Wave Distortion for Magnetic Moment Effects in Nucleon-Nucleon Scattering*

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The paper is concerned with the difficulty of making reliable corrections for the effects of nucleonic magnetic moments on nucleon-nucleon scattering on account of the unknown effects of wave function distortion. A partial removal of the difficulty is accomplished by correcting for the magnetic moment effect only in the one-pion exchange (OPE) group of phase parameters. Wave-function distortion in that group is relatively small and the error committed is less serious than if the magnetic effects are used for the whole wave. Since the low L group has its phase parameters determined by adjustment to data, these parameters have to be interpreted as including the magnetic moment effects, which will eventually have to be corrected for, when the origin of the nucleon-nucleon phase shifts is understood quantitatively in other respects. Formulas for the calculation of the magnetic moment effects in the high L group are supplied and some typical cases are considered numerically. Relatively large effects are found for the polarization P and, literally speaking, the triple-scattering parameter A at very small angles. The complication arising in n-p data analysis from coupling of states with different isotopic spin is pointed out.

I. INTRODUCTION AND NOTATION

EFFECTS on nucleon-nucleon scattering arising on account of the interaction of the magnetic moments of nucleons with the electromagnetic fields produced by their partners have been first calculated by Garren[1] and then by Breit,[2] Ebel and Hull,[3] and again by Garren.[4] The treatment in references 1 and 4 assumes in the case of p-p scattering the applicability of a divergent perturbation method, the divergence being connected with the infinite total scattering cross section in a Coulomb field. The treatment in references 2 and 3 is free of this defect. In a later paper[5] it has been pointed out that under many practical circumstances the wave-function distortion caused by nucleon-nucleon interactions of nonelectromagnetic origin is so large that results in the references quoted above are subject to serious question. At scattering angles usually considered small and at incident proton energy of $\sim$150 MeV the partial wave expansion of the Coulomb wave has been pointed out to be so slowly convergent that modifications caused by wave distortion at low L affect the result appreciably. The sum of the first three terms turned out to be 30% of the whole for $\theta=10°$ and the sum of the first two terms about half this amount. The spin-orbit effects on the other hand were shown to arise from integrals such that the first maximum of the integrand falls at 2×10^{-13} cm for $L=1$, at 2.9×10^{-13} cm for $L=2$, and 3.7×10^{-13} cm for $L=3$ so that the contributions arise from regions within which mesonic effects on the wave function are appreciable. On the other hand, $L=1$ and $L=2$ contribute 0.41 of the total spin-orbit electromagnetic effect at 150 MeV and $\theta=10°$. The applicability of calculations assuming that for small-angle scattering the Coulomb wave is not distorted, an assumption common to work preceding reference 5, appeared therefore subject to serious question. The possibility of remedying the situation by dividing the phase shifts into two groups was realized [cf. first paragraph beginning on p. 1592 of reference 2] but appeared difficult to carry out.

The situation is changed, however, by the procedure of dividing the phase parameters into two groups[6–8]: one for high L and the other for low. The L of the first group are taken to be high enough to make the one-pion exchange interaction dominant over other meson-theoretical effects. The phase parameters of the low L group are searched for, employing experimental data,[7,8] the criterion being a low value of the sum of squares of deviations of measured values from those corresponding to assumed phase parameters. Wave-function distortion is largest in the low L group, but it is not necessary to calculate its effects because the phase parameters for it can be and usually are determined by the search procedure consisting in the adjustment of the phase

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[1] A. Garren, Phys. Rev. **96**, 1709 (1954).

[2] G. Breit, Phys. Rev. **99**, 1581 (1955). One of the writers (GB) would like to record the following transcription errors from calculations to the manuscript of the paper: In Eq. (18) the first factor inside the first square brackets should be E_I-M instead of E_I+M. The second in the square brackets of (18.2) which originates in the one just mentioned, is, however, correctly reproduced in print. In Eq. (18.2) the entry after the first equality sign should be $-\Delta'\alpha_4$ rather than $\Delta'\alpha_4$.

[3] M. E. Ebel and M. H. Hull, Jr., Phys. Rev. **99**, 1596 (1955).

[4] A. Garren, Phys. Rev. **101**, 419 (1956).

[5] G. Breit, Phys. Rev. **106**, 314 (1957).

[6] M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **6**, 581 (1951); J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, *ibid.* **16**, 455 (1956); Suppl. Progr. Theoret. Phys. (Kyoto) No. 3 (1956), Part II; S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958); R. Tamagaki, *ibid.* (Kyoto) **20**, 505 (1958).

[7] M. J. Moravcsik, P. Cziffra, M. H. MacGregor, and H. P. Stapp, Bull. Am. Phys. Soc. **4**, 49 (1959); P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[8] G. Breit and M. H. Hull, Jr., Nuclear Phys. **15**, 216 (1960). G. Breit, M. H. Hull, Jr., K. Lassila, and K. D. Pyatt, Jr., Phys. Rev. Letters **4**, 79 (1960); G. Breit, M. H. Hull, Jr., K. E. Lassila, and H. M. Ruppel, Proc. Natl. Acad. Sci. U. S. **46**, 1649 (1960); G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960); M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, *ibid.* **122**, 1606 (1961).

parameters to experimental data. For the high L group, which is often referred to as the OPEP or OPE group, wave-function distortion may be expected to be smaller than for the group of low L and the error caused by neglecting it may be expected to be relatively small. The justification for employing the OPE approximation is, it will be remembered,[9] also the absence of wave distortion. The gain achieved by following this plan consists in dividing the problem into two parts and postponing the complete consideration of magnetic effects to a time when the interactions for low L can be treated sufficiently reliably to make the application of corrections for magnetic effects possible and significant. The present note is concerned therefore mainly with the separation of the magnetic effects into contributions from low and high L. Subtraction of the contributions of the low L from the total leaves an effect on the scattering amplitude which should be included with that of the OPEP. The low L effect is included in the values of the searched phase parameters. Comparison of the low L phase parameters with theory is at present in a rudimentary stage and inclusion of magnetic interactions in it appears premature. When the theory of nucleon-nucleon interactions is well enough understood, a useful inclusion of magnetic effects should become possible and would then become especially significant in precise considerations concerning the spin orbit interactions.

In principle the detection of the effects of magnetic moments and their comparison with experiment should be instructive regarding nucleon structure and the combined action of electromagnetic and mesonic interactions with nucleons. In particular there is the open question of the applicability of static values of the moments obtained under essentially field-free conditions to the quite different conditions applying in nucleon-nucleon collisions. These questions may be expected to be especially serious for the low L group, a quantitative treatment of which appears remote. For the high L group the effects are less likely to be large and the detection of deviations from elementary expectation will presumably be concerned with relatively small fractional effects. It appears likely, therefore, that in their present form the magnetic moment effects will be of value mainly as a means of improving on the knowledge of phase parameters produced by mesonic effects unless comparison with experiment should indicate that the effects expected on the basis of static values of magnetic moments are not favored by the comparison.

Notation. Some of the symbols used in this paper, a misunderstanding regarding which might be especially confusing, are listed below.

H', H'', etc. are parts of Hamiltonian.
M = nucleon mass.
$\mu_0 = e\hbar/(2Mc)$ = nuclear Bohr magneton.
$\mu_a\mu_0$ = anomalous part of proton's magnetic moment.
$\mu_p\mu_0$, $\mu_n\mu_0$ = magnetic moments of proton and neutron, respectively.
θ = scattering angle in center-of-mass system.
$\mu = \cos\theta$.
E_1 = energy of one nucleon in center-of-mass system expressed in Mc^2 as a unit.
$\mathbf{p}_1$ = momentum of nucleon 1 in center-of-mass system.
un is used as a superscript to denote an unsymmetrized quantity.
a is used as a superscript to denote a quantity obtained in calculations with antisymmetrized wave functions. Further explanation available in short paragraph preceding Eq. (3).
f^{un} = correction factor for omission of low L group in calculation of magnetic moment spin-orbit interaction in unsymmetrized case.
L_m = maximum L in low L group.

