extremely convenient if three-particle states could be handled using the two-particle approximation, it seems important to continue experimental and theoretical work to check the validity of this approximation for all three-particle states in which two of the particles can resonate.

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Influence of $\bar{K}$-Nucleon Interactions on Pion-Hyperon Scattering*†

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Scattering amplitudes for coupled $\pi\Lambda$, $\pi\Sigma$, and $\bar{K}N$ channels are obtained by an extension of the method of Amati, Stanghellini, and Vitale. The method is shown to be essentially equivalent to the N/D method in the region of nonrelativistic baryon energies, under the assumption that the only important forces arise from the Born singularities. All possibilities for the $\Sigma\Lambda$ and $\bar{K}\Lambda$ relative parities are considered. The aim is to see to what extent earlier calculations, which neglect the $\bar{K}$-N interactions, are modified by its inclusion. If the $\bar{K}N$ coupling constants are as strong as the πN coupling, significant quantitative and qualitative modifications are obtained: an $I=1$, $J=3/2$ resonance with the properties of the Y_1^* may be obtained for $P(\Sigma\Lambda)=\pm1$; an $I=0$ resonance with the location and width of the Y_0^* may be obtained for $P(\Sigma\Lambda)=-1$, in the $P_{1/2}$ state, and for $P(\Sigma\Lambda)=+1$, in the $P_{3/2}$ state. If the $\bar{K}N$ couplings are significantly weaker than the πN coupling, a $P_{3/2}$ resonance with the properties of the Y_1^* is obtained only if $P(\Sigma\Lambda)=+1$ and if the $\Sigma\Sigma\pi$ coupling is very weak; in this case one obtains no $I=0$ resonance identifiable with the Y_0^*. An $I=0$, $P_{3/2}$ resonance at 1520 MeV may be obtained with a wide variety of couplings for $P(\Sigma\Lambda)=+1$; the predicted width of this resonance is very large ($\Gamma/2>50$ MeV). Resonances in other states, multichannel effects on resonance shapes, and KN elastic scattering are discussed.

I. INTRODUCTION

The problem of π-Y and $\bar{K}$-N scattering has been studied by many authors.[1] The techniques used range from a completely relativistic approach using the Mandelstam representation,[2] through static model calculations,[3–7] to phenomenological scattering length calculations.[8,9] The major result of the first approach is the determination of the analyticity properties of the various scattering and reaction amplitudes. The low-energy $\bar{K}$-N data are fitted reasonably well by the scattering length approximation, suggesting that the low-energy s-wave interaction is determined mainly by rather distant singularities. (Inclusion of a π-π interaction in the momentum transfer channel modifies the details of the cross-section fit, but does not change the gross behavior.[10]) The static model, in many forms, has been used most extensively to predict which π-Y states will be resonant and to estimate the locations and widths of the resonances. The results of this technique which are most relevant to this paper may be briefly summarized as follows: Amati *et al.*,[3] assuming global symmetry and neglecting the $\bar{K}$-N interaction, predicted a resonance in the $I=1$, $P_{3/2}$ state with energy agreeing remarkably well with the experimental mass of the Y_1^*. In addition, they find an $I=2$, $P_{3/2}$ resonance about 160 MeV higher. They also point out that if the $\Sigma\Sigma\pi$ coupling is very weak, resonances may occur in other states in addition to these two, in particular, in the $I=0$, $P_{3/2}$ state. Assuming that the $\Sigma\Sigma\pi$ coupling is weak, Franklin[6] has tried to estimate the relative locations of these three resonances by estimating the effect on the cutoff integrals of the Σ-Λ mass difference and the crossed terms. Duimio and Wolters[4] applied the

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same technique assuming odd Σ-Λ parity but still neglecting $\bar{K}$-N interactions. They found resonances to be possible in most isotopic spin states with $J=1/2$ or $3/2$ with the notable exceptions of $I=0$, $J=3/2-$ and $I=1$, $J=3/2+$ [taking $P(\Lambda)=+1$ by definition]. In particular, a Y_1^* in an $S_{1/2}$ or $P_{1/2}$ state but *not* in a $P_{3/2}$ state is consistent with their calculations.

The effects of the $\bar{K}$-N interaction on π-Y scattering have been discussed only under very restrictive assumptions. Dalitz and Tuan[8] originally pointed out how strong attractive $\bar{K}$-N interactions could generate "virtual bound state" resonances in a zero-range approximation. These resonances arise primarily from interactions in the $\bar{K}$-N channel, whereas those discussed in the preceding paragraph arise from the π-Y interactions alone. The coupling of the $\bar{K}$-N and π-Y channels may, in addition to modifying the properties of these two sorts of resonances, generate new resonances which may be thought of as neither primarily due to $\bar{K}$-N or to π-Y interactions. Islam[5] has modified the calculations of Amati *et al.*[3] to include the $\bar{K}$-N interaction and estimated the effects to be negligible; however, he restricted the $\bar{K}$-N cutoff integrals to have very small values.

In this paper, the technique of Amati *et al.*[3] is extended to include the $\bar{K}$-N interaction. This is an extension to multichannel problems of a technique introduced by Fubini[11] and by Bosco *et al.*[12] for treating π-N scattering, and we shall refer to it as the Fubini technique. Using this technique, which has proved moderately successful for low-energy π-N scattering, is essentially equivalent to assuming that all the important forces arise from the Born singularities. In addition to this assumption, the main limitation on this approach is the ambiguity involved in the cutoff integrals. There seems to be no way around this at the present but, although this model is far from a complete theory of the $\bar{K}$-N, π-Y interaction, it is another step in that direction. This calculation is an attempt to understand what features of the low energy π-Y, $\bar{K}$-N scattering processes depend primarily on these forces. The most striking features of the low-energy scattering are: (a) the large $\bar{K}$-N s-wave scattering and absorption at very low energy,[9] (b) the existence of the Y_1^* resonant state at 1385 MeV with $J>1/2$,[13,14] (c) the existence of the Y_0^* resonant state at 1405 MeV with unknown J,[15] and (d) the existence of the Y_0^{**} resonant state at 1520 MeV with $J=3/2$.[16] As mentioned before, (a) seems to depend mainly on distant singularities and is certainly outside the scope of our calculation. Since the parities of the $\bar{K}$ and Σ are not yet definitely established, the calculation has been done for all possible parity combinations. In any case, the model predicts no scattering for $l\geq 2$. The comparison of our results with experimental values for resonance energies, branching ratios, etc. may indicate various restrictions on parities, coupling constants, and cutoff integrals. Or, from a negative point of view, the comparison may indicate under what conditions some other mechanism should be considered as primarily responsible for certain resonances (or for suppression of unobserved resonances predicted by the model). As an obvious example, this model cannot predict the Y_0^{**} if it is a $D_{3/2}$ resonance, as presently seems to be the case, regardless of the $\bar{K}$ and Σ parities. In addition to the more fundamental dynamical questions, the amplitudes allow us to explore some detailed effects of multichannel processes.

In Sec. II the Fubini technique is outlined and related to the dispersion theoretic approach and the basic equations are obtained. Section III is devoted to the resonance conditions and related equations and a listing of the possible resonant states. The effects of the $\bar{K}$-N interaction on the location and decay branching ratios of the Y_1^* are presented in Sec. IV while Sec. V is concerned with properties of possible $I=0$ resonances. Section VI is devoted to a summary of the results and possible interpretations of the resonances and to a general discussion of ambiguities in and possible extensions of this calculation.

II. GENERAL TECHNIQUES; INTEGRAL EQUATION FOR THE T AND K MATRIX

In this section, the Fubini technique[11,12] for obtaining the T and the K matrix will be outlined and its relation to the more general dispersion-theoretic approach discussed. To make our notation clear from the start, the quantity referred to here as the T' matrix is related to the cross sections in a state of definite angular momentum J by the expression

$$\sigma(\alpha\rightarrow\beta)=(4\pi/q_\alpha^2)(J+\tfrac{1}{2})\,|T_{\beta\alpha}'|^2. \tag{2.1}$$

The momentum in channel α is denoted by q_α; the diagonal matrix (in channel space) of momenta will be simply denoted by q. In general, an operator A' is related to the operator A by $A'=\pi\rho^{1/2}A\rho^{1/2}$, where ρ is the density of states operator, and $\rho(q)=q/\pi$ for two-particle channels. It is convenient to number the channels and we shall number them according to increasing threshold energy, counting only channels which are allowed by the various selection rules. Only the two-particle channels $\pi\Lambda$, $\pi\Sigma$, and $\bar{K}N$ will be included.

The Fubini technique proceeds as follows: (For de-

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tails, see references 11 and 12, and Appendix A to this paper.) A static Hamiltonian with no meson-meson interaction is assumed. (See Eq. 3.12.) The normalization is chosen so that

$$H|N\rangle=0,$$
$$H|\Lambda\rangle=M_\Lambda|\Lambda\rangle, \tag{2.2}$$
$$H|\Sigma\rangle=M_\Sigma|\Sigma\rangle=(M_\Lambda+\Delta)|\Sigma\rangle.$$

The state vectors $|B\rangle$ represent the *physical* baryons. The symbol $\phi(\mathbf{q})$ will denote a *row* vector in channel space such that the αth component represents the free meson of momentum $\mathbf{q}$ plus the physical baryon appropriate to channel α, e.g., for $I=1$, $|\phi(\mathbf{q})\rangle=a_\mathbf{q}^\dagger|\Lambda\rangle$ where $a_\mathbf{q}^\dagger$ creates a π of momentum $\mathbf{q}$. (Charge indices are suppressed.)

An "out state" vector $|\psi(\mathbf{q})\rangle_+$ is constructed in such a way that in the remote past $|\psi(\mathbf{q})\rangle_+\to|\phi(\mathbf{q})\rangle$:

$$|\psi(\mathbf{q})\rangle_+=\sum_{\mathbf{q}''}\{|\phi(\mathbf{q}'')\rangle-\sum_\lambda|\psi_\lambda\rangle_+ \times{}_+\langle\psi_\lambda|\phi(\mathbf{q}'')\rangle\}X(\mathbf{q}'',\mathbf{q}). \tag{2.3}$$

$X(\mathbf{q}'',\mathbf{q})$ is a matrix in channel space to be determined so that $|\psi(\mathbf{q})\rangle_+$ actually represents the physical scattering state; λ indexes the set of all out states except those containing one meson plus one baryon. The asymptotic condition on $|\psi(\mathbf{q})\rangle_+$ requires that $X(\mathbf{q}'',\mathbf{q})$ have the form

$$X(\mathbf{q}'',\mathbf{q})=\delta(\mathbf{q}''-\mathbf{q})+\frac{F(\mathbf{q}'',\mathbf{q})}{\omega''-\omega-i\epsilon}. \tag{2.4}$$

The physical interpretation of $F(\mathbf{q}'',\mathbf{q})$ is most readily expressed if Eq. (2.4) is transformed to the total angular momentum representation. Then, on the energy shell

$$\pi F_J(\omega,\omega)=T_J'(\omega). \tag{2.5}$$

Since $|\psi(\mathbf{q})\rangle_+$ is by construction orthogonal to all out states with the exception of the two-particle states, the obvious condition to apply to obtain equations for $X(\mathbf{q}'',\mathbf{q})$ is that

$${}_+\langle\psi(\mathbf{q}')|\psi(\mathbf{q})\rangle_+=\delta(\mathbf{q}'-q). \tag{2.6}$$

A more useful set of equations is obtained if one requires instead

$$\langle\phi(\mathbf{q}')|(H-\omega)|\psi(\mathbf{q})\rangle_+=0. \tag{2.7}$$

where ω denotes the total energy of the state $|\psi(\mathbf{q})\rangle_+$. These two conditions are equivalent (see Appendix A).

In order to obtain a soluble set of equations from Eq. (2.7), it is necessary to make the "one-meson, no-crossing" approximation. This method for obtaining soluble integral equations is discussed in references 3, 11, and 12; an equivalent but somewhat different method is outlined in Appendix A. The resulting integral equation in the total angular momentum representation is

$$X_{J\pm}(\omega',\omega)=\delta(\omega'-\omega)+\frac{1}{\omega'-\omega-i\epsilon}\times\int d\omega''K_{J\pm}(\omega',\omega'')X_{J\pm}(\omega'',\omega), \tag{2.8a}$$

where

$$[K_{J\pm}(\omega',\omega'')]_{\alpha\beta}=\frac{1}{3\pi}q^{l_\alpha+1/2}\times\sum_\mu\frac{u_\alpha{}^\mu(q')(B_{J\pm}{}^\mu)_{\alpha\beta}u_\beta{}^\mu(q'')}{(\omega''-M_\mu)}\left(\frac{\omega-M_\mu}{\omega'-M_\mu}\right)q_\beta{}^{l_\beta+1/2}. \tag{2.8b}$$

The notation is defined in terms of the Born terms for the T matrix: Let the sum of all Born terms for the processes shown in Fig. 1 be written

$$T_{J\pm}{}^{\mathrm{Born}}(\omega)=\tfrac{1}{3}q_L\sum_\mu\frac{B_{J\pm}{}^\mu}{\omega-M_\mu}q_L, \tag{2.9}$$

where q_L denotes the diagonal matrix of $q_\alpha{}^{l_\alpha}$, with l_α the orbital angular momentum in channel α. The $\pm$ subscript indicates the parity of the state. [$P(\Lambda)=+1$ by convention.] $u_\alpha{}^\mu(q)$ denotes the cutoff function in channel α associated with the pole at M_μ.

