PHYSICAL REVIEW VOLUME 127, NUMBER 5 SEPTEMBER 1, 1962

Quantization of the Yang-Mills Field*

R. L. Arnowitt and S. I. Fickler†‡
Syracuse University, Syracuse, New York

(Received March 12, 1962)

Recent efforts in quantum field theory have given rise to increased interest in nonlinear, gauge-type field theories. In this paper, we examine the Yang-Mills field, which is a theory of this type, which lies between electrodynamics and general relativity in complexity. The quantum Yang-Mills field is introduced to satisfy the requirement of invariance under isotopic phase transformations of the second kind. The theory is then put into its first-order form, so as to make it amenable to quantization by the methods of the Schwinger action principle. Quantization is then carried out for the two different gauge conditions; the first (which is the analog of the radiation gauge in electrodynamics) leads to a perturbation treatment of the constraints of the theory and the second to a rigorous solution of them. Two conditions for a consistent quantization are investigated. These are (1) the requirement that the Lagrange equations be identical to the Heisenberg equations of motion (the latter being evaluated using the field commutation relations) and (2) the requirement that the constraint equations be conserved in time (which leads to a condition on double commutators in the q-number theory). In the radiation gauge, these conditions are shown to be satisfied in lowest-order perturbation theory. In the second gauge used, they are rigorously satisfied. It is also shown for that gauge that the field Hamiltonian is positive definite when coupling to Fermion isotopic currents is ignored. No investigation of consistency conditions arising from Lorentz covariance requirements are made.

I. INTRODUCTION

In recent years, there has been a revival of interest in the possibility that "gauge-type" mesons (arising from the requirement of phase invariance of the second kind) play an important role in elementary particle interactions. Thus, Sakurai[1] has introduced three such strongly interacting vector mesons coupled to the heavy-particle number, strangeness, and isotopic spin currents, respectively. Mesons with similar properties have been suggested by Gell-Mann.[2] Further, some of the observed two- and three-pion resonances represent possible candidates for such particles. The heavy-particle number and strangeness mesons are similar (except, perhaps, in Sakurai's assumption of a mass) to the Maxwell field. The isotopic meson was first introduced by Yang and Mills[3] (with essentially the same purpose in mind as in Sakurai's work).

The quantization of a theory possessing a gauge invariance involves overcoming a number of specifically *quantum* problems not found in the reduction of the corresponding classical theory to canonical form. Due to the presence of constraint variables (arising from the gauge invariance) along with the arbitrariness of other variables (due to the freedom of gauge transformations), one does not know, *a priori*, whether the Heisenberg equations (explicitly evaluated using the equal-time commutation relations) and the Lagrange equations are consistent. Similarly, the consistency between the constraint equations and the dynamical ones depends upon the form of the commutation relations (as will be discussed below). Further, the transition to a q-number theory involves a certain loss of gauge invariance since the field equations and commutation relations will, in general, not be invariant under q-number gauge transformations. On the other hand, a Lorentz transformation generally involves a concomitant q-number gauge transformation, and so the Lorentz covariance of the theory is not manifest. While attention has been paid to these problems in electrodynamics,[4] a similar discussion does not exist for the Yang-Mills theory.[5] Though the type of difficulties here are similar to those in electrodynamics, the situation is more complex due to the basic nonlinearity of the field. The latter arises from the fact that the meson carries isotopic spin and hence the field acts as its own source. In this respect, the theory is nonlinear in a manner similar to, but simpler than, general relativity. It would therefore seem to be an excellent area for testing procedures of quantization of gauge invariant theories before applying them to the more complex gravitational field.

In Sec. II, the isotopic triplet Yang-Mills (Y-M) field, $b_\mu{}^a(x)$ (where $\mu=0, 1, 2, 3$ is the spatial index and $a=1, 2, 3$, is the isotopic index), is introduced by the usual arguments involving phase transformations of the second kind. The analysis is performed for the q-number fields. The method of quantization adopted

* This work was supported by the Aeronautical Research Laboratories, Wright-Patterson Air Force Base, Ohio. Partial results of this paper were presented previously: Bull. Am. Phys. Soc. **5**, 283 (1960); *ibid.* **6**, 59 (1961).

† Part of this material appeared in a doctoral thesis (by S. I. F.) submitted in partial fulfillment of the requirements for the Ph.D. degree at Syracuse University.

‡ Now a National Academy of Science Resident Research Associate at Aeronautical Research Laboratories, Wright-Patterson Air Force Base, Ohio.