[9] First article cited in reference 8.

II. THE SPIN-ORBIT INTERACTION

On account of the effect which the spin-orbit interaction has on polarization, the influence of magnetic moment forces on the spin-orbit interaction is probably the most important of the effects considered here. For p-p scattering the contribution to the Hamiltonian according to reference 5 is

$$H'+H'' = -(\mu_0^2/\hbar r^3)(3+4\mu_a) \times\{[(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1]\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\}, \quad (1)$$

where μ_0 is the nuclear Bohr magneton, $e\hbar/(2Mc)$; μ_a the anomalous part of the proton magnetic moment in μ_0 units; $\mathbf{r}_1$ and $\mathbf{r}_2$ are displacement vectors to the position of the charges from the origin; e is the charge on each particle; $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta; $r = |\mathbf{r}_1-\mathbf{r}_2|$ is the distance between the charges; $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli's spin matrix vectors; M is the mass of each particle. Equation (1) is the result of combining Eqs. (1) or (1.2) and (3.2) of reference 5. The contribution to the scattering amplitude is then as in Eq. (3.4) of reference 5. With

$$\mathbf{s} = \sin(\theta/2) \quad (1.1)$$

and θ standing for the scattering angle in the center-of-mass system, the individual contributions of different L may be evaluated by means of (5.1) of reference 5 and the formula

$$\sum_{L=1}^{\infty} \frac{2L+1}{L(L+1)} P_L'(\mu) = \frac{1}{1-\mu} = \frac{1}{2\mathbf{s}^2}, \quad (1.2)$$

where $\mu = \cos\theta$ and $P_L(\mu)$ is the Legendre function of μ. In (1.2) the summation is taken over all integral $L \geq 1$. For p-p scattering only odd L can enter the final result because there are no triplet-even states. It is not necessary, however, to write out the modified value of the sum for this case because it is simpler to carry out the calculation first for the unsymmetrized case and to antisymmetrize the result afterwards. Employment of (1.2) involves the assumption that the Coulomb wave functions may be approximated by corresponding free-particle wave functions in (5.1) of reference 5. This assumption is rather well satisfied and it may be pointed out that the employment of (5.1) of reference 5 has been justified in reference 2. The contributions of different L in (1.2) give the relative contributions of orbital angular momenta $L\hbar$ to unsymmetrized triplet-scattering amplitudes. In the notation of BH[10] and BEH[11] the only quantities α in terms of which the scattering amplitudes are expressed which are affected by the perturbing Hamiltonian of Eq. (1) are α_4 and α_1 and direct calculation by means of Eq. (3.4) of reference 5 shows that first-order changes in these quantities corresponding to Eq. (1) are

$$(\Delta\alpha_4)_0 = -(\Delta\alpha_1)_0 = -(3+4\mu_a)(\hbar^2k^3/2M^2c^2)S^c e^{-i\Phi} \quad (1.3)$$

with

$$\Phi = \rho - \eta \ln(2\rho) + 2\sigma_0, \quad (1.4)$$

$$\rho = kr, \quad \eta = e^2/\hbar v, \quad \sigma_L = \arg\Gamma(L+1+i\eta), \quad (1.5)$$

$$S^c = -(\eta/2k\mathbf{s}^2)\exp[i(\Phi - \eta \ln\mathbf{s}^2)]. \quad (1.6)$$

The employment of the undistorted wave-function approximation is indicated in (1.3) by $(\ )_0$. For future reference the relationship of the parameters $\alpha_1, \cdots, \alpha_5$ to the elements of S as introduced in BH is reproduced below

$$-kS_{0,1}e^{-i\varphi} = kS_{0,-1}e^{i\varphi} = -2^{-1/2}\alpha_1 \sin\theta e^{i\Phi}, \quad (2.3)_{\mathrm{BH}}$$

$$k(S_{1,1}-S^c) = k(S_{-1,-1}-S^c) = \alpha_2 e^{i\Phi}, \quad (2.2)_{\mathrm{BH}}$$

$$kS_{-1,1}e^{-2i\varphi} = kS_{1,-1}e^{2i\varphi} = \alpha_3 \sin^2\theta e^{i\Phi}, \quad (2.4)_{\mathrm{BH}}$$

$$kS_{1,0}e^{i\varphi} = -kS_{-1,0}e^{-i\varphi} = 2^{-1/2}\alpha_4 \sin\theta e^{i\Phi}, \quad (2.5)_{\mathrm{BH}}$$

$$k(S_{0,0}-S^c) = \alpha_5 e^{i\Phi}. \quad (2.6)_{\mathrm{BH}}$$

The expressions for the $\boldsymbol{\alpha}$ in terms of phase shifts are not reproduced but may be found in Eqs. (2.2) to (2.6) of BH supplemented by Eqs. (4) to (6.2) of BEH which deal with modifications caused by coupling of states with different L. Equations (1.4), (1.5), (1.6) are taken from BH. The quantity Φ does not enter (1.3) if the Coulomb amplitude S^c is introduced by means of (1.6). The remaining dependence of the two $\Delta\alpha$ on η aside from the factor η in (1.6) is through $\exp(-i\eta \ln\mathbf{s}^2)$. The smallness of η makes its entrance in the exponential unimportant and Eq. (1.4) readily transferable to the *n-p* case as will be seen presently. In this approximation

$$(\Delta\alpha_4)_0 = -(\Delta\alpha_1)_0 \cong (3+4\mu_a)(\hbar^2k^2\eta/4M^2c^2\mathbf{s}^2). \quad (1.7)$$

[10] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955). This paper is referred to as BH in the text. The authors of that paper (BH) should like to record a transcription error in Eq. (1.4), the second term of which should have inserted the factor $(4\pi)^{1/2}(2L+1)^{1/2}$.

[11] G. Breit, J. B. Ehrman, and M. H. Hull, Jr., Phys. Rev. **97**, 1051 (1955). This paper is referred to as BEH in the text.

Equations (1.3) and (1.7) can also be justified by means of (1.2) and the expressions for the α_i of BH in terms of phase shifts. A few intermediate steps in this calculation and also in the derivation of (1.2) are given in the Appendix. The representation of the $(\Delta\alpha_i)_0$ by means of the series in (1.2) makes it possible to correct for the exclusion of the low L group from the magnetic spin orbit effect. The accuracy of the calculation does not completely justify the inclusion of the factor $\exp(-i\eta \ln\mathbf{s}^2)$ because the reduction to the simple form of the sum occurring in (1.2) takes place only if the η dependent quantities e_{L0} of BH occurring in the partial wave expansions of the α_i are set equal to unity. There are, besides, additional questions connected with the distortion of the wave by the Coulomb field and with the choice of the relativistic or nonrelativistic η in the exponential. These questions have been discussed in references 2 and 3. The smallness of η makes their importance secondary.

For the antisymmetrized *p-p* amplitudes the formula corresponding to the approximation of Eq. (1.7) is

$$(\Delta^a\alpha_1)_0 \cong -(3+4\mu_a)(\hbar^2k^2\eta/8M^2c^2)(1/\mathbf{s}^2 - 1/\mathbf{c}^2). \quad (1.8)$$

Here

$$\mathbf{c} = \cos(\theta/2). \quad (1.8)'$$

The convention used in employing the symbol Δ^a will be discussed presently. The inclusion of the effect of η on the phase can be approximately taken into account here by employing

$$\mathbf{s}^{-2}\exp(-i\eta \ln\mathbf{s}^3) - \mathbf{c}^{-2}\exp(-i\eta \ln\mathbf{c}^2) \quad (1.9)$$

in place of the last factor in (1.8). The latter equation is intended for use with the antisymmetrized Coulomb amplitude

$$S^{ac} = 2^{-1/2}(\eta/2k)[-\mathbf{s}^{-2}\exp(-i\eta \ln\mathbf{s}^2) + \mathbf{c}^{-2}\exp(-i\eta \ln\mathbf{c}^2)]e^{i\Phi}, \quad (2)$$

which is Eq. (4) of BH and

$$\alpha_c = \tfrac{1}{4}\eta[-\mathbf{s}^{-2}\exp(-i\eta \ln\mathbf{s}^2) + \mathbf{c}^{-2}\exp(-i\eta \ln\mathbf{c}^2)], \quad [\mathrm{BEH}(19)] \quad (2.1)$$

which is Eq. (19) of BEH. The latter enters directly the formula for $(P\sigma)_{p-p}$ as in Eq. (20) of BEH, with P standing for the polarization.