These equations are separable and so may be solved algebraically to give, with the use of Eq. (2.5),

$$T_{J\pm}'=\pi\rho(q)^{1/2}q_L\sum_\mu u^\mu(q)B_{J\pm}{}^\mu A_{J\pm}{}^\mu(\omega), \tag{2.10}$$

where the quantities $A_{J\pm}{}^\mu(\omega)$ satisfy the algebraic matrix equations:

$$A_{J\pm}{}^\mu(\omega)=\frac{1}{3}\frac{u_\mu(q)q_L\rho^{1/2}(q)}{\omega-M_\mu}+\sum_\nu \mathcal{J}_{\mu\nu}(\omega)B_{J\pm}{}^\nu(\omega)A^\nu{}_{J\pm}(\omega)(\omega-M_\nu), \tag{2.11a}$$

with

$$\mathcal{J}_{\mu\nu}(\omega)=\frac{1}{3}\int_{\omega_t}^\infty d\omega''\frac{u_\mu(q'')q_L''^2\rho(q'')u_\nu(q'')}{(\omega''-M_\mu)(\omega''-M_\nu)(\omega''-\omega-i\epsilon)}. \tag{2.11b}$$

$u_\mu(q)$ denotes the diagonal matrix of cutoff functions associated with the pole at M_μ and ω_t denotes the threshold energy for the channel corresponding to the appropriate element of $\mathcal{J}_{\mu\nu}(\omega)$.

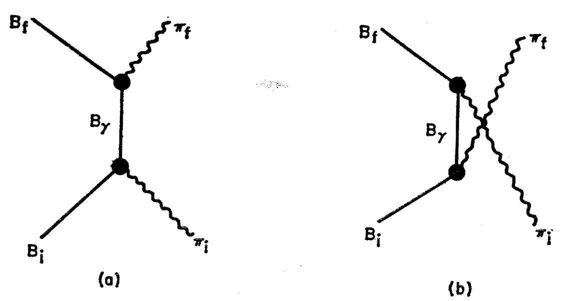

FIG. 1. Diagram for the static Born terms; B_α and π_α are generic symbols for the baryon and meson α. In (a) $M_\mu=M_\gamma$; in (b) $M_\mu=M_i+M_f-M_\gamma$.

Let us consider the relation of this method to the dispersion-theoretic approach. This is simply to put the Fubini method into this more general framework and thus to clarify the nature of the approximations involved. The analytic structure of the partial wave amplitudes has been determined by Nauenberg.[2] Considering only the two-body channels $\pi\Lambda$, $\pi\Sigma$, and $\bar{K}N$, each partial wave amplitude has, in the total energy plane, one, three, or two branch lines, for $I=2$, 1, 0, respectively, starting at the threshold for each channel and running the length of the positive real axis. The dynamical singularities are quite complicated and some of them come quite close to the physical region. The Born terms alone contribute a variety of poles and cuts. Suppose that the low-energy behavior of the amplitudes is dominated by the Born singularities. To make the term "low energy" more precise, we define it to be that region for which the baryon kinetic energy is much less than the total meson energy. To lowest order in the baryon kinetic energy divided by the meson total energy, the Born singularities are essentially the same as those determined by static calculations with correct kinematic relations. They are not exactly the same; terms arising from Fig. 1(a) require some phase space corrections. The corresponding corrections to terms arising from Fig. 1(b) are approximately canceled by other corrections which evidently arise from antibaryon contributions to the Born terms. (See Appendix A2.) The same cancellation was found in π-N scattering by Chew *et al.*[18] Since we shall be primarily interested in $P_{3/2}$ amplitudes, which arise entirely from Fig. 1(b), let us neglect these corrections in what follows:

The N/D method[19] may be used to solve the scattering problem. In order to insure the correct threshold behavior of T, define (suppressing the angular momentum index)

$$T(\omega)=q_L N(\omega)D^{-1}(\omega)q_L. \tag{2.12}$$

$N(\omega)$ is assumed to have only the dynamical, or left-hand, singularities of $T(\omega)$, while $D(\omega)$ has only the branch lines along the positive real axis. The unitarity condition requires that the discontinuity of $D(\omega)$ along this cut is given by

$$D(\omega+i\epsilon)-D(\omega-i\epsilon)=-2\pi i q_L{}^2\rho(q)\theta(\omega)N(\omega). \tag{2.13}$$

$\theta(\omega)$ is a diagonal matrix in channel space, the diagonal elements of which are equal to one above and zero below the thresholds of the corresponding channel. Then

$$D(\omega)=C-\int^{\infty} d\omega' \frac{q_L'^2\rho(q')\theta(\omega')N(\omega')}{\omega'-\omega-i\epsilon}, \tag{2.14}$$

where C is a constant matrix. The assumption that the Born singularities are the only left-hand singularities to be taken into account explicitly implies that $N(\omega)$ may be written as

$$N(\omega)=\tfrac{1}{3}\sum_\mu \frac{B^\mu}{\omega-M_\mu}D(M_\mu). \tag{2.15}$$

This approximation is certainly not valid for high energy and, in fact, if Eq. (2.15) is inserted into Eq. (2.14), the integrals diverge. Presumably, if the other left-hand singularities of $T(\omega)$ were taken into account in calculating $N(\omega)$, the integrals would converge. Since the left-hand singularities are, in general, not known, we assume that their effect in the low-energy region can be adequately accounted for by cutting the integrals off at some finite upper limit:

$$D_{\alpha\beta}(\omega)=C_{\alpha\beta}-\frac{1}{3\pi}\sum_{\mu,\gamma}\int_{\omega_{t\alpha}}^{R_\alpha} d\omega' \frac{q_\alpha'^{2l_\alpha+1}}{(\omega'-\omega-i\epsilon)(\omega'-M_\mu)} B_{\alpha\gamma}{}^\mu D_{\gamma\beta}(M_\mu). \tag{2.17}$$

$\omega_{t\alpha}$ denotes the threshold energy for channel α.

The matrix C may be eliminated by subtracting $D_{\alpha\beta}(M_\nu)$ from $D_{\alpha\beta}(\omega)$. Since the integrals all converge, it does not matter which M_ν is used for the subtraction energy; the equations obtained for $D(\omega)$ are exactly the same for each M_ν and they may be used interchangeably. It follows that $D_{\alpha\beta}(\omega)$ may be written as

$$D_{\alpha\beta}(\omega)=D_{\alpha\beta}(M_\mu)-(\omega-M_\mu)\times\sum\nolimits_{\nu,\gamma} \mathcal{g}_\alpha{}^{\mu,\nu}(\omega)B_{\alpha\gamma}D_{\gamma\beta}(M_\nu), \tag{2.18a}$$

where

$$\mathcal{g}_\alpha{}^{\mu\nu}(\omega)=\frac{1}{3\pi}\int_{\omega_{t\alpha}}^{R_\alpha} d\omega' \frac{q_\alpha'^{2l_\alpha+1}}{(\omega'-M_\mu)(\omega'-M_\nu)(\omega'-\omega-i\epsilon)}. \tag{2.18b}$$

If the diagonal matrices of integrals defined in Eq. (2.18b) are identified with those defined in Eq. (2.11b), then Eqs. (2.11a) and (2.18a) imply that

$$A^\mu(\omega)=D(M_\mu)D^{-1}(\omega)q_L\rho^{1/2}(q)/3(\omega-M_\mu). \tag{2.19}$$

A comparison of Eqs. (2.10) and (2.12) shows that the same expression for $T'(\omega)$ is obtained by both methods. We do not pretend that this justifies the application of the Fubini technique to this problem, but it does make clear the assumptions involved.

For the discussion of resonances, it is convenient to introduce the reduced K matrix, $K_r(\omega)$, defined by

$$T(\omega)=K_r(\omega)[1-i\pi\rho(q)\theta(\omega)K_r(\omega)]^{-1}. \tag{2.20}$$

$K_r(\omega)$ may be expressed in terms of $N(\omega)$, $D(\omega)$ by using Eqs. (2.12) and (2.14):

$$K_r(\omega)=q_L N(\omega)[D(\omega)+i\pi q_L{}^2\rho(q)\theta(\omega)N(\omega)]^{-1}q_L. \tag{2.21}$$

Equation (2.21) makes it clear that equations for $K_r(\omega)$ may be obtained from the equations for $T(\omega)$ by simply

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replacing $\mathcal{I}^{\mu\nu}(\omega)$ by $I^{\mu\nu}(\omega)$, where

$$I_\alpha{}^{\mu\nu}(\omega)=\frac{P}{3\pi}\int_{\omega_{t\alpha}}^{R_\alpha}d\omega'\frac{q_\alpha{}'^{2l_\alpha+1}}{(\omega'-\omega)(\omega'-M_\mu)(\omega'-M_\nu)}. \quad (2.22)$$

The same result is obtained by the Fubini technique if standing-wave boundary conditions replace outgoing-wave boundary conditions in Eq. (2.4). In general, the cutoff energy is expected to be large compared with the Σ-Λ mass difference. In that case, it is a good approximation to drop the dependence of $I_\alpha{}^{\mu\nu}(\omega)$ on μ, ν. The equations for $K_r(\omega)$ corresponding to Eq. (2.18a) may then be easily solved to give

$$K_r(\omega)=\tfrac{1}{3}q_L\left\{\sum_\mu\frac{B^\mu}{\omega-M_\mu}+(\sum_\mu B^\mu)\right.$$
$$\left.\times[1-I(\omega)\sum_\mu B^\mu(\omega-M_\mu)]^{-1}I(\omega)(\sum_\mu B^\mu)\right\}q_L. \quad (2.23)$$

It is easily seen from Eq. (2.23) that $K_r(\omega)$ is a symmetric matrix, so that the approximations made here are consistent with unitarity and time-reversal invariance.

The quantities $I(\omega)$ are the greatest source of uncertainty in this approach. For s-wave interactions, the $I(\omega)$ are not very sensitive to the cutoff but are energy dependent,[4] whereas, for p-wave interactions the $I(\omega)$ depend strongly on the cutoff but are energy independent to a good approximation for energies well below the cutoff energy. In π-N scattering, a cutoff of about 14 μ is required to give the (3,3) resonance at the observed energy, (This value is larger than the one usually quoted; this is because the crossing terms are not included explicitly here but are taken with the other left-hand singularities as determining the high-energy cutoff.) In the numerical work that follows, the p-wave integrals $I(\omega)$ are taken to have the constant value 1.6 μ;[20] this corresponds to a cutoff of about 16 μ. We have no justification for assigning this value to all of the integrals and, whenever possible, results will be phrased in such a way as to not depend on this assumption. Moderate corrections to this value will not affect the results radically. Further discussion of the cutoff is deferred until Sec. VI.

[20] The assumption that the integrals $I(\omega)$ are approximately constant will be referred to as the effective range approximation. This is not the same as the effective range approximation of M. H. Ross and G. L. Shaw, Ann. Phys. (New York) **13**, 147 (1961), but it is related to it in the same way as the Chew-Low effective range formula is related to the usual effective range expansion (for elastic scattering) of nuclear physics. The analogy can be seen most clearly if Eq. (2.23) is used to calculate K_r^{-1}, with the Σ-Λ mass difference neglected: $q_LK_r^{-1}q_L=3(\omega-M_\Lambda)[B^{-1}-I(\omega-M_\Lambda)]$. See also P. T. Mathews and A. Salam, Nuovo cimento **13**, 381 (1959). Y. Matsuzaki, in a preprint received after the bulk of this work was completed, has performed calculations similar to some of those in our paper. He too points out that this magnitude of I is consistent with the π-N results. (Islam, reference 5, uses a value about half this size for the $\bar{K}$-N integrals.) We thank Professor Dalitz for calling this paper to our attention.

III. POSSIBLE RESONANT STATES

The most natural generalization of the one-channel resonance condition is to require that one of the eigenphases goes through $\pi/2$; i.e., one of the elements of the diagonalized T'-matrix goes through i at the resonance energy. This may be neatly expressed in terms of $K_r(\omega)$: at resonance[1,21]

$$\det[K_r^{-1}(\omega)]=0. \quad (3.1)$$

This condition may be considered as a condition on $D(\omega)$: from Eqs. (2.21) and (3.1)

$$\det[\mathrm{Re}D(\omega)]=0. \quad (3.2)$$

since $\det N(\omega)<\infty$ in the physical region.

