrarily small coupling constant. As s moves from some point to the right of $4m^2$ to $+\infty$ we can again follow the trajectory from Eq. (29). The pole returns to $l=-1$ at $s=\infty$ through complex values in the upper half plane.

Some insight into what happens near $s=4m^2$ can be gained from the corresponding situation in scattering from a Yukawa potential. In that case one may see that even for arbitrarily small potential strength the first Regge pole always moves at least as far to the right as $l=-1/2$ at threshold. In the limit $g\to 0$ the pole moves discontinuously from -1 to $-1/2$ as the energy goes through the threshold value.

6. HIGH-ENERGY LIMIT OF THE STRIP APPROXIMATION

One of the main motivations for studying the complex angular momentum plane is the possible application to high-energy limits in scattering. The poles in the complex l plane are related through the formula of Regge to the high momentum transfer limit with the energy fixed.[1] This in turn is related to the high-energy limit, with fixed momentum transfer, of a certain crossed reaction.[3–5] In this section the consistency of this viewpoint is examined in perturbation theory.

We shall assume that the behavior of our amplitudes for $l\to\infty$ is such that we may neglect the contribution from the large semicircle, resulting from opening up the Watson contour in the l plane as in reference 1.

For simplicity we consider the theory of a single scalar boson with coupling $g\varphi^3$. In the ladder graphs already discussed we put $\mu=m$. The crossed graphs corresponding to the ladder graphs of Fig. 1 are the strip graphs of Fig. 2.

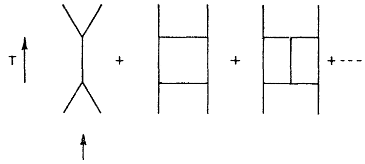

FIG. 2. The strip graphs.

[22] This formula is analogous to one derived, for scattering from a Yukawa potential, by Blankenbecler and Goldberger. See reference 6.

The question posed is that of the relevance of the perturbation expansion for the first Regge trajectory, the first two terms of which are given in Eq. (29), to the term by term asymptotic properties of the strip graphs.[23,24]

Since Eq. (29) gives a trajectory only in the vicinity of $l=-1$, we need first to use a slight modification of the Regge formula, due to Mandelstam,[17] in which the vertical contour integral in the l plane has been moved to the left of $\mathrm{Re}l=-1$.

$$T(s,\cos\theta)=-\frac{1}{8\pi^2 ip^2}\int_{-(3/2)+\epsilon-i\infty}^{-(3/2)+\epsilon+i\infty} dl(2l+1)T_l(s)Q_{-l-1}(-\cos\theta)\sec\pi l$$
$$-\frac{1}{4\pi^2p^2}\sum_{n=1}^{\infty}(-1)^{n-1}2nT_{n-(1/2)}(s)Q_{n-(1/2)}(\cos\theta)-\frac{1}{4\pi p^2}\sum_i(2\alpha_i+1)r_i(s)Q_{-\alpha_i-1}(-\cos\theta)\sec\pi l. \quad (30)$$

Here ϵ is small, real, and positive. The sum in the third term on the right of (30) is over the Regge poles to the right of $\mathrm{Re}l=-3/2$.

The asymptotic form for large $\cos\theta$, coming from the first Regge pole is of the form

$$T(s,\cos\theta)\xrightarrow[\cos\theta\to\infty]{}\frac{-1}{4\pi p^2}\frac{[2\alpha(s)+1]}{\cos\pi\alpha(s)}r(s)Q_{-\alpha(s)-1}(-\cos\theta),$$

where $r(s)$ is the residue of the first Regge pole at $l=\alpha(s)$. The asymptotic form for high energy in the crossed channel is now obtained from the replacement,

$$s\longrightarrow t,\qquad -\cos\theta\longrightarrow -1-2s/(t-4m^2). \quad (32)$$

