Complex Angular Momentum in Field Theory*

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The notion of complex angular momentum is extended to relativistic scattering amplitudes satisfying the Mandelstam representation. The domain of analyticity in the complex angular momentum plane is enlarged by the use of unitarity relations, and the existence of Regge poles in a certain restricted domain is established.

I

THE concept of complex angular momentum, first introduced by Regge in connection with non-relativistic potential scattering[1] has recently been applied to field theory.[2,3] The poles in the complex angular momentum plane that naturally arise in this approach, or the so-called Regge poles, may very well be of great importance in explaining the large number of experimentally observed resonances and also the high-energy behavior of cross sections.[2–4] In the case of potential scattering, the continuation in the complex angular momentum plane can be effected by simply considering the solutions of the Schrödinger equation for complex values of l. The proper analyticity domain in the l plane and the existence of the Regge poles can then be proved if the potential is restricted to a linear superposition of Yukawa potentials, and also some useful information about Regge poles can be derived by studying the properties of the Schrödinger equation in detail.[1,5] In field theory, it has been found that a suitable continuation in the complex angular momentum plane can be defined if the validity of the Mandelstam representation is assumed.[6–9] About the existence and the properties of Regge poles in field theory, however, so far nothing has been established rigorously except for some tentative results in the strip approximation.[7,8] In this paper, starting from the Mandelstam representation, we will show how a unique continuation in the l plane can be defined, and we will derive the form of the two-particle unitarity relation in the complex l plane. Then, using a slightly generalized form of Froissart's result[10] about the unitarity limit on the scattering amplitude and a simple analytic completion procedure, we will be able to enlarge the previous domain of analyticity in the l plane. We will show that this domain can further be enlarged except for Regge poles if one utilizes the two-particle unitarity relation below the inelastic threshold. Finally, we will discuss the physical consequences of these results. It must be emphasized that our results are rigorously valid if the Mandelstam representation is correct and there have been no approximations involved.

II

In what follows, we will restrict ourselves to the scattering of identical pseudoscalar particles of mass $m>0$. Although there is no real difficulty in extending our results to scalar particles, the absence of the Born term for pseudoscalar case will slightly simplify matters. At the end of the paper, we will also say a few words about the case of several particles of different masses and particles with spin. We now start with the following dispersion relation:

$$f(s,t)=\sum_{p=0}^{n}\rho_p(s)t^p+t^K\int_{4m^2}^{\infty}dt'\frac{A(s,t')}{t'^K(t'-t)}+u^K\int_{4m^2}^{\infty}du'\frac{A(s,u')}{u'^K(u'-u)}. \quad (1)$$

where $u=4m^2-s-t$, K is a suitable integer that will make the integrals convergent, and the mass spectrum starts at $4m^2$. The same spectral function occurs in both integrals because of crossing symmetry. Using the definition of partial-wave amplitudes, we get, with $z=\cos\theta=1+2t/(s-4m^2)$,

$$\begin{aligned}a_l(s)&=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)f\{s,\tfrac{1}{2}(z-1)(s-4m^2)\}\\&=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)\bigg\{\sum_{0}^{n}\rho_p(s)\Big[\frac{z-1}{2}(s-4m^2)\Big]^p\\&\quad+[\tfrac{1}{2}(z-1)(s-4m^2)]^K\\&\quad\times\int_{4m^2}^{\infty}dt'\frac{A(s,t')}{t'^K[t'-\frac{1}{2}(z-1)(s-4m^2)]}\\&\quad+[-\tfrac{1}{2}(z+1)(s-4m^2)]^K\\&\quad\times\int_{4m^2}^{\infty}du'\frac{A(s,u')}{u'^K[u'+\frac{1}{2}(z+1)(s-4m^2)]}\bigg\}. \quad (2)\end{aligned}$$

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[1] T. Regge, Nuovo cimento **14**, 951 (1959).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[3] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).
[4] M. Gell-Mann, Phys. Rev. Letters **6**, 263 (1962).
[5] A. Bottino, A. Longoni, and T. Regge (to be published).
[6] M. Froissart, invited paper to La Jolla Conference on Strong and Weak Interactions.
[7] K. Bardakci (to be published).
[8] A. O. Barut and D. E. Zwanziger (to be published).
[9] E. J. Squires (to be published).
[10] M. Froissart, Phys. Rev. **123**, 1053 (1961).