It is convenient to introduce the matrix Γ defined by

$$K_r'(\omega)=\Gamma(\omega)/2(\omega_r-\omega), \quad (3.3)$$

where ω_r denotes the resonance energy. If only one eigenphase goes through $\pi/2$ at ω_r, then there are some rather severe restrictions on the elements of Γ. First, of course, each element of Γ is finite. Furthermore,

$$\det\Gamma(\omega)=2^n(\omega_r-\omega)^n\det K_r'(\omega)\underset{\omega\to\omega_r}{\longrightarrow}O(\omega-\omega_r)^{n-1}, \quad (3.4)$$

where n is the number of channels. For a two-channel system this is the only condition. In general, the fact that only one element of the diagonalized Γ is nonzero at resonance implies that each element of the adjoint matrix $\Gamma^A(\omega)$ goes to zero at least linearly as $\omega\to\omega_r$ and further that at resonance

$$\Gamma_{ik}(\omega_r)/\Gamma_{jk}(\omega_r)=\Gamma_{il}(\omega_r)/\Gamma_{jl}(\omega_r). \quad (3.5)$$

If the nonresonant phase shifts are small at $\omega=\omega_r$, then Eq. (3.5) implies that the branching ratios for the decay of the resonant state are independent of how the state is formed. Each resonance predicted by our model is associated with a simple zero of Eq. (3.2) and so the conditions Eqs. (3.4) and (3.5) hold.[22]

The relations between the elements of Γ and the elements of the open-channel T' matrix for the various situations we shall consider explicitly are as follows:

(a) Two channels, both open

$$T_{ij}'(\omega)=\frac{(\Gamma_{ij}/2)+i[\det\Gamma/4(\omega-\omega_r)]\delta_{ij}}{\omega_r-\omega+[\det\Gamma/4(\omega-\omega_r)]-i\,\mathrm{Tr}(\Gamma/2)}; \quad (3.6a)$$

[21] R. Oehme, Nuovo cimento **20**, 334 (1961).

[22] The adjoint matrix Γ^A is the transpose of the matrix of minors of Γ; i.e., $(\Gamma\Gamma^A)_{ij}=\delta_{ij}\det\Gamma$. There is one amusing effect that results if the condition $\det\Gamma(\omega_r)=0$ at resonance does not hold for the two-channel case. This means that each eigenphase passes through i at ω_r. It follows that the off-diagonal elements of T' vanish at ω_r while the two diagonal elements take on the value i; i.e., at ω_r the elastic scattering cross sections reach the maximum value allowed by unitarity while the inelastic scattering cross section vanishes.

(b) Two channels, one open

$$T_{11}'(\omega)=\frac{\Gamma_{11}/2}{\omega_r-\omega-i(\Gamma_{11}/2)}; \qquad (3.6b)$$

(c) Three channels, two open

$$T_{ij}'(\omega)=\frac{(\Gamma_{ij}/2)+i[\Gamma_{33}{}^A/4(\omega-\omega_r)]\delta_{ij}}{\omega_r-\omega+[\Gamma_{33}{}^A/4(\omega-\omega_r)]-i(\Gamma_{11}+\Gamma_{22})/2}. \qquad (3.6c)$$

Consider the two-channel cases. Let U, defined by

$$U=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},$$

be unitary matrix which diagonalizes the 2×2 matrix $T'(\omega)$, such that

$$(UT'U^{-1})_{11}=\frac{\Gamma_e/2}{\omega_r-\omega-i(\Gamma_e/2)},$$
$$(UT'U^{-1})_{22}=\frac{c}{1-ic}. \qquad (3.7)$$

The elements of $T'(\omega)$ are, in terms of Γ_e, c, and θ,

$$T_{11}'(\omega)=\frac{\Gamma_e/2}{\omega_r-\omega-i(\Gamma_e/2)}\cos^2\theta+\frac{c}{1-ic}\sin^2\theta,$$
$$T_{22}'(\omega)=\frac{\Gamma_e/2}{\omega_r-\omega-i(\Gamma_e/2)}\sin^2\theta+\frac{c}{1-ic}\cos^2\theta, \qquad (3.8)$$
$$T_{12}'(\omega)=-\sin\theta\cos\theta\left[\frac{\Gamma_e/2}{\omega_r-\omega-i(\Gamma_e/2)}-\frac{c}{1-ic}\right].$$

In general,

$$\Gamma_e/2+c(\omega_r-\omega)=(\Gamma_{11}+\Gamma_{22})/2; \qquad (3.9a)$$

in the event that $|c|\ll 1$, which is usually the case, we have

$$c\approx \det\Gamma/2(\omega_r-\omega)\ \mathrm{Tr}\Gamma \qquad (3.9b)$$

for case (a) and a corresponding expression for case (c). This makes explicit the relation between the nonresonant part of the scattering and the quantities which distort Eqs. (3.6) from the usual Breit-Wigner form.

Specifically, the equation we shall use to determine the resonance energy is [cf. Eqs. (2.18a), (2.21), and (3.2)]

$$\mathfrak{D}(\omega_r)\equiv\det[1-I\textstyle\sum_\mu B^\mu(\omega_r-M_\mu)]=0. \qquad (3.10)$$

Note that in the effective-range approximation, the degree of this equation is equal to the number of channels. Hence, the number of resonances possible in this approximation is less than or equal to the number of channels.[24] The matrix Γ *at resonance* is given by [cf. Eqs. (2.23) and (3.3)]

$$\tfrac{1}{2}\Gamma(\omega_r)=-(\pi/3)[d\mathfrak{D}/d\omega]^{-1}\rho^{1/2}q_L(\textstyle\sum_\mu B^\mu)$$
$$\times[1-I\textstyle\sum_\mu B^\mu(\omega-M_\mu)]^A$$
$$\times I(\textstyle\sum_\mu B^\mu)q_L\rho^{1/2}|_{\omega=\omega_r}. \qquad (3.11)$$

These expressions are too long to present for every case. Those of direct interest will be presented in the subsequent sections and other relevant expressions will be found in Appendix B.

To make it quite clear which coupling constants are being referred to in the following, we write down explicitly the general interaction Hamiltonian of the static model:

$$H_{\rm int}=(4\pi)^{1/2}\int d^3x\ \{f_N\chi_N{}^\dagger\tau_i\boldsymbol{\sigma}\cdot\nabla\phi_i(x)\chi_N v_N(x)$$
$$-i\epsilon_{ijk}f_\Sigma\chi_{\Sigma_i}{}^\dagger\boldsymbol{\sigma}\cdot\nabla\phi_k(x)\chi_{\Sigma_j}v_\Sigma(x)$$
$$+f_\Lambda\chi_{\Sigma_i}{}^\dagger O_\Lambda\phi_i(x)\chi_\Lambda v_\Lambda(x)+\text{H.c.}$$
$$+g_\Lambda\chi_N{}^\dagger O_\Lambda' K(x)\chi_\Lambda v_\Lambda'(x)+\text{H.c.}$$
$$+g_\Sigma\chi_N{}^\dagger\tau_i O_\Sigma' K(x)\chi_{\Sigma_i}v_\Sigma'(x)+\text{H.c.}\}, \qquad (3.12)$$

where $v(x)$, $v'(x)$ are the Fourier transforms of the cutoff functions and O, O' denote either 1 or $\boldsymbol{\sigma}\cdot\nabla$, depending on the parities of the particles. $\phi(x)$ denotes the π field, $K(x)$ the K field, and χ_B the baryon spinors. The pion mass μ is always taken equal to unity so $f_N^2=0.08$. (The ΞYK couplings are neglected. See Sec. IV.)

The effect of introducing the $\bar{K}$-N interaction is to depress the energy of resonances arising from the π-Y interaction alone (in agreement with expectations based on second-order perturbation theory[1]) and to introduce new resonances. The new resonances always occur at higher energy than any in the same state arising from the π-Y couplings alone. It should be pointed out that an arbitrary increase in coupling constants may push a resonance to *higher* energy if no new channels are coupled; it is the inclusion of new channels (open or closed) which depresses the resonance energy. This can already be seen in the results of Amati *et al.*[3] in which, for $f_\Sigma=0$, there is a $J=3/2$, $I=0$ resonance which moves out to ∞ when $f_\Sigma^2=f_\Lambda^2/2$.

Table I lists the possible resonances. (The $I=2$ case is not affected by the $\bar{K}$-N coupling but is included for completeness.[25]) Some of these require certain restric-

[23] It may appear that $T_{11}'(\omega)$ given by Eq. (3.6b) is not the analytic continuation below the threshold of channel 2 of $T_{11}'(\omega)$ given by Eq. (3.6a). This is only apparent, caused by the nonanalyticity of the elements of Γ; it may be directly verified that the two expressions are connected by analytic continuation, as must be the case.

[24] The same limitation on the number of resonances is obtained in the effective-range expansion of Ross and Shaw, reference 20.

[25] The results obtained here for $P(\Sigma)=-1$ and vanishing $\bar{K}$-N couplings do not agree with those given in reference 4 for the cases $I=2$, $J=1/2+$ and $I=1$, $J=1/2+$.

TABLE I. Resonances for various parities of Σ and K.

	$I=2$	$I=1$	$I=0$
I. $P(\Sigma)=+1$, $P(K)=+1$	$\frac{3}{2}+$	$\frac{3}{2}+$, $\frac{1}{2}+$	$\frac{3}{2}+$, $\frac{1}{2}+$
II. $P(\Sigma)=+1$, $P(K)=-1$	$\frac{3}{2}+$	$\frac{3}{2}+$, $\frac{1}{2}+$	$\frac{3}{2}+$, $\frac{1}{2}+$
III. $P(\Sigma)=-1$, $P(K)=+1$	$\frac{3}{2}-$, $\frac{1}{2}+$	$\frac{3}{2}-$, $\frac{1}{2}+$, $\frac{1}{2}-$	$\frac{1}{2}-$, $\frac{1}{2}+$
IV. $P(\Sigma)=-1$, $P(K)=-1$	$\frac{3}{2}-$, $\frac{1}{2}+$	$\frac{3}{2}+$, $\frac{3}{2}-$, $\frac{1}{2}+$, $\frac{1}{2}-$	$\frac{1}{2}-$

tions on the coupling constants for their existence and others lie at very high energies (more than 3 μ above the π-Λ threshold) unless some of the couplings strengths are unreasonably large. The most interesting possibilities are discussed in the next two sections.

It should be noted that none of these resonances arises predominantly from forces in the $\bar{K}$-N channel; i.e. in the limit of zero coupling between the π-Y and $\bar{K}$-N channels, there are no $\bar{K}$-N resonances. The fact that the $\bar{K}$-N Born terms are repulsive is responsible for this. However, many of the resonances result from the coupling between the Π-Y and $\bar{K}$-N channels and fail to exist when the channels are decoupled as, for example, in the $J=3/2+$, $J=1$ state for case IV of Table I.

IV. EFFECTS OF THE $\bar{K}$-N CHANNEL ON THE Y_1^*

The two original interpretations of the Y_1^* resonant state were based on calculations that were published before the Y_1^* was discovered. The first calculation, by Amati *et al.*,[3] was based on the assumption of global symmetry and predicted a $J=3/2$, $I=1$ resonance, analogous to the (3,3) resonance. Gell-Mann had previously pointed out that exact global symmetry required such a resonance.[26] One of the sets of scattering lengths obtained by Dalitz and Tuan[8] gave a $J=1/2$ $I=1$ resonance, as a virtual $\bar{K}$-N bound state. The revised scattering lengths of Ross and Humphrey[9] make the latter interpretation unlikely. In addition, the results of Ely *et al.*[13] suggest that the spin of the Y_1^* is greater than 1/2, consistent with the first interpretation. The main difficulty with the first interpretation is that global symmetry predicts a branching ratio

$$r=(Y_1^*\rightarrow\pi+\Sigma)/(Y_1^*\rightarrow\pi+\Lambda)$$

of about 0.11,[27] whereas the experimental value is less than 0.05 and is consistent with zero. One of the main objectives of this section is to see if, with the $\bar{K}$-N interaction taken into account, an $I=1$ resonance with $J=3/2$ and $r<0.05$ is predicted in the correct energy range.

The recent results on the Λ decay parameters[28] favor $P(K)=-1$. The results obtained here may also be taken as favoring $P(K)=-1$, within the limitations of our model. In case I the $\bar{K}$-N interaction has no effect on states of $J=3/2$. Thus, the results for case I of Table I, $J=3/2$ are identical to those already discussed by Amati *et al.*[3] The only way to change r is to vary f_Λ or f_Σ. The difficulties with this are discussed in regard to case II below. In case III there is no π-Λ scattering in the $J=3/2$ state and so certainly no resonance. Thus, we consider $P(K)=+1$ to be very unlikely and will have very little more to say about it; unless otherwise explicitly stated, we henceforth take $P(K)=-1$.

The possibility that $P(\Sigma)=-1$ is appealing for many reasons,[29] one of them being the small value of r. The results of Duimio and Wolters[4] show, however, that a $J=3/2$, $I=1$ resonance is not possible on the basis of π-Y interactions alone. When the $\bar{K}$-N interaction is included, we find that such a resonance is possible (case IV); it arises from the interaction between the π-Λ and $\bar{K}$-N channels. (Each element of the K-matrix is proportional to $f_N g_\Lambda$.) Since there is no d-wave scattering in this model, r is identically zero. The resonance energy is given by

$$\omega_r=[2\sqrt{2}f_N g_\Lambda(I_1 I_3)^{1/2}]^{-1}+M_\Lambda. \quad (4.1)$$

If one substitutes into Eq. (4.1), $\omega_r=1385$ MeV and $I_1=1.6$, $I_3=1.6$, one obtains $g_\Lambda^2\approx 2f_N^2$. Since the $\bar{K}$ couplings seem to be weaker than the π couplings,[26] this is probably an unreasonably large value for g_Λ. If a more reasonable value is chosen, $g_\Lambda^2=f_N^2/4$, the resonance is located at about 1900 MeV. The half-width of the resonance is given by

$$\Gamma_{11}/2=(\sqrt{2}/3)f_N g_\Lambda q_1^3. \quad (4.2)$$

(Note that Γ_{11} is independent of the cutoff. This is quite general if the same cutoff is assumed in all

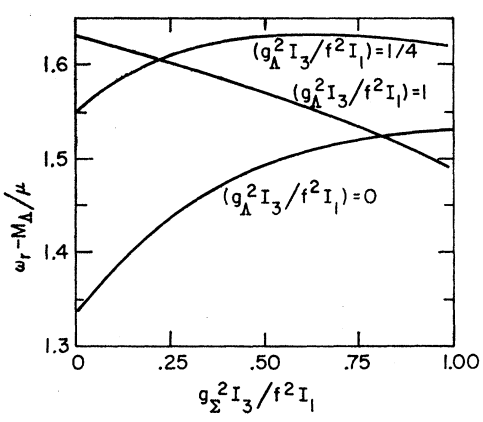

FIG. 2. The energy of the $I=1$, $J=3/2$ resonance for case II and $f_\Sigma^2=f_\Lambda^2$, as a function $g_\Sigma^2 I_3/f_\Lambda^2 I_1$ for various values of $g_\Lambda^2 I_3/f_\Lambda^2 I_1$.