[1] J. J. Sakurai, Ann. Phys. (New York) **11**, 1 (1960).

[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[3] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954). The heavy particle meson was first discussed by C. N. Yang and T. D. Lee, Phys. Rev. **98**, 1501 (1955).

[4] See, for example, B. Zumino, J. Math. Phys. **1**, 1 (1960).

[5] Recently, one of us (S. I. F.) and R. P. Kerr have shown the Lorentz covariance of the $b_3=0$ gauge (see Sec. V) for both the free and coupled Yang-Mills field, Bull. Am. Phys. Soc. **7**, 298 (1962).

is that of the Schwinger action principle,[6] which brings forth the consistency requirements most clearly. Section III is devoted to the form of the action principle for the free Y-M field and the consistency conditions to be examined. Canonical quantization cannot be carried out without invoking a gauge condition. Two conditions are examined: The radiation gauge, $\nabla\cdot\mathbf{b}^a=0$ in Sec. IV and the gauge $b_3{}^a=0$ in Sec. V. The questions of the consistency between the Lagrange equations, Heisenberg equations, and commutation relations and also whether the constraint equations are preserved in time by the dynamical equations are examined in both gauges. To analyze those points, one needs the constraint variables expressed in terms of the canonical ones. In the radiation gauge, this can only be done in a perturbation analysis, and the above consistency requirements are verified to lowest order. Also, the Hamiltonian is shown to be positive definite to second order. The second gauge allows a rigorous solution of the constraint equations, and the consistency requirements are verified rigorously. Here, the positive definiteness of the Hamiltonian is rigorously established. No attempt is made in this paper to examine requirements arising from Lorentz covariance conditions, nor is any investigation made of the deeper consistency questions involved in whether a local field theory in a Hilbert space is actually well defined.

II. THE QUANTUM YANG-MILLS FIELD

In this section[7] we give a brief description of the basic properties of the quantum Yang-Mills field. The field is introduced by the usual requirement of phase invariance of the second kind for isotopic spin rotations. For this purpose, let us consider the free-particle, Dirac Lagrangian density for the nucleon field[8]:

$$\mathfrak{L}_D=-\frac{1}{4}\left[\bar{\psi},\gamma^\mu\frac{1}{i}\partial_\mu\psi\right]+\frac{1}{4}\left[\frac{1}{i}\partial_\mu\bar{\psi},\gamma^\mu\psi\right]-(m/2)[\bar{\psi},\psi]. \quad (2.1)$$

In Eq. (2.1), $\psi(x)$ and $\bar{\psi}(x)=\psi^\dagger(x)\gamma^0$ are nucleon two component isotopic spinors, and γ^μ are Dirac matrices obeying $\{\gamma^\mu,\gamma^\nu\}=-2\eta^{\mu\nu}$. (Thus, $\gamma^{i\dagger}=-\gamma^i$, $\gamma^{0\dagger}=\gamma^0$). The commutators have been introduced into the quantum Lagrangian $\mathfrak{L}_D$ to correctly describe the Fermi statistics obeyed by the nucleon field. Let us now consider an infinitesimal isotopic spin transformation on the field variables:

$$\begin{aligned}\delta\psi&=ig\omega_a(x)\tau^a\psi\\ \delta\bar{\psi}&=-ig\omega_a(x)\bar{\psi}\tau^a,\quad a=1,2,3.\end{aligned} \quad (2.2)$$

Here, τ^a are the usual 2×2 isotopic spin matrices, $\omega_a(x)$ are three arbitrary infinitesimal *c-number functions*, and the constant g is the "isotopic charge." The Lagrangian $\mathfrak{L}_D$ is not invariant under the transformation (2.2). One finds that

$$\delta\mathfrak{L}_D=-\tfrac{1}{2}g\partial_\mu\omega_a[\bar{\psi},\gamma^\mu\tau^a\psi], \quad (2.3)$$

where the c-number nature of ω_a has explicitly been used. The Yang-Mills field may be introduced by imposing the condition that the *total* Lagrangian be invariant under these general isotopic spin rotations. This may be accomplished by introducing an auxiliary Hermitian field $b_\mu{}^a(x)$ and adding an interaction term to the Lagrangian of a form similar to that arising in Eq. (2.3):

$$\mathfrak{L}_I=\tfrac{1}{4}g\{b_\mu{}^a,[\bar{\psi},\gamma^\mu\tau^a\psi]\}. \quad (2.4)$$

The anticommutator has been introduced in Eq. (2.4) to guarantee the Hermiticity of $\mathfrak{L}_I$ [9] (i.e., we will see that $b_\mu{}^a$ does not generally commute with the Dirac fields due to the existence of constraints in the system).