The relation of Eq. (1.8) to Eq. (1.7) is

$$\Delta^a\alpha_1(\theta) = \tfrac{1}{2}[\Delta\alpha_1(\theta) - \Delta\alpha_1(\pi-\theta)], \quad (2.2)$$

which is the same as that of (2.1) to the quantity which has to be added to α_2 when one calculates $S_{-1,-1}$. According to Eqs. (2.2) and (2.9) of BH this is

$$a_c^{un} = kS^c e^{-i\Phi} = -(\eta/2\mathbf{s}^2)\exp(-i\eta \ln\mathbf{s}^2), \quad (2.3)$$

with superscript *un* indicating an unsymmetrized quantity. The validity of (1.8) and (2.2) follows from the fact that all $L \geq 1$ are included in (1.2) while in the evaluation of α_1 and α_4 by means of Eqs. (2.3) and (2.5) of BH it is convenient to use only odd L in the p-p case. Subtraction of $\alpha_1(\pi-\theta)$ removes even L and doubles the contributions of odd L. Division by 2 leaves in the sum the contributions of odd L alone. The Δ^a convention is thus such as to make the quantity directly additive to one containing odd L alone. The object of including in the present discussion Eqs. (2) and (2.3) is to show the correspondence of antisymmetrized with unsymmetrized quantities for the ordinary Coulomb amplitude together with that for the spin-orbit interaction. The same correspondence can be inferred from Eq. (3.4) of reference 5, viz.,

$$S^{c''}=-(\eta/2k\mathbf{s}^2)\exp[i(\Phi-\eta\ln\mathbf{s}^2)] \times[1+(3+4\mu_a)i(\hbar^2/4M^2c^2)(\mathbf{k}_f\times\mathbf{k}_i)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)].$$

This formula includes both effects for the unsymmetrized case. Particle interchange leaves the vector product unchanged and the two parts must be changed by the same linear formula as is seen by comparing (2), (2.1), and (2.3) with (2.2).

The employment of superscript a in the symbol Δ^a is meant in a different sense from that of the same superscript in Eq. (2) above or in Eq. (7.7) of BH and other equations of that paper as well as in BEH. According to these the true antisymmetrized $\alpha_i^a=2^{1/2}\alpha_i$. The quantity $\Delta^a\alpha_1$ in (2.2) is not meant to be $\Delta(\alpha_1^a)$ in the above sense. It is intended for use with the unsymmetrized α_i as an addition to them. The reason for this preference is that the effect of antisymmetrization on a measurable quantity can usually be readily taken into account at the end of a calculation as, e.g., in Eq. (6.1) of BH. In repetitive numerical work the more efficient procedure is to work with the unsymmetrized α_i and the quantities in the present paper are intended for that purpose.

According to Eq. (20) of BEH and Eq. (1.3) above, the change in the polarization P can be obtained as

$$\Delta\{k^2(P\sigma)_{p-p}\}=2\,\mathrm{Im}\{-(\alpha_2+\alpha_c{}^T)+\alpha_3\sin^2\theta-(\alpha_5+\alpha_c{}^T)\}\Delta^a(\alpha_1\sin\theta). \quad (3)$$

The polarization P is here used in the sense of $\langle\sigma_y\rangle$ for a right-handed coordinate system with incidence along z, scattering in xz plane, i.e., the usual azimuthal angle $\varphi=0$. The superscript T on α_c has been inserted in order to indicate that the α_c appropriate for triplets rather than singlets is meant. This α_c is given by Eq. (19) of BEH, reproduced as Eq. (2.1) above. The effect of the spin-orbit interaction on the differential cross section is usually small and Eq. (3) is then sufficient for the calculation of the effect on P. The calculation of $\Delta^a\alpha_1$ to be inserted on the right-hand side of (3) can be made by means of the undistorted wave approximation (2.2) or (1.8) provided the omission of the low L group in (1.2) is allowed for. According to (1.2) the correction factor for the absence of the low L group is

$$f^{un}=1-2\mathbf{s}^2\sum_1^{L_m}\frac{2L+1}{L(L+1)}P_L'(\mu) \quad (3.1)$$

and

$$\Delta\alpha_i=f^{un}(\Delta\alpha_i)_0 \quad (i=1,4). \quad (3.2)$$

This may be used directly in (2.2) with the $(\Delta\alpha_i)_0$ from (1.3) or (1.7). If preferred, a correction factor for the antisymmetrized result can be used directly as

$$f^a=1-\frac{4}{\mathbf{s}^{-2}-\mathbf{c}^{-2}}\Big(\sum_1^{L_m}\Big)_{\mathrm{odd}\,L}\frac{2L+1}{L(L+1)}P_L'(\mu) \quad (3.3)$$

together with

$$\Delta^a\alpha_i=f^a(\Delta^a\alpha_i)_0. \quad (3.4)$$

At the smaller angles for which the effects of the magnetic spin orbit interaction are of main interest $\mathbf{c}^{-2}$ is small compared to $\mathbf{s}^{-2}$. Thus, the values of $(\mathbf{s}^{-2},\mathbf{c}^{-2})$ are (132, 1.01) for $\theta=10°$, (58.7, 1.02) for $\theta=15°$, and (33.2, 1.03) for $\theta=20°$. Therefore, the correction for distortion may be estimated in this case by means of (3.1). For $\theta=20°$ the sum of the first three terms of the series in Eq. (1.2) with $P_L'(\mu)$ replaced by $dP_L/d\theta$ is -2.339 while the whole series is -5.67 and $f^{un}=0.59$. Similar numbers have already been mentioned in[5] but the possibility of using them in conjunction with a separation into OPE and non-OPE groups of L values was not realized. Values of f^{un} from $\theta=5°$ to 175° are collected in Table I. Since the OPEP group of phase parameters often starts at $L=4$, or more, the correction factor is not negligible except at quite small scattering angles.

For n-p scattering the spin-orbit part of the Hamiltonian is

$$H_{n-p}'=-4\mu_n(\mu_0{}^2/r^3)(\mathbf{L}\cdot\boldsymbol{\sigma}_n), \quad (4)$$

where $\mu_n\mu_0$ is the magnetic moment of the neutron, $\hbar\boldsymbol{\sigma}_n/2$ is its spin, and the notation is otherwise as previously. The scattering amplitude corresponding to S^c and arising in the present case entirely from the spin-orbit interaction is

$$S'=4i\mu_n\mu_0{}^2(e^{ikr}/Mv^2\mathbf{s}^2)(\mathbf{k}_i\times\mathbf{k}_f)\cdot\boldsymbol{\sigma}_n. \quad (4.1)$$