[26] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957).

[27] This result is more general than the model calculation of Amati *et al.* (see reference 3) might suggest. See T. D. Lee and C. N. Yang, [Phys. Rev. **122**, 1954 (1961)] where it is obtained from the properties of the global symmetry group.

[28] E. F. Beall, B. Cork, D. Keefe, P. G. Murphy, and W. A. Wenzel, Phys. Rev. Letters **8**, 75 (1962).

[29] J. J. Sakurai and Y. Nambu, Phys. Rev. Letters **6**, 377 (1961); S. Barshay, Nuclear Phys. **13**, 435 (1959) and Phys. Rev. Letters **1**, 97 (1958). See, however, reference 32.

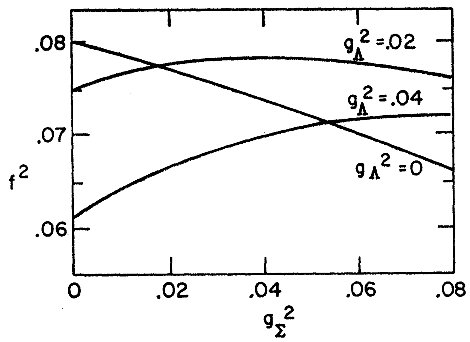

FIG. 3. The value of the coupling constant f^2 required for the $I=1$, $J=3/2$ resonance (case II) to be located at 1385 MeV as a function of g_Σ^2 for various values of g_Λ^2.

channels.[30]) For $g_\Lambda^2=2f_N^2$, Eq. (4.2) gives a half-width of 24 MeV, a value consisent with present experimental data.

With regard to case II, several authors have pointed out that the existence of an $I=1$, $J=3/2$ resonance, with $P(\Sigma)=+1$, does not depend upon the assumption of global symmetry and that it should exist even for $f_\Sigma\approx 0$.[3,6,31] Franklin[6] has emphasized that a small f_Σ would bring r into better agreement with experiment; however, it appears that, if only π-Y couplings are used, the resonance would lie about 3 μ above the $\pi\Lambda$ threshold rather than 1 μ as observed. We shall now see that, for $f_\Sigma=0$, moderate $\bar{K}$-N couplings will move the resonance into the correct energy region whereas, for global symmetry, moderate $\bar{K}$-N couplings will have a small effect on ω_r. In the following numerical results, it is always assumed that $f_\Lambda^2=f_N^2$ unless otherwise specifically stated.

First consider what happens in the global symmetry case. Equation (3.10) becomes a cubic equation as a result of including the $\bar{K}$-N interaction. The cubic terms and terms containing Δ are not small in the energy region of interest and should not be neglected. Amati *et al.*[3] obtained the Y_1^* at the observed energy by taking the π-Y cutoff integrals from the location of the (3,3) resonance. Figure 2 shows how the resonance energy depends on $g_\Sigma^2 I_3$ for several value of $g_\Lambda^2 I_3$. As another way of illustrating the effects of the $\bar{K}$-N coupling, Fig. 3 shows how the global symmetry coupling constant f^2 must vary as the $\bar{K}$-N couplings are increased in order to maintain $\omega_r=1385$ MeV. [In Fig. 3 all three cutoff integrals have been set equal to the (3,3) cutoff integral for simpler comparison with the previous work.] It is evident from these figures that the $\bar{K}$-N couplings must be quite large compared with the π-Y couplings to shift the resonance appreciably; the maximum shift shown in Fig. 2 is only about 40 MeV. In addition, these results illustrate the remark made earlier [cf. the paragraph following Eq. (3.12)]: if only one of the g's is nonzero, the resonance energy decreases as g increases; however, if both g_Λ and g_Σ are nonzero they may interfere and the resonance energy may increase as either or both are increased (though never to a value greater than its value for $g_\Lambda=g_\Sigma=0$).

Figure 4 illustrates how much more sensitive the resonance energy is to the $\bar{K}$-N coupling in the case $f_\Sigma=0$. [The main reason for this is that the only term in Eq. (3.10) linear in the energy is proportional to f_Σ^2. This makes the global symmetry case much more stable against variations of g_Λ and g_Σ.] Note that now the resonance energy always decreases as g_Λ, g_Σ are increased; this is because g_Σ is the only coupling constant connecting the $\pi\Sigma$ channel to the other two channels. It is clear that moderate $\bar{K}$-N couplings will give the resonance in the correct energy range. Without the $\bar{K}$-N interaction, a value of $f_\Lambda^2 I_1$ nearly twice as large as the (3,3) value is required to give the resonance energy correctly.

With regard to the branching ratio r, one would like to know (a) if $f_\Lambda\approx f_\Sigma$ will moderate $\bar{K}$-N couplings reduce r from the 0.11 global symmetry value to a value consistent with experiment or, (b) if $f_\Sigma\approx 0$ will the $\bar{K}$-N couplings increase r to a value larger than the experimental upper limit? (It should be noted that the widths depend on f^2, g^2 as well as f^2I_1, g^2I_3 so that the quantities compared are not the same as those compared in discussing the resonance energy. For definiteness, we shall assume $I_1=I_3$, keeping in mind that the values of the coupling constants given below must be modified accordingly if the equality does not hold.) In general, the nonresonant background is very small, mainly because

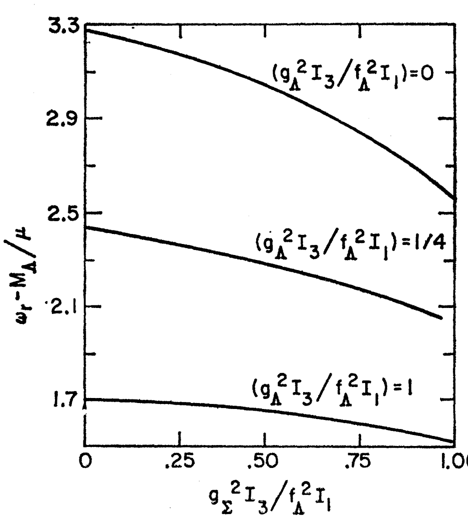

FIG. 4. The energy of the $I=1$, $J=3/2$ resonance for case II and $f_\Sigma=0$ as a function of $g_\Sigma^2 I_3/f_\Lambda^2 I_1$ for various values of $g_\Lambda^2 I_3/f_\Lambda^2 I_1$.

[30] R. C. Hwa and D. Feldman, Nuovo cimento 23, 914 (1962), seem to have obtained a more general result in a noncutoff model: the width depends only on the locations of the resonance and on the locations and residues of the Born terms. This is not true for our model; one can easily construct examples for the two-channel case with $I_1\neq I_2$ such that the result depends on I_1/I_2.

[31] G. Wentzel, Phys. Rev. 125, 771 (1962).

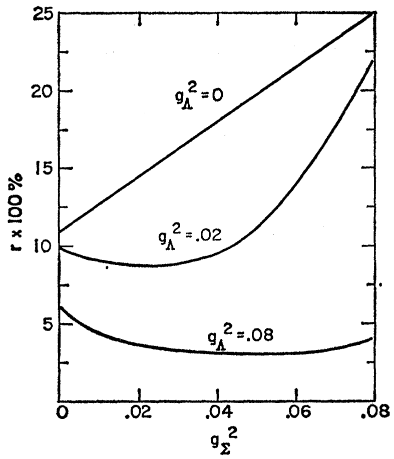

FIG. 5. The branching ratio r for the decay of the $I=1$, $J=3/2$ resonance (case II) for $f_\Sigma^2=f_\Lambda^2$ as a function of g_Σ^2 for various values of g_Λ^2.

of the low momentum in the π-Σ channel. Thus, we may very accurately take

$$r=|\Gamma_{12}(\omega_r)/\Gamma_{11}(\omega_r)|^2=|\Gamma_{22}(\omega_r)/\Gamma_{12}(\omega_r)|^2. \quad (4.3)$$

Values of r for $f_\Sigma=f_\Lambda$ are plotted in Fig. 5 as a function of g_Σ^2 for various values of g_Λ^2. Evidently, for strong $\bar{K}N\Lambda$ coupling, $g_\Lambda^2\approx f_\Lambda^2$, r may be reduced to less than 0.04 while for moderate coupling, $g_\Lambda^2\lesssim f_\Lambda^2/4$, the branching ratio is reduced to 0.08 at best.

Sakurai[32] has suggested that the small value of r may result from symmetry of the interaction under the operation R,[33]

$$R:\ p\leftrightarrow\Xi^-,\quad n\leftrightarrow\Xi^0,\quad \Sigma^+\leftrightarrow\Sigma^-,\quad \Sigma^0\leftrightarrow\Sigma^0,\quad \Lambda\leftrightarrow\Lambda,\quad K^+\leftrightarrow K^-,\quad K^0\leftrightarrow\bar{K}^0,\quad \pi^+\leftrightarrow\pi^-,\quad \pi^0\leftrightarrow\pi^0. \quad (4.4)$$

Rigorous invariance under R implies that r vanishes. Our model, with $f_\Sigma=g_\Sigma=g_\Lambda=0$, is invariant under R; with these couplings $r=0$. When the $\bar{K}NY$ couplings are introduced the symmetry is broken unless the ΞKY coupling is simultaneously included with the proper coupling constants. Even if it is included, the ΞN mass difference breaks the symmetry. Note, however, that the symmetry is unaffected by Δ and imposes no relation between g_Λ and g_Σ. For nonzero g_Λ, g_Σ the branching ratio is given by (neglecting Δ)

$$r=\left(\frac{q_2}{q_1}\right)^3\frac{32g_\Lambda^2g_\Sigma^2I^2(\omega_r-M_\Lambda)^2}{[1+2f_\Lambda^2I(\omega_r-M_\Lambda)+4g_\Lambda^2I(\omega_r-M_\Lambda)+8f_N^2(g_\Lambda^2-2g_\Sigma^2)I^2(\omega_r-M_\Lambda)^2]^2}. \quad (4.5)$$

for $f_\Sigma=0$. The equation resulting from Eq. (3.10) is actually cubic in the product $I(\omega_r-M_\Lambda)$ (assuming $I_1=I_2=I_3$), so for definite values of g_Λ, g_Σ the product $I(\omega_r-M_\Lambda)$ is fixed. Thus, I may be varied as g_Λ, g_Σ vary so as to maintain ω_r at 1385 MeV. If this is done in the calculation of r from Eq. (4.5), we obtain, for $g_\Lambda^2=g_\Sigma^2=f_\Lambda^2$ $(I=1.25)$, a branching ratio of about 8% and, for $g_\Lambda^2=g_\Sigma^2=f_\Lambda^2/4$ $(I=1.97)$, a branching ratio of about 1%. Thus, although the $R<$symmetry is drastically broken by inclusion of $\bar{K}NY$ couplings, if the coupling strength is less than about $f_\Lambda^2/4$ the predicted value of r remains quite small, well within the experimental limits.

It is interesting to see that moderate $\bar{K}$ couplings have a very small effect on the resonance widths. The dependence of the width on the $\bar{K}N$ coupling constants is illustrated in Fig. 6 for $f_\Lambda=f_\Sigma$. (The f^2I are adjusted according to Fig. 3 so that ω_r remains constant.) Even for $f_\Sigma=0$, the effect is quite small: in the absence of $\bar{K}$-N interactions the half-width at resonance is $(2/3)f_\Lambda^2q^3$, or about 24 MeV, whereas for $g_\Sigma=f_\Lambda$, $g_\Lambda=f_\Lambda/2$ the half-width is reduced only to about 20 MeV. (These numbers include Δ corrections.)

Thus, in the $I=1$, $J=3/2$ state, the effects of moderate $\bar{K}$-N couplings $g^2\lesssim f_\Lambda^2/4$) are unimportant except with regard to the location of the resonance for $f_\Sigma\approx0$. On the other hand, if the $\bar{K}$-N couplings are comparable to the π-Λ coupling strength, they will drastically affect any conclusions based on the π-Y interaction alone.