The isotopic transformation properties of the $b_\mu{}^a$ field is now determined by the requirement that the sum $\mathfrak{L}_D+\mathfrak{L}_I$ be invariant under isotopic rotations

$$\delta\mathfrak{L}_D+\delta\mathfrak{L}_I=0. \quad (2.5)$$

Writing the general transformation law for $b_\mu{}^a$ as

$$\delta b_\mu{}^c=U_{abc}b_\mu{}^a\omega^b+V_{bc}\partial_\mu\omega^b, \quad (2.6)$$

condition (2.5) determines the factors U_{abc} and V_{bc}. Thus using Eqs. (2.2) and (2.6) and the fact that the coefficients of ω^b and $\omega^b{}_{,\mu}$ in Eq. (2.5) must vanish separately, one finds

$$\delta b_\mu{}^c=2gb_\mu{}^a\omega^b\epsilon_{abc}+\omega^c{}_{,\mu}, \quad (2.7)$$

where ϵ_{abc} is the totally antisymmetric three-dimensional tensor (with $\epsilon_{123}=+1$).

It remains now to determine the free Lagrangian density $\mathfrak{L}_B$ for the $b_\mu{}^a(x)$ field. In general, one may write

$$\mathfrak{L}_B=\mathfrak{L}_B(b_\mu{}^a,b_\mu{}^a{}_{,\nu}). \quad (2.8)$$

The requirement that the total $\mathfrak{L}$ be invariant under the "isotopic gauge-phase" transformations implies that $\mathfrak{L}_B$ is a gauge scalar:

$$\delta\mathfrak{L}_B\equiv\frac{\partial\mathfrak{L}_B}{\partial b_\mu{}^a}\delta b_\mu{}^a+\frac{\partial\mathfrak{L}_B}{\partial b_\mu{}^a{}_{,\nu}}\delta b_\mu{}^a{}_{,\nu}=0. \quad (2.9)$$

Since $b_\mu{}^a(x)$ and $b_\mu{}^a{}_{,\nu}(x)$ are quantum fields and the $\delta b_\mu{}^a$ of Eq. (2.7) are q-numbers (even though $\omega^a(x)$

[6] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953). A brief survey of the action principle is given in R. Arnowitt and S. Deser (unpublished).

[7] The material in this section is a quantum treatment of the method introduced by Utiyama for classical field theories: R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[8] Natural units $\hbar=c=1$ will be used throughout. Also, Greek indices run from 0 to 3, and Latin indices (for both isotopic and spatial indices) run from 1 to 3 except where otherwise noted (i.e., in Sec. V). The signature of the Lorentz metric $\eta^{\mu\nu}$ is $-+++$. The symbols $\partial_\mu A(x)$ and $A(x)_{,\mu}$ both mean $\partial A(x)/\partial x^\mu$. Isotopic indices will be written as superscripts or subscripts, depending on which is typographically more convenient.

[9] Other Hermitian possibilities for $\mathfrak{L}_I$ exist, e.g., $\frac{1}{2}(\bar{\psi}\gamma^\mu b_\mu{}^a\tau^a\psi+\text{H.c.})$. Such a structure destroys the antisymmetrization of the fermion fields and is not commonly considered in corresponding analyses in electrodynamics.

are c numbers), the derivatives in Eq. (2.9) must be given the appropriate quantum ordering. Thus, the variations in Eq. (2.9) are to be inserted in the position where the corresponding $b_\mu{}^a$ and $b_\mu{}^a{}_{,\nu}$ stood in the nonvaried Lagrangian. This implies that $(\partial\mathcal{L}_B/\partial b_\mu{}^a)\delta b_\mu{}^a$ is shorthand for $\mathcal{L}_B(b_\mu{}^a+\delta b_\mu{}^a,\, b_\mu{}^a{}_{,\nu})-\mathcal{L}_B(b_\mu{}^a,b_\mu{}^a{}_{,\nu})$. A similar meaning is assumed for the other partial derivatives arising below. One may now introduce Eq. (2.7) into Eq. (2.9). Three equations result, from the vanishing of the coefficients of ω^b, $\omega^b{}_{,\mu}$, and $\omega^b{}_{,\mu\nu}$:

$$(\partial\mathcal{L}_B/\partial b_\mu{}^c)b_\mu{}^a\epsilon_{abc}+(\partial\mathcal{L}_B/\partial b_\mu{}^c{}_{,\nu})b_\mu{}^a{}_{,\nu}\epsilon_{abc}=0 \quad (2.10)$$

$$\partial\mathcal{L}_B/\partial b_\mu{}^b+2g(\partial\mathcal{L}_B/\partial b_\nu{}^c{}_{,\mu})b_\nu{}^a\epsilon_{abc}=0 \quad (2.11)$$

$$\partial\mathcal{L}_B/\partial b_\mu{}^c{}_{,\nu}+\partial\mathcal{L}_B/\partial b_\nu{}^c{}_{,\mu}=0. \quad (2.12)$$

Equation (2.12) states that the derivatives of $b_\mu{}^a$ appear only in the antisymmetrized form. Alternately, one can then require that these derivatives appear in $\mathcal{L}_B$ through the Hermitian, antisymmetric tensor $f_{\mu\nu}{}^c$:

$$f_{\mu\nu}{}^c(x)=b_\mu{}^c{}_{,\nu}-b_\nu{}^c{}_{,\mu}-g(b_\mu{}^ab_\nu{}^b-b_\nu{}^ab_\mu{}^b)\epsilon_{abc}, \quad (2.13)$$

and thus $\mathcal{L}_B$ may now be viewed as a function of $b_\mu{}^a$ and $f_{\mu\nu}{}^a$, i.e., $\mathcal{L}_B=\mathcal{L}_B(b_\mu{}^a,f_{\mu\nu}{}^a)$. From the change of $b_\mu{}^a$ under the gauge transformation, i.e., Eq. (2.7), one finds directly that

$$\delta f_{\mu\nu}{}^c=2gf_{\mu\nu}{}^a\omega^b\epsilon_{abc}, \quad (2.14)$$

i.e., $f_{\mu\nu}{}^c$ is an isotopic vector for c number $\omega^b(x)$. Using $b_\mu{}^a$ and $f_{\mu\nu}{}^a$ as the independent variables, Eq. (2.11) reads

$$\frac{\partial\mathcal{L}_B}{\partial b_\mu{}^b}+\frac{1}{2}\frac{\partial\mathcal{L}_B}{\partial f_{\sigma\rho}{}^a}\frac{\partial f_{\sigma\rho}{}^a}{\partial b_\mu{}^b}=2g\frac{\partial\mathcal{L}_B}{\partial f_{\mu\nu}{}^d}b_\nu{}^a\epsilon_{abc}. \quad (2.15)$$

Again, the definition concerning the position of quantities in the quantum derivatives appearing in Eq. (2.15) should be kept carefully in mind. Using Eq. (2.13), one may carry out the indicated differentiation of $f_{\sigma\rho}{}^a$ appearing in Eq. (2.15). Equation (2.15) then becomes simply: $\partial\mathcal{L}_B/\partial b_\mu{}^b=0$. Thus, one has the well-known result that $\mathcal{L}_B$ depends on the Yang-Mills field only through the combination $f_{\mu\nu}{}^a$. Finally, inserting these results into Eq. (2.10), one finds

$$(\partial\mathcal{L}_B/\partial f_{\mu\nu}{}^a)\delta f_{\mu\nu}{}^a=0, \quad (2.16)$$

which merely says that $\mathcal{L}_B$ is an isotopic scalar formed from $f_{\mu\nu}{}^a$. The simplest gauge and Lorentz invariant choice for $\mathcal{L}_B$ is, of course, the quadratic function:

$$\mathcal{L}_B=-\tfrac{1}{4}f_{\mu\nu}{}^af^{\mu\nu}{}_a. \quad (2.17)$$

The above discussion shows that when due care is taken, the standard classical formulas hold for the quantum Yang-Mills field. However, one distinction should be emphasized. While the classical theory is invariant under all gauge transformations including those where $\omega^a(x)$ is any functional of the dynamical variables $b_\mu{}^a(x)$, $\psi(x)$, and $\bar\psi(x)$, in the quantum theory such $\omega^a(x)$ will be q numbers and the theory will not, in general, be invariant under q-number gauge transformations. Indeed, the derivations of this section break down at all stages for q-number $\omega^a(x)$. Hence, the quantum theory has a much smaller gauge group than the classical one. Similar phenomena, of course, occur also in quantum electrodynamics. However, the situation is more aggravated in the Yang-Mills theory. Thus, in electrodynamics, for c-number gauge functions $\omega(x)$, the change of the vector potential $\delta A_\mu=\omega_{,\mu}$ is a c number. However, from Eq. (2.7) one sees that $\delta b_\mu{}^a$ is not a c number, even if the $\omega^a(x)$ are. One may thus expect more complicated quantum ordering questions to arise even when one restricts oneself (as we shall do here) to c-number gauge transformations.