Resolving this expression into parts symmetric and antisymmetric in the spins of the two particles, there is obtained the part responsible for triplet scattering. This part is

$$S'^T=2i\mu_n\mu_0{}^2(e^{ikr}/Mv^2\mathbf{s}^2)(\mathbf{k}_i\times\mathbf{k}_f)\cdot(\boldsymbol{\sigma}_n+\boldsymbol{\sigma}_p), \quad (4.2)$$

with superscript T indicating triplets. In the undistorted wave-function approximation this part contributes changes to the α_i as follows:

$$(\beta)_0\equiv(\Delta^{TT}\alpha_{4,n-p})_0=-(\Delta^{TT}\alpha_{1,n-p})_0 =4\mu_n\mu_0{}^2(k^3/Mv^2\mathbf{s}^2) \quad (4.3)$$

and the change in neutron polarization is found to be

$$\Delta(k^2P\sigma)_n=(\beta/2)\,\mathrm{Im}(2\alpha_2+\alpha_5+\alpha_0)\sin\theta, \quad (4.4)$$

where

$$\alpha_0=\sum\nolimits_L(2L+1)P_LQ(K_L) \quad (4.4\mathrm{a})$$

in the notation of BH and

$$\beta = f^{un}(\beta)_0. \tag{4.5}$$

The change in proton polarization is obtainable from

$$\Delta(k^2 P\sigma)_p = (\beta/2)\,\mathrm{Im}\{-2\alpha_3 \sin^2\theta + \alpha_5 - \alpha_0\}\sin\theta. \tag{4.6}$$

The polarization P is defined as $\langle\sigma_y\rangle$ for the azimuthal angle $\varphi=0$. In the derivation of these results it is essential to include the effect of off-diagonal elements between singlets and triplets. This part of the matrix of S' of (4.1) is

$$S'^{(ST)} = \frac{(\beta)_0 \sin\theta}{2^{1/2}k} e^{ikr} \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{4.7}$$

the rows and columns being labeled here in the order of magnetic quantum numbers 0_0, 1, 0, -1 from top to bottom and from left to right, respectively. The 0_0 refers to the singlet state. The spin functions in (4.7) are used in the convention

$$\chi_0{}^0 = (\alpha_n\beta_p - \alpha_p\beta_n)/2^{1/2}, \quad \chi_1 = \alpha_n\alpha_p,$$
$$\chi_0 = (\alpha_n\beta_p + \alpha_p\beta_n)/2^{\frac{1}{2}}, \quad \chi_{-1} = \beta_n\beta_p. \tag{4.8}$$

The transition from $(\beta)_0$ to β, i.e., the employment of the wave distortion correction factor, requires justification additional to that for the triplets described in connection with (1.2) and (3.1). This is found in the consideration of matrix elements of $(\mathbf{L}\cdot\boldsymbol{\sigma}_n)/r^3$ which enters (4) between initial and final states of plane wave functions. Taking the incident wave to propagate along the positive z axis, the quantity of interest is

$$\mathfrak{M}_m = (\chi_0{}^0 \exp(i\mathbf{k}_f\cdot\mathbf{r}),\quad r^{-3}(\mathbf{L}\cdot\boldsymbol{\sigma}_n)\chi_m \exp(ikz)). \tag{5}$$

Expressing

$$\exp(i\mathbf{k}_f\cdot\mathbf{r}) = 4\pi \textstyle\sum_{Lm} i^L Y_{Lm}{}^*(\hat{k}_f) Y_{Lm}(\hat{r}) F_L/\rho, \tag{5.1}$$

where the dependence of the spherical harmonics Y_{Lm} on polar angles with respect to the z axis is indicated symbolically by the insertion of the corresponding unit vector and the function $F_L(\rho)$ is the usual r times the regular radial function for orbital angular momentum $L\hbar$ normalized to be asymptotic to $\sin(kr-L\pi/2)$ at large r. Taking $m=1$ and expanding e^{ikz} in the Y_{L0}, there appears a $[L(L+1)]^{1/2}Y_{L1}$ in the $\chi_0{}^0$ part of $(\mathbf{L}\boldsymbol{\sigma}_1)Y_{L0}\chi_1$ and $(2L+1)^{1/2}$ in the expansion of e^{ikz}. When Y_{L1} is expressed in terms of $P_L'(\cos\theta)$, there occurs the factor $(2L+1)^{1/2}[L(L+1)]^{-1/2}$. Combining all L-dependent factors mentioned so far there remains $(2L+1)P_L'(\mu)$. Besides, there is $1/[L(L+1)]$ from integrating $F_L{}^2/\rho^3$. The combination under the summation sign in (1.2) is thus reproduced. A similar counting up of factors leads to the same conclusion for $\mathfrak{M}_{-1}$ and no contribution is obtained to $\mathfrak{M}_0$. The whole effect may thus be considered as a sum of contributions from different L in the same proportions as for the other spin orbit interactions and the employment of (4.5) is justifiable.

TABLE I. Values of f^{un}, the unsymmetrized correction factor for wave-function distortion.

L_m \ θ	5°	10°	20°	30°	150°	160°	170°	175°
1	0.99	0.98	0.91	0.80	−1.80	−1.91	−1.98	−1.99
2	0.98	0.94	0.77	0.51	2.24	2.65	2.91	2.98
3	0.97	0.89	0.59	0.19	−2.25	−3.15	−3.78	−3.94
4	0.95	0.82	0.38	−0.11	1.84	3.37	4.56	4.89
5	0.93	0.75	0.18	−0.32	−1.12	−3.31	−5.23	−5.80

The spin-orbit interaction in the n-p case couples pairs of states with the same J and parity but different isotopic spin T. In the main analysis of n-p data performed so far it is assumed that the T=0 and T=1 states are uncoupled. This approximation is probably justified by the relative smallness of the coupling caused by the magnetic spin-orbit interaction as well as the relatively incomplete state of the experimental n-p scattering data as compared with p-p data. While in a general way the magnetic coupling may be relatively harmless, it should be remarked that it affects the interesting spin-orbit features of the interactions. A somewhat different way of saying this is that the employment of the T=1 value as obtained from p-p scattering for the $^3P_1$ phase shift and an independent search for a T=0 phase shift for $^1P_1$ is not strictly justifiable, the two states being coupled to each other. The possibility remains open that mesonic nucleon-nucleon interactions are not completely charge independent and that coupling of $J=L$ states to each other may actually be stronger than the magnetic interaction alone would lead one to expect. If this should be the case, further modifications of the n-p phase parameter analysis might become necessary.

It may be of interest to point out a few quantities which are not sensitive to the spin-orbit interaction and to the off-diagonal matrix elements connecting singlets with triplets. The change in the cross section is

$$\begin{aligned}(\Delta\sigma/\sigma)_{n-p} &= 4\Delta\{\mathrm{Tr}(S^\dagger S)\}/\mathrm{Tr}(S^\dagger S) \\ &= [2\beta\sin^2\theta\,\mathrm{Re}(\alpha_4-\alpha_1)]/[|\alpha_0|^2 \\ &\quad + |\alpha_1|^2\sin^2\theta + 2|\alpha_2|^2 + |\alpha_4|^2\sin^2\theta \\ &\quad + 2|\alpha_3|^2\sin^4\theta + |\alpha_5|^2],\end{aligned} \tag{6}$$

and is small. If there were no other cause for scattering than the magnetic spin-orbit interaction, the differential cross section would be for unpolarized particles

$$\tfrac{1}{4}(4\mu_n\mu_0{}^2/Mv^2)^2\mathbf{s}^{-4}\,\mathrm{Tr}\{[(\mathbf{k}_i\times\mathbf{k}_f)\cdot\boldsymbol{\sigma}_n]^2\} = [\mu_n e^2/(2Mc^2)]^2\mathbf{c}^2/\mathbf{s}^2. \tag{6.05}$$

Since, as will be seen below, the spin-orbit effects are much larger at small θ than the spin-spin effects the above expression gives the dominant term of the cross section as a function of θ at small θ. The entrance of the small length $e^2/(Mc^2)$ is noteworthy as an indicator of the smallness of the effects.