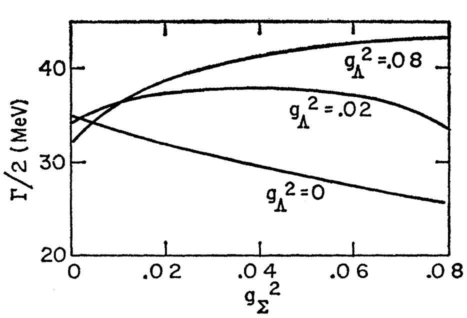

FIG. 6. The value of $\Gamma/2$ at resonance for the $I=1$, $J=3/2$ resonance (case II) for $f_\Sigma^2=f_\Lambda^2$ as a function of g_Σ^2 for various values of g_Λ^2.

V. THE $I=0$ RESONANCES

The experimental situation regarding the $I=0$ resonances is not yet completely clear. There is Y_0^* at 1405 MeV with half-width of about 10 MeV and uncertain spin, and Y_0^{**} at 1520 MeV with half-width about 8 MeV and spin 3/2. The simplest and most natural interpretation of the Y_0^{**} is that it is a $D_{3/2}$ resonance analogous to the second πN resonance. Tripp *et al.*[13] emphasize that any interpretation of it as a $P_{3/2}$ resonance would require very complex behavior of

[32] J. J. Sakurai, Phys. Rev. Letters 7, 426 (1961).
[33] M. Gell-Mann, California Institute of Technology Report CTSL-20, 1961 (unpublished).

the other partial waves in a wide energy range about the resonance. If it is indeed a $D_{3/2}$ resonance, some mechanism other than the one considered here must be responsible for its existence. Sakurai[32] has suggested a virtual $\rho\Sigma$ bound state as a possible mechanism; a Peierls type mechanism[34] or a Ball-Frazer mechanism[35] are also possible explanations. Since it is possible that the simplest interpretation of the Y_0^{**} is not the correct one, we shall consider here the possibility that it is a $P_{3/2}$ resonance and see how it fits into our model. It is in fact possible that neither of these two $I=0$ resonances result from the forces considered here. Several authors have pointed out that a Y_0^* resulting from a virtual bound $\bar{K}N$, $J=1/2$ state is consistent with the data.[36,37] Indeed, it is possible that the Y_0^* is not a true resonance at all. However, since so little is known about this resonance, it is desirable to consider other ways in which it may arise. From Table I, we see that there are a number of possibilities within this model.

Let us first consider the Y_0^{**}. From Table I, it is clear that such a resonance is possible only if $P(\Sigma)=+1$. We will continue to confine ourselves to the case $P(K)=-1$. [The $\bar{K}N$ couplings do not affect the Y_0^{**} for $P(K)=+1$; as a result, very large π couplings would be required for the resonance to have the correct energy.] In case II the only $\bar{K}$ coupling constant which enters is g_Σ. g_Σ may be expressed in terms of the other coupling constants and the resonance energy by Eq. (3.10) as

$$g_\Sigma^2 I_3 = \frac{1+2f_\Lambda^2 I_1 \Delta + 2I_1(2f_\Sigma^2 - f_\Lambda^2)(\omega_r - M_\Sigma)}{24 f_N^2 I_1 (\omega_r - M_\Sigma)^2}. \quad (5.1)$$

In order to get some idea of the magnitude of g_Σ required for $\omega_r = 1520$ MeV, take $f_N = f_\Lambda$, $I_1 = I_2$. Then for two extreme possibilities for f_Σ, Eq. (5.1) gives

(a) $f_\Sigma^2 = f_\Lambda^2$, $g_\Sigma^2 = 0.068$,

(b) $f_\Sigma^2 = 0$, $g_\Sigma^2 = 0.022$.

The cross sections $\sigma(\bar{K}N \to \bar{K}N)$ and $\sigma(\bar{K}N \to \pi\Sigma)$ corresponding to these two possibilities are plotted in Fig. 7. It is notable that these cross sections have very broad peaks; in fact, especially in case (b), they have little or no resemblance to resonance cross-sections. *A priori*, this can arise from a number of factors. First, the large momenta available, especially in the $\pi\Sigma$ channel, tend to make the width very large. Second, the fact that the momenta in the two channels are comparable means the nonresonant background may not be negligible. Finally, Eq. (3.7) indicates that the diagonal T-matrix element for the resonant channel does not go to zero at the $\bar{K}$-N threshold. This requires that $\sin^2\theta$ vanish at the $\bar{K}$-N threshold, according to Eq. (3.8). Thus, it may be expected that as the energy increases the $\bar{K}$-N cross sections contain relatively more and more of the resonant part, thus making the peak skew toward higher energies. These remarks are illustrated in

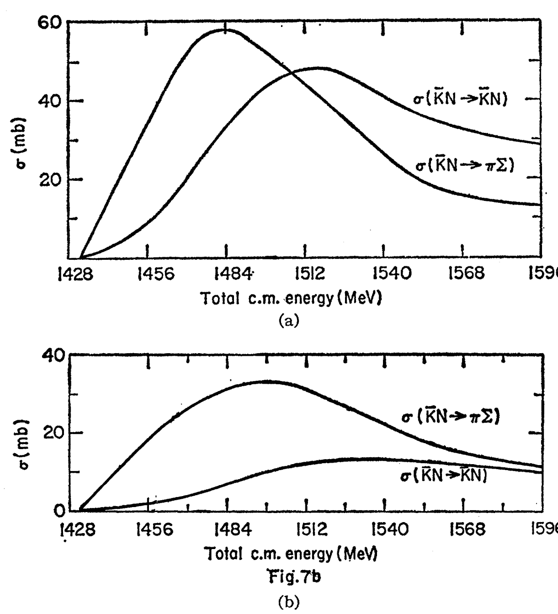

FIG. 7. The $I=0$, $J=3/2$ resonance cross section for case II, for elastic and inelastic scattering as a function of total center of mass energy: (a) $f_\Sigma^2=0.08$, $g_\Sigma^2=0.068$, (b) $f_\Sigma^2=0$, $g_\Sigma^2=0.022$.

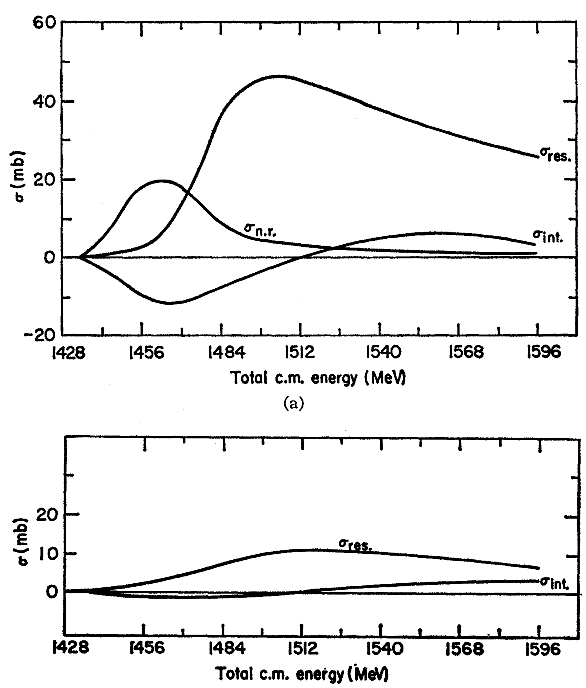

FIG. 8. The $I=0$, $J=3/2$ elastic cross section of Fig. 7 analyzed into resonant, nonresonant, and interference (between resonant and nonresonant) contributions.

[34] R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961); S. F. Tuan, Phys. Rev. **125**, 1761 (1962).

[35] J. Ball and W. Frazer, Phys. Rev. Letters **7**, 204 (1961).

[36] J. Franklin, R. C. King, and S. F. Tuan, Phys. Rev. **124**, 1995 (1961).

[37] M. Ross and G. Shaw, Bull. Am. Phys. Soc. **6**, 509 (1961).

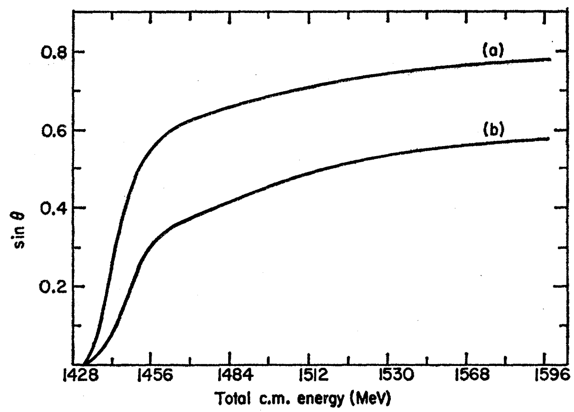

FIG. 9. The quantity $\sin\theta$ as a function of total center-of-mass energy for the $I=0$, $J=3/2$ resonance of Fig. 7.

Figs. 8 and 9. In Fig. 8 the elastic cross sections are analyzed into resonant, nonresonant, and interference parts. [For case (b) the nonresonant cross section is always less than about 1 mb and so it is not plotted.] Figure 9 shows the dependence of $\sin\theta$ on energy. It is clear from these figures that the primary reason for the distortion from a Breit-Wigner form is the large and rapidly increasing value of the width and not the mixing of the nonresonant part. Of course, the fact the width is so large is partly a result of the presence of two channels: In case (a) the $\pi\Sigma$ width contributes about half the value of the total width at resonance, while in case (b) it contributes about 0.7 the total width. In conclusion, this resonance has no resemblance to the observed one and it is very unlikely that the Y_0^{**} is generated by the forces considered here.

Turning now to the question of the Y_0^*, we ask first if the resonances given by Eq. (5.1) may be the Y_0^*. The minimum value of $g_\Sigma^2 I_2$ is obtained for $f_\Sigma=0$ and in general, for the resonance to be at 1405 MeV with the same values of f_Λ and I_1 as above, it is required that $g_\Sigma^2 I_2 \geq 0.104$ or, with $I_1=I_2$, $g_\Sigma^2 \geq 0.065$. Of course if either I_1 or I_2 is larger than this value the resonance could be obtained for smaller g_Σ^2. Franklin[6] has suggested that this resonance with $g_\Sigma=f_\Sigma=0$ is the Y_0^*. From Eq. (5.1) we see this requires $f_\Lambda^2 I_1=0.52$ or, for $f_\Lambda^2=0.08$, $I_1 \approx 6.5$. To get such a large value for I_1 requires a cutoff of about 65 μ. It is even more unlikely that the $J=1/2+$ resonance for case II could be the Y_0^*. To get a rough idea about the magnitude of g_Σ^2 required, we neglect Δ, set $I_1=I_2$ and $f_\Sigma=f_\Lambda$. (For $f_\Sigma<f_\Lambda$ the value of g_Σ^2 required is even larger.) Then

$$g_\Sigma^2=\frac{2.6}{2.9(1+x)^2-14.7x^2}, \tag{5.2}$$

where $x=g_\Lambda/g_\Sigma$. This shows that if x is too large, no value of g_Σ^2 will do and, at best, the value of g_Σ^2 required is enormous.

The strength of the $N\Sigma\bar{K}$ coupling required to give a $P_{1/2}Y_0^*$ in case IV are comparable to those required for a $P_{3/2}Y_0^*$ in case II. Equation (3.10) for the $P_{1/2}$ resonance may be written

$$g_\Sigma^2 I_2=\frac{1-2f_\Sigma^2(\omega_r-M_\Sigma)I_1}{18f_N^2 I_1(\omega_r-M_\Sigma)^2}. \tag{5.3}$$

For $f_\Sigma=f_N$, this gives $g_\Sigma^2 I_2=0.114$ and for $f_\Sigma=0$, it gives $g_\Sigma^2 I_2=0.187$. Note that this state is associated with the $S_{1/2}\,\bar{K}$-N channel. Thus, g_Σ is a scalar coupling constant and I_2 an s-wave integral, so in solving Eq. (5.3) the energy dependence of I_2 should be taken into account. Rather than do this, we simply note that the numbers given above require g_Σ of order unity. For $f_\Sigma=0$, the width can be expressed quite simply by

$$\Gamma_{11}/2=q_1^3/6I_1(\omega_r-M_\Sigma), \tag{5.4}$$

which gives a half-width of 12 MeV. This is slightly less than the 16 MeV width obtained for the $P_{3/2}Y_0^*$ in case II and slightly greater than the observed width.

VI. SUMMARY AND DISCUSSION

The main purpose of the preceding calculations is to see what features of the low energy πY, $\bar{K}N$ interaction can be understood on the basis of forces arising solely from the Born singularities. It is evident that this understanding is limited by the uncertainty about the various coupling constants and parities, in addition to the cutoff uncertainty inherent in this model. In particular, if the $\bar{K}$-N coupling constants are comparable to the π-N coupling constant, a large variety of interpretations of the observed resonances is possible. The various possibilities may be summarized as follows: [For $P(\Sigma)=+1$ it is always assumed that $f_\Lambda^2 \approx f_N^2$; in addition, by strong $\bar{K}$-N couplings, it is meant that $g^2 I_3 \approx f_N^2 I_1$ and, by moderate $\bar{K}$-N couplings, it is meant that $g^2 I_3 \lesssim f_N^2 I_1/4$. g^2 denotes g_Λ^2 or g_Σ^2.]