III. ACTION PRINCIPLE FORMALISM AND CONSISTENCY REQUIREMENTS

We examine now the quantization of the Y-M field by means of the Schwinger action principle.[6] The beginning of this section is devoted to a brief summary of the fundamental results of the technique. The general analysis is then applied to the explicit case of the Y-M field and the consistency conditions mentioned before are deduced.

Let $|a'\sigma_1\rangle$ and $|b''\sigma_2\rangle$ be two eigenkets of two complete sets of operators on the respective space-like surfaces σ_1 and σ_2. The basic postulate of the action principle is that

$$\delta\langle a'\sigma_1|b''\sigma_2\rangle=i\langle a'\sigma_1|\delta W_{12}|b''\sigma_2\rangle, \quad (3.1)$$

where W_{12} is the quantum action integral,

$$W_{12}=\int_{\sigma_2}^{\sigma_1}d^4x\,\mathcal{L}(x). \quad (3.2)$$

The Lagrangian density $\mathcal{L}(x)$, which must be Hermitian, is taken to be in the generalized Kemmer or "first-order" form where derivatives appear linearly:

$$\mathcal{L}=\tfrac{1}{4}[\chi A^\mu\partial_\mu\chi-\partial_\mu\chi A^\mu\chi]-K(\chi_a)+\partial_\mu W^\mu(\chi_a). \quad (3.3)$$

In Eq. (3.3), χ is a column symbol whose components $\chi_a(x)$ are the field variables (in first-order form), A^μ are a set of numerical matrices (i.e., generalized Kemmer matrices), $K(\chi_a)$ is a function of the fields (but not their derivatives) containing the mass and interaction terms, and $\partial_\mu W^\mu$ represents the usual arbitrary divergence one is free to add to a field Lagrangian. On the left-hand side of Eq. (3.1) the variations to be considered are the changes of the kets under the infinitesimal unitary transformations of changes of bases. Thus, let the varied ket $|\bar a'\sigma\rangle$ equal $U^{-1}|a'\sigma\rangle$, where $U\simeq 1+iG$, and $G=G^\dagger$ is the generator of the unitary transformation. Two types of unitary transformations may be considered. First, one may make changes of the complete set keeping the surface fixed. We denote the generators of such changes by G_χ. Second, the dynamical motion of the system (corresponding to rigid translations and rotations of σ) is also a unitary transformation (whose generator we will call G_x). One has, then, in general, that

$$\delta\langle a'\sigma_1|b''\sigma_2\rangle = i\langle a'\sigma_1|G(\sigma_1)-G(\sigma_2)|b''\sigma_2\rangle, \quad (3.4)$$

where $G(\sigma)=G_\chi(\sigma)+G_x(\sigma)$. For local systems, $G(\sigma)$ must depend only on the fields on surface σ and hence comparison with Eq. (3.1) yields the quantum Hamilton's principle:

$$\delta W_{12}=G(\sigma_1)-G(\sigma_2). \quad (3.5)$$

Equation (3.5) says that the variation of the action depends only on the endpoints. The explicit variations of W_{12} to be carried out are similar to those used in classical physics. One makes an "intrinsic" field variation, $\delta_0\chi_a(x)$ (which corresponds to the classical virtual displacement) and a coordinate variation, $\delta x^\mu(x)$. The $\delta x^\mu(x)$ are arbitrary c numbers except that on the endpoint surfaces, σ_1 and σ_2, they are restricted to be rigid motions:

$$\delta x^\mu(\sigma_{1,2})=\epsilon^\mu{}_{(1,2)}-\epsilon^\mu{}_{\nu(1,2)}x^\nu, \quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu}. \quad (3.6)$$