In Eq. (6) the quantity α_0 is defined in the same way with respect to the singlet scattering matrix which is denoted by s_{00} in Eq. (5) of BH as the other α_i are defined in terms of the elements of the triplet scattering matrix. Explicitly for the n-p case $\alpha_0=\sum_L(2L+1)P_LQ_L$ with Q_L computed employing the singlet phase shifts K_L. For p-p scattering the corresponding formula is

$$(\Delta\sigma/\sigma)_{p-p}=[2(\Delta^a\alpha_4)\sin^2\theta\,\mathrm{Re}(\alpha_4-\alpha_1)]/[|\alpha_0+\alpha_c{}^S|^2+2|\alpha_2+\alpha_c{}^T|^2+2|\alpha_3|^2\sin^4\theta+|\alpha_5+\alpha_c{}^T|^2+(|\alpha_1|^2+|\alpha_4|^2)\sin^2\theta]. \quad (6.1)$$

This formula gives $(\Delta\sigma/\sigma)_{p-p}=0.012$ and 0.016 for $\theta=10°$ and 20°, respectively, in the case of the fit[12] YLAM at an incident energy of 140 MeV. In Eq. (6.1) the quantities $\alpha_c{}^S$ and $\alpha_c{}^T$ are the additions which must be made to the α_i in order to account for the effect of the Coulomb field on the $S_{\mu\nu}$ for singlets and triplets, respectively. The numbers just quoted are too small to make $\Delta\sigma$ worth taking into account in calculations of ΔP. It may also be remarked that the α_c are important in (6.1) only at the lowest energies and could be omitted for most of the energy range considered in reference 12.

The polarization correlation coefficient C_{nn} is also insensitive to the magnetic spin-orbit interaction in spite of the fact that C_{nn} can be used to separate triplet from singlet effects and the coupling between triplets and singlets caused by the magnetic $\mathbf{L}\cdot\mathbf{S}$ effect. This may be seen from

$$1-C_{nn}=1-\langle\sigma_{py}\sigma_{ny}\rangle=2[|S_{0,0}{}^0|^2+|S_{1,1}+S_{1,-1}|^2]/\sum_{\mu\nu}|S_{\mu\nu}|^2, \quad (6.2)$$

a formula readily derived from the matrices for σ_{ny} and σ_{py} in the singlet-triplet reference frame. Comparison with (4.7) shows that the nonvanishing elements of $S'^{(ST)}$ are not involved. Neither are α_1 and α_4 which are concerned with the (1,0), (0,1), (0, −1) and (−1, 0) elements of S^T. The only effect is therefore in the denominator and the effect is similar to that for the cross section σ. In (6.2) the singlet matrix element of S is distinguished by the superscript 0.

The expressions dealt with above are nonrelativistic and are closely related to the pictorial representation of spin-orbit interactions used in atomic physics. A complete theory of relativistic effects would involve a consideration similar to that in reference 2 so as to avoid lack of convergence caused by infinite Coulomb scattering but extended to the effect of the anomalous part of the nucleon magnetic moment. In view of the necessity of dividing the effect according to contributions from partial waves so as to separate out the low L portion of the effect, a more elaborate calculation would be needed. For the Diracian part of the moment, most of the work has already been performed by Ebel and Hull.[3] No similar treatment appears to be available for the effect of the anomalous part of the moment. On the other hand, the computational complications arising from the employment of exact equations for the first-order effects appear unwarranted at this time for various reasons as follows. In the first place, the whole effect while not negligible is small. Secondly, some wave-function distortion is present in the OPEP (high L) group of phase parameters and its effects may be more serious than the slight inaccuracies in relativistic corrections present in the discussion below. Thirdly, the inaccuracies in relativistic corrections arise because the effect of the low L group is being subtracted from the whole magnetic spin-orbit effect. The uncertainties usually arise, therefore, only when the magnetic V_{LS} entering the calculations is even smaller in absolute value than the already small $(V_{LS})_{\mathrm{mag}}$ calculated for high and low L together as though there were no wave distortion. It appears satisfactory, therefore, to employ the same relativistic correction factor for $(V_{LS})_{\mathrm{mag}}$ as though there were no separation into high and low L groups.

Employing Eq. (18) of reference 2 for the contribution arising from the Diracian part of the moment[13] and performing a calculation along the lines of Garren's[1,4] for the anomalous part, the contribution to $(S^c)_{p-p}$ arising from both causes is

$$\Delta(S^c)_{p-p}{}^{un}=-[(ie^2/\mathbf{s}^2)/8Mc^2]\times\{\exp[i(\Phi-\eta\ln\mathbf{s}^2)]\}\times\{2E_1+1+2E_1(E_1+1)\mu_a-\mathbf{s}^2[E_1-1+2(E_1{}^2-1)\mu_a]\}\times[(\hat{k}_f\times\hat{k}_i)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)]/[E_1(E_1+1)]. \quad (7)$$

Here E_1 is the energy of one of the protons in the center of mass system expressed in Mc^2 units. In the nonrelativistic limit this becomes Eq. (3.4) of[5] which is reproduced above as an unnumbered equation following Eq. (2.3). The nonrelativistic value of this quantity is

$$[(\Delta S^c)_{p-p}]_{nr}{}^{un}=-[(ie^2/\mathbf{s}^2)/16Mc^2]\times\{\exp[i(\Phi-\eta\ln\mathbf{s}^2)]\}(3+4\mu_a)\times[(\hat{k}_f\times\hat{k}_i)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)] \quad (7.1)$$

and the ratio of the two is the relativistic correction factor for the unsymmetrized case. Defining $\Delta\alpha_1$ and $\Delta\alpha_4$ as in BH, the quantity to be used with the unsymmetrized α_i in a calculation with antisymmetrized functions is again given by (2.2). One has from (7) and the

[12] Fourth article cited in reference 8.

[13] The first factor in the equation quoted should be E_I-M, an error having occurred in transcribing the calculations for publication.

last two equations

$$[(\Delta\alpha_4)_0]_r = -[(\Delta\alpha_1)_0]_r = [(ke^2/\mathbf{s}^2)/4Mc^2] \times[\exp(-i\eta \ln\mathbf{s}^2)]\{2E_1+1+2E_1(E_1+1)\mu_a - \mathbf{s}^2[E_1-1+2(E_1^2-1)\mu_a]\} / [E_1(E_1+1)]. \quad (7.2)$$

Applying (2.2) there is available the relativistic modification of the α_i for p-p scattering, neglecting wave distortion. According to (7) and (7.1) the relativistic correction factor for the unsymmetrized $\Delta\alpha$ is

$$(f_r^{un})_{p-p} = \{2/[E_1(E_1+1)]\}\{2E_1+1+2E_1(E_1+1)\mu_a - \mathbf{s}^2[E_1-1+2(E_1^2-1)\mu_a]\}/(3+4\mu_a). \quad (7.3)$$

This applies to the first term in brackets in (2.2). The relativistic correction factor for the second term is obtained from that of the first term by changing $\mathbf{s}$ to $\mathbf{c}$. The relativistic correction factor for the n-p case is, according to (7.3),

$$(f_r)_{n-p} = 1-[(E_1-1)/E_1]\mathbf{s}^2. \quad (7.4)$$

Employment of the relativistic correction factors as in (7.3) and (7.4) presupposes that the kinematic relativistic corrections have been made and that the values of k and η have been appropriately calculated. The meaning of $\Delta^a\alpha_i$ has been discussed in the short paragraph preceding that containing Eq. (3).

III. THE SPIN-SPIN INTERACTION.

The nonrelativistic addition to the Hamiltonian arising from the magnetic spin-spin interaction is

$$H_{SS} = (1+\mu_a)^2(e\hbar/2Mc)^2[(\nabla_2\boldsymbol{\sigma}_1)(\nabla_1\boldsymbol{\sigma}_2)(1/r)] = -(1+\mu_a)^2\mu_0^2 S_{12}/r^3, \quad (8)$$

where

$$S_{12}(\hat{r}) = 3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2 - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \quad (8.1)$$

is the usual tensor force operator. In (8) the [ ] indicates termination of the differential operations at the end of the last bracket.