For $l \geqq K$, the orders of integration in (2) can be interchanged, as the resulting integral is then convergent. If we also take $l>n$, the subtraction polynomial does not contribute. For $l>\max\{K,n\}=L$, we therefore have

$$Q_l(s)=-[1+(-1)^l]R(s,l),$$

where

$$R(s,l)=\frac{2}{s-4m^2}\int_{4m^2}^{\infty} dt\, Q_l\left(1+\frac{2t}{s-4m^2}\right)A(s,t). \quad (3)$$

$Q_l(x)$, the Legendre function of the second kind, asymptotically approaches a function of the form Cx^{-l-1}, where C depends only on l, for all x except for $x=\mp 1$.[11] It therefore follows that $Q_l(1+2t/(s-4m^2))$ is bounded by an expression of the form Ct^{-l-1} for large positive t. Hence, the integral in (3) converges for $l \geqq K$, as asserted previously. Furthermore, if N is the greatest lower bound of the values of Rel for which the integral in (3) converges, where $N \leqq K$ necessarily, then the function $R(s,l)$ is analytic for Re$l>N$, since the function Q_l is an analytic function in this region. We now turn to the analyticity properties of $R(s,l)$ as a function of s, keeping Re$l>N$. The region of singularities of $R(s,l)$ is then the union of the regions of singularities of $Q_l(1+2t/(s-4m^2))$ and $A(s,t)$. From the Mandelstam representation, it follows that $A(s,t)$ is analytic in the s plane except for cuts extending from $4m^2$ to ∞ and from $-4m^2$ to $-\infty$. $Q_l(1+2t/(s-4m^2))$ contributes a cut extending from $s=4m^2$ to $s=-\infty$. Therefore, $R(s,l)$ is cut along the real axis from $-\infty$ to ∞. The part of the cut extending from $s=0$ to $s=4m^2$ is, however, purely kinematical and can easily be removed. To this end, we define:

$$R(s,l)=\left(\frac{s-4m^2}{4m^2}\right)^l T(s,l), \quad (4)$$

where the factor $[(s-4m^2)/4m^2]^l$ is defined to have a branch cut along the negative real axis. The nature of the branch cut of $Q_l(1+2t/(s-4m^2))$ is such that part of it is eliminated by this procedure, and $T(s,l)$ has cuts extending from $-\infty$ to 0 and from $4m^2$ to $+\infty$. Furthermore, since the l'th partial wave vanishes at least like $(s-4m^2)^l$ at $s=4m^2$, this definition does not give rise to poles for $T(s,l)$ for integer values of l.

Next we come to the important question of the asymptotic behavior of $R(s,l)$ as $|l|\to\infty$. In reference 1, the asymptotic properties of $R(s,l)$ were obtained by Regge through a detailed analysis of the Schrödinger equation for nonrelativistic scattering, and his asymptotic estimate enabled him to write down a Watson-Sommerfeld representation for the scattering amplitude. In the relativistic case, Froissart[6] has shown that the existence of a dispersion relation like (1) in linear momentum transfer is sufficient to establish an asymptotic bound for $R(s,l)$ which leads to a slightly modified Watson-Sommerfeld integral. At this point, we would like to stress on the importance of the proper asymptotic behavior of this function, since if one relaxes this condition, given an arbitrary set of partial waves, it is always possible to interpolate them by an infinite number of analytic functions. In what follows, we restrict ourselves to real $s>4m^2$. Using the standard estimate[11]

$$Q_l(x)\leqq (C/|l|^{1/2})\exp\{-[x+(x^2-1)^{1/2}]\,\mathrm{Re}l\}, \quad (5a)$$

where $x=$real and >1 and C is independent of l, we obtain:

$$\begin{aligned}|R(s,l)| &\leqq \frac{2}{s-4m^2}\int_{4m^2}^{\infty} dt\,|A(s,t)|\left|Q_l\left(1+\frac{2t}{s-4m^2}\right)\right| \\ &\leqq \frac{2}{s-4m^2}\int_{4m^2}^{\infty} dt\, C_1 t^N \frac{C_2}{|l|^{1/2}}\exp\left\{-\mathrm{Re}l\left[1+\frac{2t}{s-4m^2}+\left[\left(1+\frac{2t}{s-4m^2}\right)^2-1\right]^{1/2}\right]\right\} \\ &\leqq \frac{C_3}{|l|^{1/2}}\exp\left\{-\mathrm{Re}l\left[1+\frac{8m^2}{s-4m^2}+\left[\left(1+\frac{8m^2}{s-4m^2}\right)^2-1\right]^{1/2}\right]\right\}. \quad (5b)\end{aligned}$$

where all the constants in question have no l dependence. In deriving the final result, we treated $A(s,t)$ as an ordinary function and ignored all distribution theoretical subtlities. The result can, however, be justified by a more careful treatment.[10]

Equation (5a) shows that $R(s,l)$ goes to zero as $|l|\to\infty$ for physical s, although it should be noted that this is in general not true for nonphysical values of s. One can then derive the Watson-Sommerfeld representation exactly as in reference 1, with two minor modifications: There is an additional crossed channel and the line of integration is moved to Re$l=L$. The result is, with the restriction $s>4m^2$,

$$f(s,t)=\sum_{i=0}^{L}(2i+1)a_i(s)P_{(i)}(z)+\frac{1}{2i}\int_{L-i\infty}^{L+i\infty} dl\frac{2l+1}{\sin\pi l}R(s,l) \times[P_l(z)+P_l(-z)]. \quad (6)$$

This in turn implies Eq. (1), so that analyticity in the linear momentum space with only a finite number

[11] *Higher Transcendental Functions*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 1, Chap. 3.

of subtractions at infinity is completely equivalent to analyticity and asymptotic boundedness in the angular momentum space.

Next we turn to the questions of uniqueness of $R(s,l)$ and the two-particle unitarity relation. For this purpose, we need the following theorem from the complex variable theory.[12]

Given a function $h(w)$, which is analytic for $\text{Re}w \geqq 0$ with the following additional restrictions:

(a) $H(w)$ is bounded as $|w| \to \infty$ uniformly in the angle $-\pi/2 \leqq \arg w \leqq \pi/2$.

(b) $h(n)=0$ for all positive integer n. Then it follows that $h(w)=0$ identically.

It is now easy to see that a function which interpolates the partial waves and which is asymptotically bounded and has the proper domain of analyticity in the l plane is necessarily unique, since if there were two such functions, their difference would vanish by the above theorem. As another application, let us write the two-particle unitarity relation in terms of the function $R(s,l)$ in the interval $4m^2 < s < b$, where b is the threshold for inelastic processes:

$$R(s+i\epsilon,\, l) - R(s-i\epsilon,\, l) + i[(s-4m^2)/s]^{1/2},\\ R(s+i\epsilon,\, l) R(s-i\epsilon,\, l) = 0. \quad (7)$$

This relation was originally true only for integer l. The left side of it, however, satisfies the conditions of our theorem and the equation stays valid for complex values. Equation (7) and some other results derived so far have independently been obtained in references 8 and 9 by different methods.