(a) *The Y_0^{**} resonant state.* It is almost certain that the observed Y_0^{**} depends on some other mechanism than the one considered here (which is, of course, necessary if it is confirmed to be a d-wave resonance). This model predicts no $J=3/2$, $I=0$ resonance for any values of the coupling constants if $P(\Sigma)=-1$. For $P(\Sigma)=+1$, any resonance in the region of the Y_0^{**} mass is found to be much broader than the observed resonance. Of course, it is possible that such a $P_{3/2}$ resonance exists and lies near the observed Y_0^{**} mass, but is so broad as not to be observed as a resonance. [See Fig. 7(b).]

(b) *The Y_1^* resonant state.* If $P(\Sigma)=-1$, an $I=1$, $P_{3/2}$ resonance is obtained in the correct energy range only if the $\bar{K}\Lambda N$ coupling is strong. The branching ratio r for the decay of this resonance is identically zero. If $P(\Sigma)=+1$ and $f_\Sigma^2 \approx f_\Lambda^2$ an $I=1$, $P_{3/2}$ resonance is obtained in the correct energy range for a wide variation of $\bar{K}$-N coupling strength. (See Fig. 2). However, r is predicted to be much larger than the observed Y_1^* decay branching ratio unless $g_\Lambda^2 \approx f_\Lambda^2$. If $P(\Sigma)=+1$ and $f_\Sigma \approx 0$, an $I=1$, $P_{3/2}$ resonance is obtained in the correct energy range for moderate $\bar{K}$-N couplings. The value of r predicted with these couplings is about 0.01, a value consistent with experiment. In addition, the resonance width obtained with these couplings is in better agreement with experiment than those obtained with $f_\Sigma^2 \approx f_\Lambda^2$.

(c) *The Y_0^* resonant state.* For both parity assumptions, strong $\bar{K}$-N couplings are required in order to obtain an $I=0$ resonant state in the vicinity of 1405 MeV. In the case that $P(\Sigma)=-1$, the resonance occurs in the $P_{1/2}$ state and for $P(\Sigma)=+1$, the resonance occurs in the $P_{3/2}$ state, provided the $\bar{K}\Sigma N$ coupling is strong. In both cases, the predicted width agrees moderately well with the experimental width.

Thus, we may conclude that if $P(\Sigma)=-1$, this mechanism may generate both the Y_1^* and Y_0^* only if the $\bar{K}N$ couplings are strong. If the $\bar{K}$N couplings are moderate, the only case which gives agreement with the location and branching ratio of the Y_1^* is $P(\Sigma)=+1$ and $f_\Sigma \approx 0$. In this case, the Y_0^* cannot be generated by this mechanism. This latter interpretation is consistent with the recent experimental results of Ferro-Luzzi *et al.*, which suggest that the $\bar{K}\Sigma$ relative parity is odd.[38]

It is usually stated that the $\bar{K}$ coupling constants are much smaller than the π coupling constants. The primary argument for this is based on the application of the Kroll-Ruderman Theorem to photoproduction of K mesons.[39] However, as Gell-Mann has pointed out,[39] this is not nearly so reliable as in π photoproduction because of the much greater distance to zero K meson mass and the additional complication of possible πN intermediate states. One might also consider the fact that there are no resonances in KN scattering as evidence for small KN coupling constants: In the static approximation the Born terms for the p-wave T' matrix are:

$$P(\Sigma)=+1\begin{cases} T_{3/2}'(I=1)=\frac{2}{3}q^3\left(\dfrac{g_\Lambda^2}{\omega+M_\Lambda}+\dfrac{g_\Sigma^2}{\omega+M_\Sigma}\right), \\ T_{3/2}'(I=0)=\frac{2}{3}q^3\left(-\dfrac{g_\Lambda^2}{\omega+M_\Lambda}+\dfrac{3g_\Sigma^2}{\omega+M_\Sigma}\right), \\ T_{1/2}'(I=1)=-\frac{1}{2}T_{3/2}'(I=1), \\ T_{1/2}'(I=0)=-\frac{1}{2}T_{3/2}'(I=0); \end{cases} \tag{6.1a}$$

$$P(\Sigma)=-1\begin{cases} T_{3/2}'(I=1)=\frac{2}{3}q^3\dfrac{g_\Lambda^2}{\omega+M_\Lambda}, \\ T_{3/2}'(I=0)=-\frac{2}{3}q^3\dfrac{g_\Lambda^2}{\omega+M_\Lambda}, \\ T_{1/2}'(I=1)=-\frac{1}{2}T_{3/2}'(I=1), \\ T_{1/2}'(I=0)=-\frac{1}{2}T_{3/2}'(I=0). \end{cases} \tag{6.1b}$$

It is clear that, for either $P(\Sigma)=\pm 1$, no matter what values the coupling constants assume, some of the Born terms are attractive and can lead to a resonance. For example, for $P(\Sigma)=-1$, the resonance condition, Eq. (3.10) is for the $P_{3/2}$, $I=1$ state,

$$1-2(\omega_r+M_\Lambda)g_\Lambda^2 I_{KN}=0. \tag{6.2}$$

This gives a resonance energy of

$$\omega_r=\frac{1}{2g_\Lambda^2 I_{KN}}-M_\Lambda. \tag{6.3}$$

Thus, if $g_\Lambda^2 I_{KN} \approx f_N^2 I_{(3,3)}$ a low energy resonance is predicted in K^+p scattering which is not observed. This sort of evidence for small g_Λ, g_Σ is limited by the ambiguity of the cutoff integrals. If $I_{KN} \ll I_{(3,3)}$, g_Λ and g_Σ can be quite large without contradicting the K^+p scattering data. It may be that, for example, the inclusion of the π-π interaction in the momentum transfer channel would suppress the KN integrals and enhance the $\bar{K}N$ integrals. In fact, Dalitz has shown how the ρ and ω exchange may contribute potentials of opposite sign to KN and $\bar{K}N$ systems.[1] Then g_Λ, g_Σ could have moderate values while appearing weak in KN interactions and strong in $\bar{K}N$ interactions.

Further suggestions regarding the magnitude of the coupling constants may be gotten by looking at some of the states which have not been discussed in the preceding sections. The $I=2$ states depend only on f_Σ and f_Λ. Equation (3.10) for the $P_{3/2}$ state is

$$\begin{aligned} P(\Sigma)=+1:\quad & 1-2(f_\Lambda^2+f_\Sigma^2)I(\omega-M_\Lambda) \\ & \qquad +2(f_\Lambda^2+f_\Sigma^2)I\Delta=0; \\ P(\Sigma)=-1:\quad & 1-2f_\Sigma^2 I(\omega-M_\Lambda)+2f_\Sigma^2 I\Delta=0. \end{aligned} \tag{6.4}$$

For $P(\Sigma)=+1$ and $f_\Sigma^2=f_\Lambda^2$ this resonance lies at a rather low energy, about 160 MeV above the Y_1^*; if $P(\Sigma)=+1$, $f_\Sigma \approx 0$ or if $P(\Sigma)=-1$ and $f_\Sigma^2 I \lesssim f_N^2 I_{(3,3)}$ the resonance energy is considerably higher, more than 3.5 μ above the $\pi\Lambda$ threshold. Thus, a $P_{3/2}$, $I=2$ resonance in the low energy region is indicative of $P(\Sigma)=+1$, $f_\Sigma^2 \approx f_\Lambda^2$. The only other possible $I=2$ resonance occurs in the $S_{1/2}$ state for $P(\Sigma)=-1$. For this case Eq. (3.10) is

$$1-3f_\Lambda^2 I(\omega-M_\Sigma-\Delta)=0. \tag{6.5}$$

Note that here f_Λ is a scalar coupling constant and I an s-wave cutoff integral. A reasonable estimate for

[38] R. Tripp, M. Watson, and M. Ferro-Luzzi, Phys. Rev. Letters 8, 175 (1962).

[39] M. Gell-Mann, 1958 *Annual International Conference on High-Energy Physics at CERN* (CERN Scientific Information Service, Geneva, 1958).

$f_\Lambda{}^2 I$ is 0.2 for strong $\Lambda\Sigma\pi$ coupling; with this value a resonance is obtained at about $1.2\,\mu$ above the $\pi\Sigma$ threshold; however, the half-width of this resonance is given by $qf_\Lambda{}^2$ which is enormous at this energy. Thus, should this resonance exist, it would be very difficult to detect. It is clear that the existence of an $I=2$, $J=3/2$ resonance in the low-energy region suggests that $P(\Sigma)=+1$ and $f_\Sigma{}^2\approx f_\Lambda{}^2$; on the other hand, if no $I=2$ resonance is observed, this fact by itself would be indicative of very little.

The remaining states which have not been discussed are the $J=1/2$, $I=1$ states. These states are interesting for two reasons: (a) the existence or nonexistence of resonances in these states may give some information regarding the magnitude of the $\bar{K}$-N couplings and (b) it is still possible that the $Y_1{}^*$ has spin 1/2. With regard to the first question, if the $\bar{K}N$ couplings are moderate the possible resonances in these states all lie more than $3\,\mu$ above the $\pi\Lambda$ threshold. Even if this model could be trusted at such high energies, the resonances would be even broader than the one shown in Fig. 7(b), and hence difficult to observe. On the other hand, if the $\bar{K}N$ couplings are strong, these resonances may move into the low-energy region. Specifically, for $P(\Sigma)=-1$ there is a $J=1/2-$ and, if $f_\Sigma\approx f_N$, a $J=1/2+$, $I=1$ resonance for strong $\bar{K}\Sigma N$ couplings; for $P(\Sigma)=+1$ there is a $J=1/2+$, $I=1$ resonance if both $\bar{K}N$ couplings are strong. If we accept one of the $J=3/2$ interpretations of the $Y_1{}^*$, the fact that no additional $I=1$ resonances are observed in the low energy region indicates that (a) if $P(\Sigma)=-1$ the $\bar{K}\Sigma N$ coupling is not strong or (b) if $P(\Sigma)=+1$ both $\bar{K}N$ couplings cannot be strong.

Turning now to the possibility that the $Y_1{}^*$ has spin 1/2, we see that $P(\Sigma)=+1$ is very unlikely. In that case, large $\bar{K}N$ couplings are required; however, regardless of the value of f_Σ there is also a $J=3/2$, $I=1$ resonance in the low energy region if the $\bar{K}N$ couplings are large. Since only one resonance is observed, this possibility seems remote. If $P(\Sigma)=-1$, it is necessary that the $\bar{K}\Sigma N$ coupling be strong and that the $\bar{K}\Lambda N$ coupling be moderate or weak to give a $J=1/2$ but no $J=3/2$, $I=1$ resonance. With $g_\Lambda\approx 0$, the $J=1/2+$ resonance can be immediately ruled out since it is predominantly a $\pi\Sigma$ resonance. For example, if $f_\Sigma=0$, the $\pi\Lambda$ channel is completely decoupled from the resonant state and hence the branching ratio r is infinite, while if $f_\Sigma\approx -f_N\approx\pm f_\Lambda$ r is about 3. The $J=1/2-$ resonance is quite a good possibility; this resonance results primarily from the coupling between the $\pi\Lambda$ and $\bar{K}N$ channels and so r should be quite small. In fact, $r=0$ if $f_\Sigma=0$, regardless of the value of g_Λ or g_Σ. Thus, if it turns out that the $Y_1{}^*$ has spin 1/2 the only possibility within this model is that $P(\Sigma)=-1$ with strong $\bar{K}\Sigma N$ coupling.

A next step in improving these calculations would be to include effects of other nearby singularities, those arising from crossing and from π-π and π-K interactions. If the properties of the resonant states were well known, the crossed terms could be included by assuming the resonant contributions to the crossed terms dominate and neglecting all others. Bosco *et al.*[12] have used this procedure in calculating the crossing effects in π-N scattering. (They find that the cutoff integral must be reduced by 20% from its no-crossing value to give the resonance energy correctly.)