Defining $\tilde{T}$ as the T matrix for the crossed graphs of Fig. 2 we obtain, using the asymptotic form for $Q_l(z)$,[25]

$$\lim_{s\to\infty}\tilde{T}(st)=\beta(t)s^{\alpha(t)}, \quad (33)$$

where

$$\beta(t)=\{r(t)[2\alpha+1]4^{\alpha}/\pi(t-4m^2)^{\alpha+1}\cos\pi\alpha\}\times[\pi^{1/2}\Gamma(-\alpha)/\Gamma(\tfrac{1}{2}-\alpha)]. \quad (34)$$

We now expand β and α in perturbation series,

$$\beta(t)=g^2\beta_1(t)+g^4\beta_2(t)+\cdots,\qquad \alpha(t)=-1+g^2\alpha_1(t)+\cdots. \quad (35)$$

The expanded form of Eq. (33) is

$$\lim_{s\to\infty}\tilde{T}(s,t)=g^2\beta_1(t)s^{-1}+g^4\beta_1(t)\alpha_1(t)(\ln s)s^{-1}+g^4\beta_2(t)s^{-1}+O(g^6). \quad (36)$$

It is seen from Eq. (36) that α_1 and β_1 suffice to determine the second-order high-energy limit and that part of the fourth order which goes as $s^{-1}\ln s$. From Eqs. (10), (29), and (34) we find

$$\beta_1=1/(2\pi)^4,\qquad \alpha_1(t)=\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty}\frac{dt'}{(t'-t)[t'(t'-4m^2)]^{1/2}}. \quad (37)$$

The high-energy limit including the leading, logarithmic term of the fourth order is,

$$\lim_{s\to\infty}\tilde{T}(s,t)=\frac{g^2}{(2\pi)^4s}+\frac{\ln s}{s}\frac{g^4}{2(2\pi)^6}\times\int_{4m^2}^{\infty}\frac{dt'}{t'-t}\frac{1}{[t'(t'-4)]^{1/2}}+O(g^4/s)+O(g^6). \quad (38)$$

It may easily be seen that this agrees with the high-energy limit computed directly from the first two Feynman graphs of Fig. 2, which are the crossed counterparts of the first two ladder graphs. This is the result which was anticipated. It indicates the consistency of a perturbation approach to the calculation of Regge trajectories.

The expansion in Eq. (36) is of course only a formal one. In the limit $s\to\infty$ the $(2n)$th-order term goes as $s^{-1}\ln^n s$ and the perturbation series must fail to converge. The proper approach to high-energy limits in perturbation theory is clearly through the Regge poles. Note that given $\alpha(s)$ to the second order we can compute the coefficient of the term of order $s^{-1}\ln^n s$ in the $(2n)$th-order strip graph.

We have been loosely referring to the graphs of Fig. 2 as the strip graphs. However the strip approximation usually means an approximation in which only two-particle intermediate states are retained in a crossed channel. As was seen from the discussion of Sec. 4, the ladder graphs of Fig. 1 contain, in fact, many particle intermediate states. So we do not claim to have solved exactly the problem of the high-energy behavior of the strip approximation.[23,24] Nevertheless the Bethe-Salpeter solution would seem to contain all the strip terms plus additional contributions from many particle states. Moreover, for the Bethe-Salpeter equation, we begin with a proof that a solution exists. No such proof has been written down for the strip model.

[23] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 5, 580 (1960).
[24] D. Amati, S. Fubini, A. Stanghellini, and M. Torin (to be published).
[25] *Higher Transcendental Functions*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Sec. 3.9.2.

7. INCLUSION OF A SPECTRAL FUNCTION

One generalization of the problem which is of some interest is the replacement of the exchanged particle of mass μ by a continuous distribution of masses. In the operators B and K of Eq. (6) we make the replacement,

$$\frac{1}{(p'-p)^2+\mu^2-i\epsilon} \to \int_{\xi_0}^{\infty} \frac{\sigma(\xi)d\xi}{(p'-p)^2+\xi-i\epsilon}. \quad (39)$$

We have investigated the Regge poles only in the Born approximation for D_l,

$$D_l = 1-\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty} \frac{ds'}{[s'(s'-4m^2)]^{1/2}} \frac{1}{s'-s-i\epsilon} \times \int_{\xi_0}^{\infty} d\xi\, Q_l\left(1+\frac{2\xi}{s'-4m^2}\right)\sigma(\xi). \quad (40)$$