So far, we have established that $R(s,l)$ is analytic in the product of a cut plane in the variable s and the region $\text{Re}l > N$ in the l plane. However, Froissart has shown that the scattering amplitude is bounded by an expression of the form $Cs \ln^2 s$ for real $t \leqq 0$ and large s.[10] Using crossing symmetry, it follows that $A(s,t)$ is bounded by $Ct \ln^2 t$ for real negative s, so that the integral in (3) converges for $\text{Re}l > 1$ if s is real and negative. This result can be extended to complex s as follows:

The scattering amplitude for large s is bounded by an expression $Cs^{[1+(N-1)\,\text{Re}(t/4m^2)^{1/2}+\epsilon]}$ for all $\epsilon > 0$ and for a fixed t which falls inside the ellipse with foci ∓ 1 and semimajor axis $1+8m^2/(s-4m^2)$ in the $z=\cos\theta$ plane for large enough s. Correspondingly, $R(s,l)$ is analytic in the region $\text{Re}l > 1+(N-1)\,\text{Re}(s/4m^2)^{1/2}$, for $\text{Re}(s/4m^2) \leqq 1$. (The square root is defined to have always a non-negative real part). For a proof of this result, we refer the reader to the Appendix.

The domain given above is certainly not the best possible domain, since it is not even a natural domain of holomorphy. To see this, we note that for a large domain in the s plane, the corresponding domain in the l plane is $\text{Re}l > N$. When s crosses the curve $\text{Re}(s/4m^2)^{1/2} = 1$, the domain in the l plane suddenly starts getting larger. Such a discontinuous behavior, however, violates the continuity theorem for the surface of singularities of an analytic function of several complex variables,[13] therefore, it cannot be a natural domain of holomorphy. We now proceed to construct the required domain of holomorphy.

The method essentially consists of mapping the cut plane with s variable, into a simpler domain conformally. In order to apply one of the standard results of analytic completion, we use the following conformal transformations:

$$\omega = (2/\pi) \arc \sin[(s-2m^2)/2m^2], \quad (8)$$

where the function "arc sin" has a cut extending from $s=0$ to $s=-\infty$ and from $s=4m^2$ to $s=\infty$ and is defined to be $\pi/2$ at $s=4m^2$. Equation (8) maps the s-plane cut from $s=0$ to $s=-\infty$ and from $s=4m^2$ to $s=+\infty$ into the strip between the lines $\text{Re}(w)=-1$ and $\text{Re}(w)=1$ in the w plane. We now need a slightly modified form of a standard result in the theory of functions of several complex variables.[14]

Suppose that a function of two complex variables z_1 and z_2 is analytic in the union of two domains (A) and (B), where (A) is the product of a small strip containing the line $\text{Re}z_1 = a_1$ in the z_1 plane and the region satisfying $\text{Re}z_2 > a_2$ in the z_2 plane, and (B) is the product of the strip $a_1 \leqq \text{Re}z_1 < b_1$ in the z_1 plane and the region $\text{Re}z_2 > b_2$ in the z_2 plane, with the real numbers a_1, b_1, a_2, b_2 satisfying $a_1 < b_1$ and $a_2 < b_2$. Every such function is also analytic in the larger region given by

$$a_1 \leqq \text{Re}z_1 < b_1, \quad \text{Re}z_2 > a_2,\\ (b_2-a_2)\,\text{Re}(a_1-z_1) + (b_1-a_1)\,\text{Re}(z_2-a_2) > 0.$$

Moreover, this is a natural domain of holomorphy.

If we consider $T(s,l)$ defined in (4) as a function of l and w as given in (8), we see that it satisfies the conditions of the above theorem with $z_1=w$, $z_2=l$, $a_1=-1$, $b_1=1$, $a_2=1$, $b_2=N$, and the existence of neighborhood of analyticity around $\text{Re}w=-1$, or equivalently, around $s \leqq 0$ is proved in the Appendix. Transforming back to the variable s, it follows that $T(s,l)$ is analytic in the domain given by $\text{Re}l > 1$, with s not on the cuts from $s=0$ to $s=-\infty$ or from $s=4m^2$ to $s=\infty$, and

$$2\,\text{Re}(l-1) - (N-1)\\ \times \text{Re}\left\{1+\frac{2}{\pi} \arc\sin\left(\frac{s-2m^2}{2m^2}\right)\right\} > 0. \quad (9)$$

[12] E. C. Titchmarsh, *Theory of Functions* (The Clarendon Press, Oxford, 1932), p. 186.