Special care must be exercised regarding the employment of this interaction energy in the case of s states, Eq. (8) as written being inapplicable in this case. Since wave-function distortion cannot be neglected for s states, they will always be in the low L group and the inapplicability of Eq. (8) is therefore immaterial in the application being made here. Arguments similar to those in[2] show that the modification of S^c can be performed approximately through the employment of plane wave expressions in combination with S^c as a factor. Straightforward calculation then shows that the change in S^c caused by H_{SS} is

$$\Delta_{SS}S^c = \tfrac{1}{3}(1+\mu_a)^2(\hbar/2Mc)^2q^2S_{12}(\hat{q}), \quad (8.2)$$

where

$$\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f, \quad (8.3)$$

so that $\hbar\mathbf{q}$ is the momentum transfer. The matrix elements of $S_{12}(\hat{q})$ are the same as those of the matrix appearing in Eq. (19.1) of reference 2 which represents there the quantity

$$[3((\mathbf{p}'-\mathbf{p})\cdot\mathbf{S})^2 - (\mathbf{p}'-\mathbf{p})^2\mathbf{S}^2]/(2p^2\mathbf{s}^2) \quad (8.4)$$

and is $3\mathfrak{M}_2$ of that paper. The matrix elements in the absence of wave distortion are thus readily available. It is convenient to introduce a constant C by means of

$$\Delta_{SS}Q_{L,J} = CS_{12,J}^{LL}\int_0^\infty (F_L^2/\rho^3)d\rho, \quad (9)$$

where $S_{12,J}^{LL}$ are the matrix elements of the tensor force operator, diagonal in L. Denoting the additions to the α_i caused by the spin-spin interaction as $\Delta_{SS}\alpha_i$ and taking into account the effect of the coupling of states with different L but the same J, the values of the $\Delta_{SS}\alpha_i$ are readily found in the form of sums over the L which reduce to the simpler form of the matrix corresponding to (8.4) above. These sums will be written out below so as to enable a simple removal from the sum of the terms corresponding to the low l group. The contributions to the sums arising from matrix elements diagonal in L are written employing L as the summation index. Contributions originating in the coupling of states with $L=l$ and $L=l+2$, corresponding to $J=l+1$, are written using l as the summation index. The expansions are

$$(\Delta_{SS}\alpha_1)_0/C = (\Delta_{SS}\alpha_4)_0/C = 3\sum_{L=1}^\infty A(L)P_L' - \sum_{l=0}^\infty B(l) \times[(l+1)P_l' - (l+2)P_{l+2}'] = 1, \quad (9.1)$$

where

$$A(L) = \frac{2L+1}{L(L+1)(2L-1)(2L+3)}, \qquad B(l) = \frac{1}{(l+1)(l+2)(2l+3)}, \quad (9.2)$$

$$-2(\Delta_{SS}\alpha_2)_0/C = (\Delta_{SS}\alpha_5)_0/C = -\sum_{L=1}^\infty \frac{2(2L+1)}{(2L-1)(2L+3)}P_L + \sum_{l=0}^\infty \frac{1}{2l+3}(P_l+P_{l+2}) = (1-3\cos\theta)/3, \quad (9.3)$$

$$(\Delta_{SS}\alpha_3)_0/C = -3\sum_{L=2}^\infty A(L)P_L'' - \tfrac{1}{2}\sum_{l=0}^\infty B(l)(P_l''+P_{l+2}'') = -1/(4\mathbf{s}^2). \quad (9.4)$$

TABLE II. Values of correction factor to the unsymmetrized spin-spin effect arising from the omission of the low L and l for three cases A, B, C corresponding to $(L_m,l_m)=(3,3)$, $(4,3)$, and $(4,4)$, respectively.

Quantity	Case	$\theta=5°$	10°	20°	30°	150°	160°	170°	175°
α_2 and α_5	A	0.37	0.32	0.16	−0.04	−0.02	−0.03	−0.03	−0.02
	B	0.03	0.01	−0.02	−0.14	−0.02	0.06	0.12	0.14
	C	0.28	0.23	0.06	−0.20	0.01	0.02	0.02	0.02
α_3	A	0.98	0.94	0.77	0.54	1.38	1.74	2.00	2.07
	B	0.98	0.92	0.69	0.39	−0.71	−0.90	−1.02	−1.05
	C	0.98	0.90	0.65	0.34	−1.34	−1.96	−2.43	−2.56
α_1 and α_4	A	0.68	0.63	0.48	0.27	−0.08	−0.06	−0.03	−0.03
	B	0.50	0.47	0.34	0.18	0.01	0.07	0.13	0.15
	C	0.63	0.57	0.37	0.13	0.06	0.05	0.03	0.02

The unsymmetrized correction factors for wave distortion to the $(\Delta_{SS}\alpha_i)_0$ are obtainable from these expressions by subtracting in each case the sum of the terms belonging to the low L group from the right-hand side and dividing by the right-hand side. This procedure is the same as that used to obtain f^{un} of Eq. (3.1) from the series of Eq. (1.2). The verification of the summation of the series in Eqs. (9.1), (9.3), (9.4) involves only the grouping of terms in the L and l parts of the series. The value of the constant C is immaterial for the application of the equations. Since (8) is symmetric in the two spins, it has no matrix elements between the singlets and triplets and it is not necessary therefore to obtain correction factors for wave distortion to quantities additional to the α_i. The relation of the $\Delta_{SS}{}^a\alpha_i(\theta)$ to the $\Delta_{SS}\alpha_i(\theta)$ is again as in (2.2) and the meaning of the Δ^a is the same as previously, viz., the addition to the unsymmetrized α_i for use in equations relying on symmetry properties of the parts of the α_i arising from the triplet part of p-p scattering containing only odd L and l.

For n-p scattering the triplet amplitudes contain both odd and even L and l. The question of antisymmetrization does not arise, however. The equations apply as they stand except for the replacement

$$(1+\mu_a)^2 \to \mu_p\mu_n, \tag{9.5}$$

where μ_p, μ_n are, respectively, the proton and neutron magnetic moments expressed in μ_0 as a unit.

Although $(\Delta_{SS}\alpha_3)_0$ becomes large at small θ, this quantity occurs in the scattering matrix with the factor $\sin^2\theta$. The effects of the spin-spin interaction at small angles are not large, therefore, and are much smaller than those of the magnetic spin-orbit interaction. In view of this the omission of relativistic correction factors for this quantity appears justifiable and the whole spin-spin effect may often be neglected altogether.

Values of the correction factor, taking into account the omission of the first few values of L up to and including L_m and the first few values of l up to and including l_m, are listed in Table II. This correction factor is meant to be applied to the unsymmetrized spin-spin effect corresponding to Eq. (8.2) in the p-p case and to Eqs. (9.1), (9.3), (9.4). Since according to Eq. (2.2) the antisymmetrization brings in quantities for $\pi-\theta$ in addition to those for θ, the table contains in addition to the small scattering angles 5°, 10°, 20°, 30°, also their supplementary angles 175°, 170°, 160°, 150°. The corrections for wave distortion are large on a fractional basis. Since the whole effect is small, they are not nearly as important as the table would indicate if considered by itself. The three cases A, B, C correspond, respectively, to the maximum values of L and l included in the low angular momentum group being, respectively, $(L_m,l_m)=(3,3)$, $(4,3)$, $(4,4)$. Without going into complete detail, these pairs of values correspond to searches YLAM, YRB1 for T=1 parameters and YLAN3M for those with T=0 at laboratory energies >150 MeV.