The critical point will be the behavior of $\sigma(\xi)$ at infinity. The interesting range of behaviors at infinity is,

$$\lim_{\xi\to\infty}\sigma(\xi) = C\xi^{-1+\eta}, \quad 0<\eta<1. \quad (41)$$

From Eq. (40) one may see that in this case D_l will have a fixed, simple pole at $l=\eta-1$, and the first Regge pole will begin at $l=\eta-1$ at $s=-\infty$. For $\eta=1$ the integrals in Eq. (40) no longer converge for any l. This is unfortunately the case for the spectral function describing double pion exchange in the renormalized $\lambda\varphi^4$ theory. For $\eta\leq 0$ the first singularity of D_l is at $l=-1$, as in the case of a simple Yukawa potential.

These results are exact analogs of corresponding results for superpositions of Yukawa potentials in nonrelativistic theory. For the potential,

$$V(r) = \frac{1}{r}\int_{\mu_0}^{\infty} e^{-\mu r}\sigma(\mu^2)\mu d\mu \quad (42)$$

with the behavior of $\sigma(\mu^2)$ at infinity given by Eq. (41), it is possible to show that the first Regge pole begins at $l=\eta-1$ for infinite energy. However for potential theory the method fails for $\eta\geq 1/2$, which corresponds to a potential more singular than r^{-2} at the origin. In field theory there is enough additional convergence [from $(s')^{-1/2}$ in Eq. (36)] to allow values of η between 1/2 and 1.

8. REMARKS

The ND^{-1} method which has been used is not limited to the ladder graphs. One could probably extend the results to include more complicated field-theoretic effects by including a higher-order kernel in the Bethe-Salpeter equation. There is no obvious source of singularities other than poles in the complex angular momentum plane.

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APPENDIX

First we consider the range, $0<s<4m^2$, and note that Wick's change of contour for the ω integrals in Eq. (11) is allowed.[11] The singularities in the ω plane of the factor $F^{-1}(q,\omega,s)$ [Eq. (4)] are confined to poles in the second and fourth quadrants. This suggests a counterclockwise rotation of the ω integration contour from the real to the imaginary axis. Considering now the complete kernel [Eq. (8)],

$$\langle q,\omega|K_l|q'\omega'\rangle = \frac{g^2}{i(2\pi)^3}F^{-1}(q,\omega,s) \times Q_l\left(\frac{q^2+q'^2+\mu^2-i\epsilon-(\omega-\omega')^2}{2qq'}\right), \quad (A1)$$

we note that as both the ω and ω' contours are simultaneously rotated counterclockwise, the quantity, $-(\omega-\omega')^2$, develops a negative imaginary part. Hence, the branch points of Q_l, which are for real argument, are avoided in the rotation. The vanishing of the contributions from the circular segments at infinity follows from consideration of Eq. (11).

The transformed kernel is

$$\langle q\omega|K_l'|q'\omega'\rangle = \frac{g^2}{(2\pi)^3 i}F^{-1}(q,i\omega,s) \times Q_l\left(\frac{q^2+q'^2+\mu^2+(\omega-\omega')^2}{2qq'}\right). \quad (A2)$$

As before, we have

$$D_l(s) = \mathrm{Det}(1-K_l'). \quad (A3)$$

Equations (A2) and (A3) define a function which is manifestly analytic in s in the region in which $F(q,i\omega,s)$ does not vanish,

$$|\mathrm{Re}\sqrt{s}|<2m.$$

Next we define a symmetrized kernel which yields the same function $D_l(s)$ when the determinant is computed,

$$\langle q\omega|K_l^s|q'\omega'\rangle = \langle q\omega|K_l'|q'\omega'\rangle F^{-1/2}(q',i\omega',s) \times F^{1/2}(q,i\omega,s). \quad (A4)$$