[13] See Wightman's lectures on Analytic Functions of Several Complex Variables in *Relations de dispersion et particules élémentaries* (Hermann et Cie, Paris, 1960).

[14] H. Behnke and P. Thullen, *Theorie Der Funktionen Mehrerer Komplexer Veränderlichen* (Verlag Julius Springer, Berlin, 1934), Chap. 4. One can also get the same result by setting Levi's determinant (see reference 13) equal to zero. [M. Froissart (private communication)].

The next step is to enlarge the domain given by (9) using the two-particle unitarity relation. The main point is that it is possible to construct families of functions $K(s,l)$ with the property that the elastic part of the cut is eliminated, a result familiar from continuation to the second energy sheet. One such function is

$$K(s,l)=\frac{1}{T(s,l)}\times\frac{1}{2\pi}\int_{4m^2}^{b}\frac{ds'}{s'-s}\times\left(\frac{s'-4m^2}{s'}\right)^{1/2}\left(\frac{s'-4m^2}{4m^2}\right)^{l}, \quad (10)$$

where the integral is analytic for $\mathrm{Re}l>0$. Since the part of the branch cut in the s plane from $s=4m^2$ to $s=b$ is absent in $K(s,l)$, as can easily be checked using (7), we can now replace $4m^2$ by b in (8) and use the mapping,

$$w=(2/\pi)\arcsin[(2s-b)/b]. \quad (11)$$

Before applying the analytic completion procedure, one must realize that $K(s,l)$ may have poles. Such isolated poles do not change the conclusion of the theorem, however,[15] and solving for $T(s,l)$ in terms of $K(s,l)$, we get:

The function $T(s,l)$ is meromorphic in the domain given by

$$2\,\mathrm{Re}(l-1)-(N-1)\times\mathrm{Re}\left[1+\frac{2}{\pi}\arcsin\left(\frac{2s-b}{b}\right)\right]>0, \quad (12)$$

when $\mathrm{Re}l>1$ and s is not on the usual cut in the s plane. Combining this with the previous result about the domain of analyticity, we see that the strip in between the lines

$$2\,\mathrm{Re}(l-1)-(N-1)\,\mathrm{Re}\left[1+\frac{2}{\pi}\arcsin\left(\frac{s-2m^2}{2m^2}\right)\right]=0$$

and

$$2\,\mathrm{Re}(l-1)-(N-1)\,\mathrm{Re}\left[1+\frac{2}{\pi}\arcsin\left(\frac{2s-b}{b}\right)\right]-0,$$

is the region where the Regge poles can occur.

IV

Let us summarize some of the more important results of the preceding sections. The double dispersion relation enables one to prove analyticity in the l plane to the right of a certain line $\mathrm{Re}l=N$. It is not possible to say anything more without invoking unitarity. The unitarity limit enables one to extend this domain to that given by (9), and the application of the two-particle unitarity relation further enlarges this region to that given by (12), except for Regge poles. If one could continue this procedure to the inelastic portion of the branch cut, one probably would be able to push the domain of analyticity up to the line $\mathrm{Re}l=1$. At this point, however, such a program seems very difficult to carry out due to our lack of knowledge of inelastic processes.

It is of some interest to note that (9) sets an upper bound on the real part of the angular momentum of a Regge pole in terms of its energy. The relation is not, however, very useful since it contains an undetermined constant.

We now turn to some possible generalization of our results. In the case of scattering of scalar particles, one can divide $R(s,l)$ by the Born term and thereby eliminate the additional singularity due to that term, and the problem reduces to the one treated here with some minor modifications. When one deals with the scattering of particles with unequal masses, however, there are some nontrivial complications. If the double dispersion relation is valid, one can define two functions $R_1(s,l)$ and $R_2(s,l)$ corresponding to the t and u channels, respectively, and in general they can be analytic to the right of two different lines $\mathrm{Re}l=N_1$ and $\mathrm{Re}l=N_2$. The two particle unitarity relations in the s channel satisfied by these functions are:

$$\begin{aligned}&R_1(s+i\epsilon,l)-R_1(s-i\epsilon,l)\\&\quad+i[(s-4m^2)/s]^{1/2}[R_1(s+i\epsilon,l)R_1(s-i\epsilon,l)\\&\qquad+R_2(s+i\epsilon,l)R_2(s-i\epsilon,l)]=0,\\&R_2(s+i\epsilon,l)-R_2(s-i\epsilon,l)\\&\quad+i[(s-4m^2)/s]^{1/2}[R_1(s+i\epsilon,l)R_2(s-i\epsilon,l)\\&\qquad+R_2(s+i\epsilon,l)R_1(s-i\epsilon,l)]=0.\end{aligned} \quad (13)$$

The result (9) is still obtained when applied to each R separately, with N replaced by N_1 and N_2, respectively, and $4m^2$ replaced by the two-particle mass threshold in the s channel. If one further defines a matrix (R), given by

$$(R)\equiv\begin{pmatrix}R_1+R_2 & 0\\ 0 & R_1-R_2\end{pmatrix}, \quad (14)$$

the unitarity relation (7) can be written in the familiar form in terms of (R). Then the line of reasoning that led to (12) can be repeated, and (12) remains valid if N is replaced by the larger of N_1 and N_2 and $4m^2$ is replaced by the proper two-particle threshold in the s channel, provided this threshold is not an anomalous one.

[15] Here we are assuming that the analytic completion procedure previously used for a domain of holomorphy equally well applies to a domain of meromorphy. This has been proved by Kneser with the condition that the boundary of the domain of meromorphy be twice differentiable in a piece-wise fashion. [M. Froissart (private communication)]. Also, in the case of a finite number of poles, we can consider the intersection of the original domain with a domain of the form $|K(s,l)|\leq M$ (again a natural domain of holomorphy), where the positive constant M can be taken arbitrarily large. Unfortunately, the theorem has not apparently been proved in all generality. It is the feeling of the author that our discussion covers the physically interesting cases.

The situation for particles with spin is not clear, and to the author's knowledge, no clear-cut mathematical definition of complex angular momentum has so far been given in this case.

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APPENDIX

Here, we want to sketch very briefly the generalization of Froissart's result to complex values of t, since the argument is a quite straightforward extension of that given in reference (10). Exactly as in (10), analyticity in the t plane yields the following upper bound for the partial waves:

$$|a_l(s)|\leqq\frac{C}{s-4m^2}s^N\left[1+\frac{8m^2}{s-4m^2}+\left[\left(1+\frac{8m^2}{s-4m^2}\right)^2-1\right]^{1/2}\right]^{-l}, \quad \text{(A1)}$$

which, for large s, goes over to

$$|a_l(s)|\leqq Cs^{N-1}\exp[-2l(4m^2/s)^{1/2}]. \quad \text{(A2)}$$

On the other hand, the unitarity relations imply

$$|a_l(s)|\leqq 1. \quad \text{(A3)}$$

In the expansion

$$f(s,t)=\sum_{p=0}^{\infty}(2l+1)a_l(s)P_l(z), \quad \text{(A4)}$$

convergent for z inside the ellipse with foci at ∓ 1 and semimajor axis equal to $1+8m^2/(s-4m^2)$, we use the bound (A2) for $l>(s/8m^2)^{1/2}[\ln C+(N-1)\ln s]$ and bound (A3) otherwise. Combining this with the estimates:

$$|P_l(z)|\leqq D\max\{|[z+(z^2-1)^{1/2}]^{\pm l}|\}, \quad \text{(A5)}$$

or

$$|P_l(z)|\leqq D|\exp[2l(t/s)^{1/2}]|, \quad \text{(A6)}$$

again valid for large s, we obtain:

$$f(s,t)\leqq Es^{\{1+(N-1)\,\mathrm{Re}(t/4m^2)^{1/2}+\epsilon\}}. \quad \text{(A7)}$$

Here ϵ is an arbitrarily small positive quantity and it is used to get rid of polynomials in $\ln(s)$ irrelevant for our purpose. The result given in (A7) is of course only valid in the ellipse of convergence of (A4).