IV. APPROXIMATE MAGNITUDES AND DISCUSSION

Values of the relativistic correction factor to the spin-orbit effect at five representative energies are shown in Table III. At each energy the correction factor to the unsymmetrized effect at the specified angle is shown in the top line and the corresponding factor to the unsymmetrized quantity for the supplementary angle is shown in the second line. The two factors needed for the two quantities in the square brackets in (2.2) are thus available. This table is intended as an aid in reaching a decision as to whether to include the relativistic correction factor in analyzing one or another set of data. In Table IV there are shown values of spin-orbit magnetic moment effects for p-p scattering at a bombarding energy of 147 MeV for the polarization $P(\theta)$ and the triple scattering parameters $R(\theta)$, $A(\theta)$, and $D(\theta)$. For comparison there are also shown in parentheses and directly below each entry the values of these quantities for the phase parameter fit YLAM. The values listed in the table include the relativistic correction factor applied separately to the two contributions in Eq. (2.2) and also the $\exp(-i\eta\ln s^2)$ and $\exp(-i\eta\ln c^2)$ factors, as previously described.

At 147 MeV the values of the correlation coefficient C_{kp} in p-p scattering are appreciably affected at small angles by the magnetic moment corrections. Thus at $\theta=10°$, 20°, and 30° for YLAM they are, respectively, 0.076, 0.30, and 0.51 while the increases in them are roughly 0.020, 0.014, and 0.004. Although at very small θ the effects are large in fractional measure, the difficulties in detecting these effects are obviously great. The spin-spin effects are small. For p-p scattering at bombarding energy of 147 MeV and without the correction for wave distortion, i.e., without subtraction of effects caused by the low L, the calculated changes in YLAM values of P, R, A, and D are, respectively,

$$\begin{array}{llll} 0.00016, & -0.00057, & -0.00009, & 0.00052 \text{ at } \theta=30°; \\ 0.00014, & -0.00068, & 0.00000, & 0.00044 \text{ at } \theta=20°; \\ 0.00004, & 0.00010, & 0.00007, & 0.00025 \text{ at } \theta=10°; \\ 0.00000, & 0.00018, & 0.00001, & 0.00003 \text{ at } \theta=\ 5°. \end{array}$$

The numbers may have rounding off errors of roughly ± 0.00001. Since the whole effect is much smaller than the experimental error, this inaccuracy should be immaterial. Similar small numbers have been obtained up to $\theta=50°$ with no indication of a significantly rapid increase. The correction factor for the omission of low L is available in Table II for the amplitudes from which the experimental quantities can be calculated.

At 147-MeV bombarding energy and in the range $5°<\theta<30°$, the spin-spin effect on $\mathrm{Re}\alpha_1(\theta)$ is nearly (within 0.3%) the same as on $\mathrm{Re}\alpha_1(\pi-\theta)$ but $\mathrm{Im}\Delta\alpha_1(\pi-\theta)/\mathrm{Im}\Delta\alpha_1(\theta)$ varies from 0.00003 at 5° to 0.026 at 30°; $\mathrm{Re}\Delta\alpha_2(\pi-\theta)/\mathrm{Re}\Delta\alpha_2(\theta)$ varies from -2.01 to -2.25 for the same values of θ; $\mathrm{Im}\Delta\alpha_2(\pi-\theta)/\mathrm{Im}\Delta\alpha_2(\theta)$ from -0.00006 to -0.058; $\mathrm{Re}\Delta\alpha_3(\pi-\theta)/\mathrm{Re}\Delta\alpha_3(\theta)$ from 0.002 to 0.072, and $\mathrm{Im}\Delta\alpha_3(\pi-\theta)/\mathrm{Im}\Delta\alpha_3(\theta)$ from 6×10^{-7} to 0.0018. These numbers are quoted because they show that some of the quantities are of very little consequence. It will be useful to recall that for the spin-spin effect $\Delta\alpha_1=\Delta\alpha_4$ and $\Delta\alpha_5=-2\Delta\alpha_2$.

At 147-MeV bombarding energy for the magnetic spin-orbit effect in p-p scattering, the ratio $\mathrm{Re}\Delta\alpha(\pi-\theta)/\mathrm{Re}\Delta\alpha(\theta)$ has the values -0.0078, -0.0326, -0.1668, -0.8663, at $\theta=5°$, 10°, 20°, 30°, respectively, while the corresponding values of $\mathrm{Im}\Delta\alpha(\pi-\theta)/\mathrm{Im}\Delta\alpha(\theta)$ are -0.000002, -0.00005, -0.0015, -0.0220. These numbers are supplied because they show the angular ranges within which the supplementary angle contribution may be neglected.

TABLE III. Values of relativistic correction factor to the unsymmetrized spin-orbit interaction effect. For each energy the upper number is the value for angle θ, the lower for $\pi-\theta$.

Energy (MeV) \ θ	5°	10°	20°	30°
9.69	0.999	0.999	0.999	0.999
	0.997	0.997	0.997	0.997
98.0	0.994	0.994	0.993	0.992
	0.974	0.974	0.974	0.975
147.0	0.991	0.991	0.990	0.989
	0.961	0.961	0.962	0.963
210.0	0.987	0.987	0.986	0.984
	0.946	0.946	0.947	0.949
345.0	0.980	0.979	0.978	0.976
	0.915	0.916	0.917	0.920

TABLE IV. Values of changes in $P(\theta)$, $R(\theta)$, $A(\theta)$, and $D(\theta)$ caused by magnetic spin-orbit effect at bombarding energy of 147 MeV for p-p scattering with $L_m=3$ and of values of these quantities according to fit YLAM.

θ	$\Delta P(\theta)$ ($P(\theta)$)	$\Delta R(\theta)$ ($R(\theta)$)	$\Delta A(\theta)$ ($A(\theta)$)	$\Delta D(\theta)$ ($D(\theta)$)
5°	0.0036	−0.0002	0.0328	0.0002
	(0.0006)	(0.9184)	(−0.0151)	(0.9208)
10°	0.0214	0.0021	0.0093	0.0146
	(0.0647)	(−0.0864)	(0.0439)	(0.0315)
20°	0.0083	0.0029	−0.0141	0.0137
	(0.1831)	(−0.4458)	(−0.2189)	(−0.0421)
30°	0.0018	0.0002	−0.0035	0.0050
	(0.2255)	(−0.2984)	(−0.3614)	(0.1219)

The inclusion of $\exp(-i\eta \ln s^2)$ in the calculations may often be omitted. Its effect on $P(\theta)$ for YLAM is estimated at 9.69 MeV to be 0.0007, 0.0011, 0.0011 at $\theta=5°$, 10°, 15°, respectively. At 147 MeV the corresponding values are 0.0037, 0.0037, 0.0009; at 210 MeV 0.0045, 0.0031, 0.0007; at 345 MeV 0.0057, 0.0028, 0.0009. In these estimates the relativistic correction factor to the spin-orbit effect has not been included but the wave-function distortion correction (exclusion of low L) has been made. Since calculations are simpler without the $\exp(-i\eta \ln s^2)$ factor, it appeared desirable to give an idea of the often harmless effect of using the simpler formula. At $\theta\geq10°$ from 147 MeV to 345 MeV the effects are small compared with P and the $\mathbf{L}\cdot\mathbf{S}$ effect on P. At $\theta=5°$ they are relatively more important and definitely increase with E in absolute value. The effect of the increase of $|\mathrm{Im}\alpha|$ with E is important for the presence of this effect which also accounts for the increase of the influence of $\exp(-i\eta \ln s^2)$ on P from 9.69 to 147 MeV. However, at 9.69 MeV the value of $P(\theta)$ is small and small effects are relatively more important at such low energies. The experimental alignment difficulties at small angles are more serious than at larger ones and hence the large effects of $\exp(-i\eta \ln s^2)$ at $\theta=5°$ are of interest at present only in exceptional cases. In Fig. 1 there is shown a comparison of fit YLAM with proton-proton polarization data at 147-MeV bombarding energy, together with the result of applying the correction for the magnetic moment spin-orbit effect. The value of L_m used was 3. The relativistic correction factor and $\exp(-i\eta \ln s^2)$ are included but the spin-spin effects have not been, being small. Agreement with experiment[14] is seen to be improved[15] employing the

[14] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. (New York) **5**, 299 (1958). In cases of two values for the same angle the values were averaged.