The π-π cut, arising from the reaction $K+\bar{K}\rightarrow N+\bar{N}$ in the momentum transfer channel, lies very close to the physical region[2]; this suggests that the π-π interaction may have a stronger effect in this problem than in the π-N problem. The contribution of this cut as well as of the πK cut, from $\pi+K\rightarrow N+\bar{Y}$, may be estimated by assuming the dominance of the ρ, ω, and K^* resonant states. This may all be done in a completely relativistic way, by modifying the technique of Frautschi and Walecka.[40] However, the results of Frautschi and Walecka show that the unknown short-range forces are very important in determining the resonance energies; thus, it cannot be expected that trustworthy quantitative results can be obtained in this way. It is evident that no matter how the extension is made, the results must contain some parameters which characterize our ignorance of the short range forces. The spirit of the present calculation is the assumption that *all* but the Born singularities can be represented by a simple cutoff procedure. *A priori*, there is no justification for assuming that all cutoff integrals have the same value. Certainly, if these other nearby singularities make important contributions to the low-energy scattering, this assumption is not valid. Inclusion of the above mentioned singularities will remove the uncertainty of their contributions and may have important quantitative effects. However, there will still be ambiguities analogous to our cutoff due to the still more distant neglected singularities. Another factor which may be important here is the fact that the $\pi\pi\Lambda$ threshold lies in the energy region of interest. This process may have a more substantial effect on the results than π production has in π-N scattering. An indication of this is that the $\pi\pi\Lambda$ channel accounts for about 10% of the $Y_0{}^{**}$ decays. An estimate of this effect can perhaps be gotten by a static calculation, using techniques similar to those applied to $\pi+N\rightarrow\pi+\pi+N$ by Carruthers.[41]

A final comment on the resonance widths: the values that have been quoted for half widths are always the value of $\Gamma/2$ at resonance. There is some suggestion from π-N scattering that this may not be an accurate measure of the width, that the half width at half maximum is less than $\Gamma/2$ at resonance. It is well known that the (3,3) cross section drops off faster above the resonance than is predicted by the Chew-Low plot. An expression which fits the cross section very well throughout the resonance

[40] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[41] P. Carruthers, Ann. Phys. (New York) **14**, 229 (1961).

region is given by Gell-Mann and Watson[42]:

$$\sigma=\frac{2\pi}{q^2}\frac{\Gamma^2}{(\omega-\omega_r)^2+(\Gamma/2)^2},\qquad \Gamma=\frac{2(qa)^3}{1+(qa)^2}\gamma^2, \tag{6.6}$$

with $a=0.88\,\mu^{-1}$, $\gamma^2=58$ MeV and $\omega_r=299$ MeV. Roughly, the energy dependence of Γ in Eq. (6.6) differs from the dependence of the Chew-Low Γ by a factor of $[1+(qa)^2]^{-1/2}$. The half-width of the resonance given by Eq. (6.6) is about 20% smaller than the value of $\Gamma/2$ at resonance given by the Chew-Low theory. These remarks are especially relevant to the case of the Y_1^* where, for $P(\Sigma)=+1$ and $f_\Sigma\approx f_\Lambda$, the predicted width is generally greater than the experimental widths. (See Fig. 6.) It might be expected that a similar reduction should be applied to these predicted values. Unfortunately, it is not clear how one can incorporate a barrier penetration factor, a factor analogous to the $(1+q^2a^2)^{-1}$ in Eq. (6.6), into the sort of calculation done here. (There is no such factor in the relativistic treatment of Frautschi and Walecka[40]; in fact, their calculation predicts the phase shift goes through $\pi/2$ even more slowly than does the Chew-Low theory.) In the limit of exact global symmetry (the only case where the π-N parameters are directly related to the πY parameters) one can take over the values of a and γ^2 directly to the πY case. However, the Λ-N, Σ-N mass differences make this procedure ambiguous. If one assumes that the only modification necessary is that the momenta appropriate to the $\pi\Lambda$ and $\pi\Sigma$ channels at the observed Y_1^* resonance energy replace the (3,3) pion momentum then one obtains, for $f_N=f_\Sigma=f_\Lambda$, $g_\Lambda=g_\Sigma=0$,

$$\Gamma_\Sigma/\Gamma_N=\tfrac{1}{3}(q_\Sigma^3/q_N^3)[(1+q_N^2a^2)/(1+q_\Sigma^2a^2)]=0.11,\qquad \Gamma_\Lambda/\Gamma_N=\tfrac{2}{3}(q_\Lambda^3/q_N^3)[(1+q_N^2a^2)/(1+q_\Sigma^2a^2)]=0.57, \tag{6.7}$$

Then the half-width of the Y_1^* resonance is $0.68\Gamma_N/2=34$ MeV for $\Gamma_N/2=50$ MeV, which is about the same value given in Fig. 6 for the global symmetry case. Fulthermore, the branching ratio is $\Gamma_\Sigma/\Gamma_\Lambda\approx0.2$, a value in violent disagreement with experiment. The obvious reason for this is that the added barrier penetration factor suppresses the $\pi\Lambda$ channel considerably more than it does the $\pi\Sigma$ channel. Thus, it is clear that, while the π-Y resonances may be better fitted on the high energy side by an expression similar to Eq. (6.6) and hence be narrower than perdicted by the model, the parameters a and γ^2 cannot be simply carried over from the (3,3) resonance.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor R. H. Dalitz, who suggested this problem, for many suggestions and critical discussions during the course of this work.

[42] M. Gell-Mann and K. Watson, Ann. Rev. Nuclear Sci. 4, 219 (1954).

APPENDIX A

Physical Meaning of $X(q'',q)$

In references 11 and 12, the meaning of $X(\mathbf{q}'',\mathbf{q})$ for π-N scattering is determined by transforming the solution to configuration space and comparing with the usual phase shift expansion. That proof can be carried over to the many channel case. Another way to look at the problem is to consider the asymptotic behavior in time. One component of the vector given in Eq. (2.3) may be written, in the angular momentum representation,

$$|\psi_\alpha{}^J(\omega)\rangle_+'=\sum_\gamma\int d\omega'\left\{|\psi_\gamma{}^J(\omega')\rangle_+ +\sum_\beta\int d\omega''\frac{|\psi_\beta{}^J(\omega'')\rangle_{++}\langle\psi_\beta{}^J(\omega'')|V_\gamma{}^J(\omega')|B_\gamma\rangle}{\omega''-\omega-i\epsilon}\right\}\times X_{\gamma\alpha}{}^J(\omega',\omega), \tag{A1.1}$$

where the prime indicates that this is the trial state vector. Require that

$$\lim_{t\to\infty}e^{-iHt}|\psi_\alpha{}^J(\omega)\rangle_+'=\lim_{t\to\infty}e^{-i\omega t}|\psi_\alpha{}^J(\omega)\rangle_+. \tag{A1.2}$$

This implies that $X_{\gamma\alpha}(\omega',\omega)$ has the form (suppressing J)

$$X_{\gamma\alpha}(\omega',\omega)=\delta(\omega'-\omega)\delta_{\gamma\alpha}+\frac{F_{\gamma\alpha}(\omega',\omega)}{\omega'-\omega-i\epsilon}, \tag{A1.3}$$

since

$$\lim_{t\to\infty}\frac{e^{ixt}}{x+i\epsilon}=0.$$

Require in addition that

$$\lim_{t\to-\infty}e^{-iHt}|\psi_\alpha{}^J(\omega)\rangle_+'=\lim_{t\to-\infty}e^{-i\omega t}|\psi_\alpha{}^J(\omega)\rangle_+. \tag{A1.4}$$

This requires that

$$\sum_\gamma[\delta_{\gamma\alpha}+2\pi iF_{\gamma\alpha}(\omega,\omega)]\times[\delta_{\gamma\beta}+2\pi i\,{}_+\langle\psi_\beta(\omega)|V_\gamma(\omega)|B_\gamma\rangle]=\delta_{\gamma\beta}. \tag{A1.5}$$

As in the Chew-Low theory

$$\pi_+\langle\psi_\beta(\omega)|V_\gamma(\omega)|B_\gamma\rangle=-(T'^\dagger)_{\beta\gamma}, \tag{A1.6}$$

so Eq. (A1.5) may be written in matrix form:

$$[1-2iT'^\dagger(\omega)][1+2\pi iF(\omega,\omega)]=1, \tag{A1.7}$$

and, hence,

$$\pi F(\omega,\omega)=T'(\omega). \tag{A1.8}$$

Bosco *et al.*[14] have shown that the solution of Eq. (2.7) for the π-N problem is an exact solution of Eq. (2.6) by showing that it satisfies a dispersion relation deduced from Eq. (2.6). Rewriting Eq. (2.4) in the form Eq. (A1.1) makes it clear that this is true for the multichannel case as well: one is expanding the trial state in terms of exact states; thus, if a solution of Eq. (2.6) exists at all, no matter how the $X(\omega',\omega)$ are constructed so that the boundary conditions Eq. (A1.2) and Eq. (A1.4) are satisfied, all physical quantities must be the same as if Eq. (2.6) were solved exactly. This can be made explicit: Let the solution of Eq. (2.6) be written as in Eq. (A1.3) and denote the corresponding solution obtained in any other way by a superscript t. Define the matrix

$$\phi(\omega',\omega)=F(\omega',\omega)-F_t(\omega',\omega). \qquad \text{(A1.9)}$$

It follows that

$$|\psi(\omega)\rangle_+-|\psi(\omega)\rangle_+'=\int d\omega'\left\{|\psi(\omega')\rangle_+-\frac{1}{\pi}\int d\omega''\times\frac{|\psi(\omega'')\rangle_+T_{\omega''}'^{\dagger}(\omega)}{\omega''-\omega'-i\epsilon}\right\}\frac{\phi(\omega',\omega)}{\omega'-\omega-i\epsilon}, \qquad \text{(A1.10)}$$

where $T_{\omega'}'(\omega)=-\pi\langle B|V(\omega)|\psi(\omega')\rangle_+$ is a matrix in channel space. Equation (A1.4) then implies

$$[1-2iT'^{\dagger}(\omega)]\phi(\omega,\omega)=S^{\dagger}(\omega)\phi(\omega,\omega)=0. \qquad \text{(A1.11)}$$

Thus, since the S matrix does not have a pole at the physical energy ω, $\phi(\omega,\omega)=0$ and the solutions are the same.

Of course, neither Eq. (2.6) nor Eq. (2.7) may be solved exactly. The usual method used, and the one adopted here, is to make the one-meson, no crossing approximation. This may be done by utilizing the commutation relations of the meson operators with the Hamiltonian as done in references 3, 11, and 12, or by using the expression

$$|\psi_\alpha(\omega)\rangle_+=|\phi_\alpha(\omega)\rangle-\frac{1}{H-\omega-i\epsilon}V_\alpha(\omega)|B_\alpha\rangle. \qquad \text{(A1.12)}$$

This is, in fact, the expression used to obtain Eq. (A1.1) from Eq. (2.4). The reduction to the one-meson terms is then straightforward: Substitute Eq. (A1.1) in to Eq. (2.7) and note that

$$\langle\phi_\beta(\omega')|\psi_\alpha(\omega)\rangle_+=\delta_{\alpha\beta}\delta(\omega-\omega')+\frac{1}{\omega'-\omega+i\epsilon}\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_\alpha(\omega)\rangle_+. \qquad \text{(A1.13)}$$

Expand the matrix elements $\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_\alpha(\omega)\rangle_+$ over all intermediate states S:

$$\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_\alpha(\omega)\rangle_+=\sum_S\left\{\frac{\langle B_\beta|V_\alpha(\omega)|\psi_S\rangle_+{}_+\langle\psi_S|V_\beta^{\dagger}(\omega')|B_\alpha\rangle}{M_\alpha+M_\beta-\omega_S-\omega}-\frac{\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_S\rangle_+{}_+\langle\psi_S|V_\alpha(\omega)|B_\alpha\rangle}{\omega_S-\omega-i\epsilon}\right\}. \qquad \text{(A1.14)}$$

The "one-meson" approximation consists of keeping only those states S which contain a baryon or one meson plus a baryon. If, in addition, the states containing one meson plus a baryon are dropped from the first term in brackets *only*, we have the "one-meson, no-crossing" approximation:

$$\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_\alpha(\omega)\rangle_+\approx\sum_\gamma\left\{\frac{\langle B_\beta|V_\alpha(\omega)|B_\gamma\rangle\langle B_\gamma|V_\beta^{\dagger}(\omega')|B_\alpha\rangle}{M_\alpha+M_\beta-M_\gamma-\omega}-\frac{\langle B_\beta|V_\beta^{\dagger}(\omega')|B_\gamma\rangle\langle B_\gamma|V_\alpha(\omega)|B_\alpha\rangle}{M_\gamma-\omega}\right\}-\sum_\gamma\int d\omega''\frac{\langle B_\beta|V_\beta^{\dagger}(\omega')|\psi_\gamma(\omega'')\rangle_+{}_+\langle\psi_\gamma(\omega'')|V_\alpha(\omega)|B_\alpha\rangle}{\omega''-\omega-i\epsilon}. \qquad \text{(A1.15)}$$

When this approximation is made for all matrix elements entering into Eq. (2.7) the integral equations, Eq. (2.8), result.

Nonrelativistic Born Terms

In order to see what extent the static Born terms approximate the relativistic Born terms, it is most convenient to use the conventional notation of dispersion theory. (See, for example, reference 18.) Nauenberg[2] has shown that the scattering amplitude can be written, when the initial and final pairs of particles have the same intrinsic parities,

$$f_{fi}^{l\pm}=\frac{(q_iq_f)^{1/2}}{8\pi W}\left\{(E_i+M_i)^{1/2}(E_f+M_f)^{1/2}\times\left[A^l+\left(W-\frac{M_i+M_f}{2}\right)B^l\right]-(E_i-M_i)^{1/2}(E_f-M_f)^{1/2}\times\left[A^{l\pm1}-\left(W+\frac{M_i+M_f}{2}\right)B^{l\pm1}\right]\right\}. \qquad \text{(A2.1)}$$

Now the total energy is denoted by W while ω is reserved for the meson energy. f_{fi} is equivalent to T_{fi}' in our former notation. Processes of the sort shown in Fig. 1(a) have only a simple pole and the reduction to the nonrelativistic limit and comparison with the static model is rather obvious. The diagrams in Fig. 1(b) give rise to a variety of cuts and the comparison with the static model is less clear. For Fig. 1(a), neglecting isotopic spin complications,

$$B=\frac{4\pi G_iG_f}{W^2-M_\gamma^2},\qquad A=[\tfrac{1}{2}(M_i+M_f)\mp M_\gamma]B; \tag{A2.2}$$

the upper (lower) sign is taken if both couplings are pseudoscalar (scalar). G_i, G_f are the coupling constants at the initial and final vertices. Restricting the energy to the range for which $q^2/\omega M\ll 1$ in all channels, the only non-zero amplitudes arising from these diagrams are

$$f_{fi}{}^{S_{1/2}}\approx\frac{G_iG_f(q_iq_f)^{1/2}}{(1+\omega_i/M_i)^{1/2}(1+\omega_f/M_f)^{1/2}}\frac{W\mp M_\gamma}{W^2-M_\gamma^2},$$
$$f_{fi}{}^{P_{1/2}}\approx\frac{G_iG_f(q_i{}^3q_f{}^3)^{1/2}}{(1+\omega_i/M_i)^{1/2}(1+\omega_f/M_f)^{1/2}}\times\frac{1}{4M_iM_f}\frac{W\pm M_\gamma}{W^2-M_\gamma^2}. \tag{A2.3}$$

If the terms with the pole at $W=-M$ are neglected, these differ from the corresponding static amplitudes only by the phase space correction factors

$$1/(1+\omega_i/M_i)^{1/2}(1+\omega_f/M_f)^{1/2}.$$

(Of course, the correct relation between W and q is used here, and this differs from the static model also.)