To show that the series expansion of the determinant of $1-K_l^s$ converges we make changes of variable of the form,

$$q_i = f(t_i), \quad \omega_i = g(u_i). \quad (A5)$$

The kernel K_l^s is now to be replaced by $\bar{K}_l$, where

$$\langle tu|\bar{K}_l|t'u'\rangle=[f'(t)g'(u)]^{1/2}\langle f(t),g(u)|K_l^s|f(t')g(u')\rangle \times[f'(t')g'(u')]^{1/2}. \quad \text{(A6)}$$

The changes of variables (A5) are to be chosen such that the new region of integration for the evaluation of the traces is finite and such that the transformed kernel $\bar{K}_l$ of Eq. (A6) is bounded in this region. Transformations which accomplish this end are,

$$t=[1-(1+q)^{-\epsilon}]^{\epsilon},\quad u=[1+|\omega|]^{-\epsilon},\qquad \epsilon>0. \quad \text{(A7)}$$

These transformations are not single valued and the new region of integration in the t, u space will be very complicated. However, the integration region is now finite.

It may be seen that the new kernel defined from (A6) and the transformations (A7) is bounded in the entire region of integration for values of Rel greater than some minimum value which depends on the parameter ϵ in (A7). As ϵ approaches zero this minimum value of Rel approaches $-3/2$.

Convergence of the series expansion of $D_l(s)$ now follows in the region Re$l>-3/2$, $|\mathrm{Re}\sqrt{s}|<m$ standard methods being applicable to the case with a finite domain of integration and a bounded kernel.[26] Convergence of the series for $N_l(s)$ also follows, since the transformations described in this section are just those that transform a Bethe-Salpeter integral equation into an equation of Fredholm type. Moreover, $D_l(s)$ is analytic in s in this domain.

To enlarge the domain of holomorphy of $D_l(s)$ in s, we first note that for $0<s<4m^2$, the ω integration contour can be rotated from the one implied by Eq. (A2) by any angle $\varphi-\frac{1}{2}\pi$, $\epsilon<\varphi<\pi-\epsilon$, where ϵ is a small positive number. Under the simultaneous rotation of the ω and ω' contours, the kernel transforms into

$$\langle q\omega|K_l''(s)|q'\omega'\rangle =\frac{g^2}{(2\pi)^3 i}F^{-1}(q,e^{i\varphi}\omega,s) \times Q_l\left(\frac{q^2+q'^2+\mu^2-e^{2i\varphi}(\omega-\omega')^2}{2qq'}\right), \quad \text{(A8)}$$

$$D_l(s)=\mathrm{Det}(1-K_l'').$$

Equation (A8) defines a function $D_l(s)$ which is analytic in the region in which $F(q,e^{i\varphi}\omega,s)$ does not vanish:

$$|\mathrm{Re}\sqrt{s}-\cot\varphi\,\mathrm{Im}\sqrt{s}|<2m. \quad \text{(A9)}$$

The function $D_l(s)$ defined by (A8) in this region is clearly the analytic continuation of $D_l(s)$ defined by Eqs. (A2) and (A3) in the region $|\mathrm{Re}\sqrt{s}|<2m$.

Convergence of the series expansion for $D_l(s)$ in Eq. (A8), for Re$l>-3/2$, can be inferred as before, by converting the kernel in Eq. (A8) to a bounded one in a finite integration region by the transformation, Eq. (A7).

The union of the domains of s defined by Eq. (A9) as φ changes from ϵ to $\pi-\epsilon$ is the entire cut plane of s, with branch cut from $s=4m^2$ to ∞. Hence, $D_l(s)$ is analytic in the cut s plane, and meromorphic in l for Re$l>-3/2$.

We may observe in addition that the new kernel defined by Eqs. (A5), (A6), (A7) has the limit as s approaches infinity:

$$\lim_{|s|\to\infty}\langle tu|\bar{K}|t'u'\rangle=(1/\sqrt{s})R(t,u,t',u',s),$$

where the function R is bounded for all t, u, t', u' in the integration region, and for all s in the cut plane. We see, therefore, that $D_l(s)-1$ approaches zero at least as fast as $s^{-1/2}$ as s approaches infinity.

[26] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1940), 4th ed., Sec. 11.21.