[15] In a letter written during the summer of 1961 to one of the authors (GB), Dr. E. H. Thorndike, then at Harvard, has enclosed a graph related to the one in Fig. 1. Employing values of amplitudes α at 140 MeV furnished to him by the Yale group he compared $P(\theta)$ for YLAM at 140 MeV with the 147-MeV data, (see reference 14) and employing the same values of α he calculated the magnetic moment spin-orbit effect without taking into

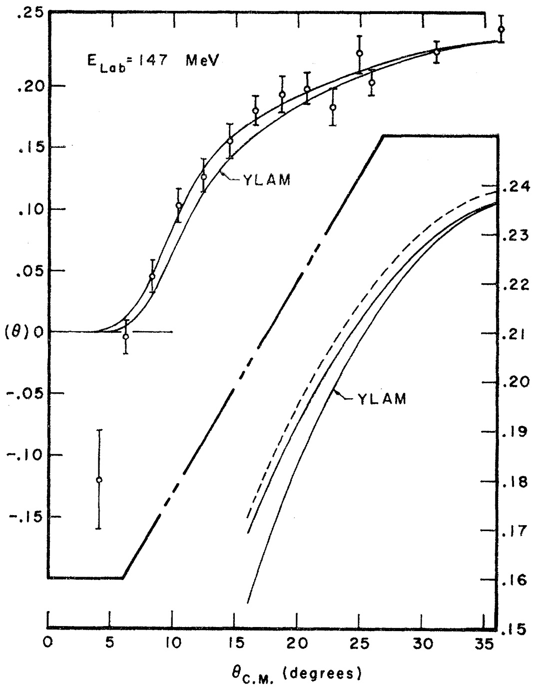

FIG. 1. Comparison of magnetic moment spin-orbit effect with experiment at bombarding energy of 147 MeV in p-p scattering. Original YLAM corresponds to the lower graph. Inclusion of the magnetic moment spin-orbit effect gives the upper curve. The inset in the lower part of figure shows on a magnified scale the curves just mentioned and a third dashed plot which is obtained if the effect of the low L is not subtracted.

correction. This improvement is marked in the angular range 8° to 20°. At 6.2° and around 4° the agreement is worsened by the correction. In this connection the private communication from Professor R. Wilson of Harvard which has been mentioned as a "Note in Proof" at the end of the fourth footnote in reference 8 should be recalled. According to it special sources of error at small angles have been found in the Harvard measurements and the negative values of $P(\theta)$ could not be considered to be reliable. The disagreement at the two smallest angles is thus not necessarily significant. It should be remarked that the wave-distortion effect is not pronounced in the region in which improvement with experiment is most marked. This circumstance cannot be considered as an argument however for not using the correction for the omission of the low L especially because the correction becomes marked at only slightly higher angles.

In the lower part of Fig. 1, the polarization is shown on a scale displaced vertically and magnified in the ratio of 4 to 1. The theoretical curves alone are shown and a third dashed curve represents the effect of adding the full magnetic spin-orbit effect without the subtraction of the effect of low L.

Attempts have been made to look for over-all improvements in p-p and n-p fits as a result of applying the magnetic spin-orbit corrections. No definite indications have been found and no striking indications of improvement in n-p analysis in special cases have been noticed. The immediate practical application of the effect appears to be more promising in cases of data selected on the basis of extra care in low-angle measurements. In such cases the confirmation of the theoretically expected effect should add confidence in the phase parameters of mesonic origin and should be helpful in improving their determination. The quantities P and A show relatively large effects. It is not clear however that measurements of A at the low angles for which the effects are pronounced can be performed with adequate accuracy.

account the omission of low L and basing himself on formulas in H. A. Bethe's well-known paper on nucleon-nucleus scattering. He noticed that the 140-MeV calculations agreed better with the 147-MeV data after the correction was made and inquired into the reasons for not including magnetic moment effects in p-p scattering analysis. In the reply to Dr. Thorndike the difficulty with wave-function distortion was mentioned and also the fact that there existed a way of effectively dealing with the difficulty. It is a pleasure to record the fact that Dr. Thorndike was the first to see an indication of improvement in the comparison with experiment arising from the magnetic moment effect.

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APPENDIX

The validity of Eq. (1.2) of the text may be verified as follows. From the known value[16] of

$$\int_{-1}^{+1}(1-\mu^2)[P_n'(\mu)]^2 d\mu=\frac{2n(n+1)}{2n+1}, \tag{A1}$$

and the fact that $(1-\mu^2)^{1/2}P_n'(\mu)$ is the φ-independent part of $Y_{n1}(\theta,\varphi)$, it follows that

$$\int_{-1}^{+1}(1-\mu^2)P_n'(\mu)P_m'(\mu)d\mu=\frac{2n(n+1)}{2n+1}\delta_{nm}; \tag{A2}$$

and hence, assuming the completeness of the $(1-\mu^2)^{1/2}P_n'(\mu)$ for the expansion, it is only necessary to test

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(1-\mu^2)^{1/2}P_n'(\mu)=\left(\frac{1+\mu}{1-\mu}\right)^{1/2}$$

by employing (A2). This is readily done by means of

$$\int_{-1}^{+1}(1+\mu)P_n'(\mu)d\mu=2, \tag{A3}$$

[16] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1920), 3rd ed., p. 311.

which is readily verified by partial integration. The above is one of many ways of arriving at Eq. (1.2). It should be observed however that the series on the left-hand side of (1.2) does not converge in a rigorous sense. In fact, employing a well-known recurrence relation and taking the sum of the first N terms

$$[1/(1-\mu^2)]-P_{N+1} = [1/(1-\mu)][1-(P_{N-1}+P_{N+1})/(1+\mu)]. \quad \text{(A4)}$$

The functions $P_{N+1}(\mu)$, $P_{N-1}(\mu)$ oscillate rapidly with an amplitude which for fixed μ is proportional to $N^{-1/2}$. If $|\mu|\neq 1$ the series converges, a definite limit being approached as $N\to\infty$. But for $\mu=\pm 1$ the limit does not exist and the convergence is nonuniform. This does not interfere with the applicability of the result since the formula is not needed for $\mu=\pm 1$. The employment of asymptotic forms of P_n for large n which gave the $N^{-1/2}$ dependence requires modification for $\sin\theta=0$ because then the amplitude of the oscillations of the dominant term in the asymptotic formula becomes infinite. This shows that the convergence of the series becomes poor as $\mu=\pm 1$ is approached, in agreement with there being no convergence at $\mu=\pm 1$.

Defining

$$K\equiv(3+4\mu_a)\mu_0^2 kM/\hbar^2 \quad \text{(A5)}$$

and employing the value of

$$\int_0^\infty (F_L^2/\rho^3)d\rho=1/[2L(L+1)],$$

which corresponds to the neglect of the effect of the Coulomb field on the radial functions, as in Eq. (1.2) of the text, the phase shift corresponding to the magnetic spin-orbit interaction in the case of p-p scattering is readily found to be

$$\delta_J{}^L=K(\mathbf{L}\cdot\mathbf{S})/[L(L+1)]. \quad \text{(A6)}$$

Working to first order in the phase shifts, substituting in Eq. (2.3) of BH, and setting in the latter $e_{L0}=1$ so as to be consistent with the neglect of the Coulomb field on the F_L, it follows that

$$\Delta\alpha_1=-K\sum_{L=1}^{\infty}(2L+1)P_L'(\mu)/[L(L+1)]. \quad \text{(A7)}$$

Employment of Eq. (1.2) and the introduction of S^c by means of Eq. (1.6) gives (1.3).