Let us take a particular reaction to illustrate what happens in the case of Fig. 1(b): $\pi+\Lambda\to\pi+\Sigma$ with a Σ intermediate state. Then[2]

$$B=-\sqrt{2}\frac{4\pi G_\Sigma G_\Lambda}{\bar{S}-M_\Sigma{}^2},\qquad A=\tfrac{1}{2}\Delta B, \tag{A2.4a}$$

and

$$B_l=\frac{4\pi\sqrt{2}G_\Sigma G_\Lambda}{2q_iq_f}Q_l(a). \tag{A2.4b}$$

The quantity a is very large for small momenta,

$$a=\frac{M_\Sigma{}^2-(E_f-\omega_i)^2+q_i{}^2+q_f{}^2}{2q_iq_f}\approx\frac{M_\Sigma\omega_i{}^*(1-\omega_i/2M_\Sigma)}{q_iq_f}, \tag{A2.5}$$

where $\omega_i{}^*=\omega_i+q_i{}^2/2M_\Lambda$ and $Q_l(a)$ may be expanded in powers of $1/a$. The ones of interest to us are

$$Q_0(a)=1/a+1/3a^3+\cdots,\quad Q_1(a)=-1/3a^3-1/5a^4+\cdots,\quad Q_2(a)=-2/15a^3+\cdots. \tag{A.26}$$

Then to lowest order in $q^2/M\omega$,

$$f_{fi}{}^{S_{1/2}}\approx\frac{\sqrt{2}}{2}\frac{G_\Sigma G_\Lambda}{M_\Sigma M_\Lambda}\frac{(q_iq_f)^{1/2}}{(1+\omega_i{}^*/M_\Lambda)(1-\omega_i{}^*/2M_\Sigma)},$$
$$f_{fi}{}^{P_{1/2}}\approx\frac{\sqrt{2}}{3}\frac{G_\Sigma G_\Lambda}{4M_\Sigma M_\Lambda}\frac{(q_i{}^3q_f{}^3)^{1/2}}{\omega_i{}^*},$$
$$f_{fi}{}^{P_{3/2}}\approx-\frac{2\sqrt{2}}{3}\frac{G_\Sigma G_\Lambda}{4M_\Sigma M_\Lambda}\frac{(q_i{}^3q_f{}^3)^{1/2}}{\omega_i{}^*}.$$

Factors of the form $[1-(\omega_i\Delta/M_\Sigma M_\Lambda)+(\omega_i{}^2/4M_\Sigma{}^2)]$ have been neglected. Thus, just as found in reference 18, the phase space correction factors are canceled out in the $P_{3/2}$ amplitude. Diagrams involving $\bar{K}$-N reduce in the same way. Since the expansion is in powers of $q^2/\omega M$, the large $\bar{K}$ mass does not affect this. For all diagrams of the type shown in Fig. 1(b), one can always arrange it so that the expansion of factors of the form $(1+\omega/M)^{-1}$ is for ω corresponding to a pion energy. This is true because there are no $\bar{K}N\to\bar{K}N$ crossed poles.

APPENDIX B

The following are expressions for

$$G=[1-I\textstyle\sum_\mu B^\mu(\omega-M_\mu)]$$

for cases II and IV. All relevant quantities can be calculated from these by using Eqs. (2.23), (3.10), and (3.11). (The angular momentum indices on the cutoff integrals are suppressed.)

Case II: $P(\Sigma)=+1$, $P(K)=-1$;

$I=2$, $J=3/2+$:

$$G=1-2(f_\Lambda{}^2+f_\Sigma{}^2)I(\omega-M_\Lambda)+2(2f_\Lambda{}^2+f_\Sigma{}^2)I\Delta;$$

$I=2$, $J=1/2+$:

$$G=1+(f_\Lambda{}^2+f_\Sigma{}^2)I(\omega-M_\Lambda)-(2f_\Lambda{}^2+f_\Sigma{}^2)I\Delta;$$

$I=1$, $J=3/2+$:

$$G_{11}=1-2I_1f_\Lambda{}^2(\omega-M_\Lambda+\Delta),$$
$$G_{12}=2\sqrt{2}f_\Lambda f_\Sigma I_1(\omega-M_\Lambda),$$
$$G_{13}=2\sqrt{2}f_Ng_\Lambda I_1(\omega-M_\Lambda),$$
$$G_{21}=2\sqrt{2}f_\Lambda f_\Sigma I_2(\omega-M_\Lambda),$$
$$G_{22}=1+I_2[2f_\Lambda{}^2(\omega-\Delta-M_\Sigma)-2f_\Sigma{}^2(\omega-M_\Sigma)],$$
$$G_{23}=4g_\Sigma f_NI_2(\omega-M_\Sigma),$$
$$G_{31}=2\sqrt{2}f_Ng_\Lambda I_3(\omega-M_\Lambda),$$
$$G_{32}=4g_\Sigma f_NI_3(\omega-M_\Sigma),$$
$$G_{33}=1;$$

$I=1,\ J=1/2+$:

$$G_{11}=1+I_1f_\Lambda^2[(\omega-M_\Lambda+\Delta)+3(\omega-M_\Sigma)],$$
$$G_{12}=2\sqrt{2}f_\Lambda f_\Sigma I_1(\omega-M_\Sigma-\tfrac{1}{2}\Delta),$$
$$G_{13}=-\sqrt{2}I_1[f_Ng_\Lambda(\omega-M_\Lambda)+3f_\Lambda g_\Sigma(\omega-M_\Sigma)],$$
$$G_{21}=2\sqrt{2}f_\Lambda f_\Sigma I_2(\omega-M_\Sigma-\tfrac{1}{2}\Delta),$$
$$G_{22}=1+[7f_\Sigma^2(\omega-M_\Sigma)-f_\Lambda^2(\omega-\Delta-M_\Sigma)]I_2,$$
$$G_{23}=-2(f_Ng_\Sigma+3f_\Sigma g_\Sigma)(\omega-M_\Sigma)I_2,$$
$$G_{31}=-\sqrt{2}[f_Ng_\Lambda(\omega-M_\Lambda)+3f_\Lambda g_\Sigma(\omega-M_\Sigma)]I_3,$$
$$G_{32}=-2(f_Ng_\Sigma+3f_\Sigma g_\Sigma)(\omega-M_\Sigma)I_3,$$
$$G_{33}=1+6g_\Sigma^2I_3(\omega-M_\Sigma);$$

$I=0,\ J=3/2+$:

$$G_{11}=1+[4f_\Sigma^2(\omega-M_\Sigma)-2f_\Lambda^2(\omega-M_\Sigma-\Delta)]I_1,$$
$$G_{12}=-2(6)^{1/2}g_\Sigma f_N(\omega-M_\Sigma)I_1,$$
$$G_{21}=-2(6)^{1/2}g_\Sigma f_N(\omega-M_\Sigma)I_2,$$
$$G_{22}=1;$$

$I=0,\ J=1/2+$:

$$G_{11}=1+[9f_\Lambda^2(\omega-M_\Lambda)+f_\Lambda^2(\omega-M_\Sigma-\Delta)-2f_\Sigma^2(\omega-M_\Sigma)]I_1,$$
$$G_{12}=(6)^{1/2}I_1[3f_\Lambda g_\Lambda(\omega-M_\Lambda)+g_\Sigma f_N(\omega-M_\Sigma)],$$
$$G_{12}=(6)^{1/2}I_1[3f_\Lambda g_\Lambda(\omega-M_\Lambda)+g_\Sigma f_N(\omega-M_\Sigma)],$$
$$G_{21}=(6)^{1/2}I_2[3f_\Lambda g_\Lambda(\omega-M_\Lambda)+g_\Sigma f_N(\omega-M_\Sigma)],$$
$$G_{22}=1+6g_\Lambda^2(\omega-M_\Lambda)I_2;$$

Case IV: $P(\Sigma)=-1,\ P(K)=-1$;

$I=2,\ J=3/2+$:

$$G=1,$$

$I=2,\ J=3/2-$:

$$G=1-2f_\Sigma^2I(\omega-M_\Sigma),$$

$I=2,\ J=1/2+$:

$$G=1-3f_\Lambda^2(\omega-M_\Sigma-\Delta)I,$$

$I=2,\ J=1/2-$:

$$G=1+f_\Sigma^2(\omega-M_\Sigma)I;$$

$I=1,\ J=3/2+$:

$$G_{11}=G_{22}=G_{33}=1,$$
$$G_{12}=G_{21}=G_{32}=G_{23}=0,$$
$$G_{13}=2\sqrt{2}f_Ng_\Lambda(\omega-M_\Lambda)I_1,$$
$$G_{31}=2\sqrt{2}f_Ng_\Lambda(\omega-M_\Lambda)I_3;$$

$I=1,\ J=3/2-$:

$$G_{22}=1-2f_\Sigma^2I(\omega-M_\Sigma),$$
$$G_{11}=G_{33}=1,$$
$$G_{12}=G_{21}=G_{13}=G_{31}=G_{23}=G_{32}=0;$$

$I=1,\ J=1/2+$:

$$G_{11}=1,$$
$$G_{12}=-(6)^{1/2}f_\Lambda f_\Sigma(\omega-M_\Lambda)I_1,$$
$$G_{13}=-\sqrt{2}f_Ng_\Lambda(\omega-M_\Lambda)I_1,$$
$$G_{21}=-(6)^{1/2}f_\Lambda f_\Sigma(\omega-M_\Lambda)I_2,$$
$$G_{22}=1+3f_\Lambda^2I_2(\omega-\Delta-M_\Sigma),$$
$$G_{23}=2\sqrt{3}g_\Sigma(f_N-f_\Sigma)(\omega-M_\Sigma)I_2,$$
$$G_{31}=-\sqrt{2}f_Ng_\Lambda(\omega-M_\Lambda)I_3,$$
$$G_{32}=2\sqrt{3}g_\Sigma(f_N-f_\Sigma)(\omega-M_\Sigma)I_3,$$
$$G_{33}=1;$$

$I=1,\ J=1/2-$:

$$G_{11}=1-3f_\Lambda^2[(\omega-M_\Lambda+\Delta)-(\omega-M_\Sigma)]I_1,$$
$$G_{12}=(6)^{1/2}f_\Lambda f_\Sigma(\omega-M_\Sigma)I_1,$$
$$G_{13}=-3\sqrt{2}f_\Lambda g_\Sigma(\omega-M_\Sigma)I_1,$$
$$G_{21}=(6)^{1/2}f_\Lambda f_\Sigma(\omega-M_\Sigma)I_2,$$
$$G_{22}=1+7f_\Sigma^2(\omega-M_\Sigma)I_2,$$
$$G_{23}=G_{32}=0,$$
$$G_{31}=-3\sqrt{2}f_\Lambda g_\Sigma(\omega-M_\Sigma)I_3,$$
$$G_{33}=1+6g_\Sigma^2(\omega-M_\Sigma)I_3;$$

$I=0,\ J=3/2-$:

$$G_{11}=1+4f_\Sigma^2(\omega-M_\Sigma)I_1,$$
$$G_{21}=G_{12}=0,$$
$$G_{22}=1;$$

$I=0,\ J=3/2+$:

$$G_{11}=G_{22}=1,$$
$$G_{12}=G_{21}=0,$$

$I=0,\ J=1/2+$:

$$G_{11}=1+3[3f_\Lambda^2(\omega-M_\Lambda)-f_\Lambda^2(\omega-M_\Sigma-\Delta)]I_1,$$
$$G_{12}=-3\sqrt{2}f_\Lambda g_\Lambda(\omega-M_\Lambda)I_1,$$
$$G_{21}=-3\sqrt{2}f_\Lambda g_\Lambda(\omega-M_\Lambda)I_2,$$
$$G_{22}=1+6g_\Lambda^2(\omega-M_\Lambda)I_2;$$

$I=0,\ J=1/2-$:

$$G_{11}=1-2f_\Sigma^2(\omega-M_\Sigma)I_1,$$
$$G_{12}=-3\sqrt{2}f_Ng_\Sigma(\omega-M_\Sigma)I_1,$$
$$G_{21}=-3\sqrt{2}f_Ng_\Sigma(\omega-M_\Sigma)I_2,$$
$$G_{22}=1.$$