Carrying out the above variations[6] leads to the usual Lagrange equations associated with Lagrangian (3.3) (from the requirement that the variations in the interior vanish). The end point terms allow one to determine the form of $G(\sigma)$. One finds

$$G(\sigma)=\int d\sigma_\mu[\chi A^\mu\bar{\delta}\chi+\bar{\delta}W^\mu]+\int d\sigma_\mu\delta x_\alpha T^{\mu\alpha}, \quad (3.7a)$$

where

$$\bar{\delta}\chi\equiv\delta_0\chi+\delta x^\mu\partial_\mu\chi+\tfrac{1}{2}i(\partial_\mu\delta x_\nu)S^{\mu\nu}\chi \quad (3.7b)$$

and $S^{\mu\nu}$ are the usual spin matrices which arise from the Lorentz transformation properties of the field $\chi(x)$. The quantity $\bar{\delta}\chi_a$ must be an infinitesimal c number for the integral spin Bose fields (which have antisymmetric A^μ matrices) and anticommute with field variables for the Fermi fields[10] (where the A^μ are symmetric). In Eq. (3.7a) $T^{\mu\nu}$ is the standard symmetric stress tensor of classical physics except that in cubic or higher terms, the order of the operators is determined by the order chosen in the quantum $\mathfrak{L}$ of Eq. (3.3). If one considers only dynamical motions, $G=G_x$. For this case one can show that $\bar{\delta}\chi$ vanishes, i.e., by Eq. (3.6),

$$\delta_0\chi=-\epsilon_\alpha\partial^\alpha\chi-\tfrac{1}{2}i\epsilon_{\alpha\beta}[i^{-1}(x^\alpha\partial^\beta-x^\beta\partial^\alpha)+S^{\alpha\beta}]\chi \quad (3.8)$$

and hence $G_x=\epsilon_\alpha P^\alpha+\frac{1}{2}\epsilon_{\alpha\beta}J^{\alpha\beta}$ (where P^α and $J^{\alpha\beta}$ are the usual expressions for energy momentum and angular momenta, respectively). Further, if one defines the change of an operator under a unitary transformation by $\Delta\chi_a\equiv-U^{-1}\chi_aU+\chi_a=-i[\chi_a,G]$, it can be shown that $\Delta\chi=\delta_0\chi$ for this situation, i.e., $i\delta_0\chi=[\chi_a,G_x]$. Equation (3.7) then gives rise to the usual formulas:

$$[\chi,P^\alpha]=-i\partial^\alpha\chi, \quad (3.9a)$$

$$[\chi,J^{\alpha\beta}]=[-i(x^\alpha\partial^\beta-x^\beta\partial^\alpha)+S^{\alpha\beta}]\chi. \quad (3.9b)$$

Equations (3.9) represent the general field commutator equations of motion. Turning next to transformations leaving σ fixed, i.e., $\delta x^\mu=0$, one has

$$G=G_\chi=\int d\sigma_\mu(\chi A^\mu\delta_0\chi+\delta_0W^\mu).$$

For the simple generator with $W^\mu=0$, the requirement $\Delta\chi=\frac{1}{2}\delta_0\chi$ and hence,

$$\left[\chi_a,\int d\sigma_\mu\,\chi A^\mu\delta_0\chi\right]=\tfrac{1}{2}i\delta_0\chi_a, \quad (3.10)$$

leads to the usual canonical commutation relations. The changes generated by G_χ with $W^\mu\neq0$ can then be determined from Eq. (3.10) [since Eq. (3.10) generally gives rise to a complete set of commutation relations].

The action principle thus yields three separate elements: the Lagrange equations, the Heisenberg Eqs. (3.9), and Eq. (3.10). If the canonical commutation relations arising from Eq. (3.10) are used to evaluate the commutators of Eq. (3.9), explicit equations of motion will result. These must be identical with the Lagrange equations determined from $\mathfrak{L}$. This represents the fundamental consistency condition imposed by the action principle. It is, in fact, the basic origin of the condition $\Delta\chi=\frac{1}{2}\delta_0\chi$ [of Eq. (3.10)] imposed for usual theories. The consistency of these three elements will be investigated (in Secs. IV and V) for the Y-M field, where the situation is more complicated due to the constraints arising from the gauge invariance. In classical theory, Eqs. (3.9) and (3.10), of course, hold with the appropriate replacement of commutators by Poisson brackets (P.B.). Indeed, the operations carried out in finding the canonical variables of the classical field (and then replacing their elementary P.B.'s by commutators) is in one-to-one correspondence with the steps involved in finding the commutation relations by the action principle. One advantage involved in using the action principle arises from the fact that it requires one to carry out the conventional analysis completely within the quantum framework, and so a consistent ordering of operator factors, as described by the order chosen in $\mathfrak{L}$, is adopted throughout.

We now examine the above formalism for the special case of the free (uncoupled) Y-M field. Two possible gauge-invariant, Hermitian Lagrangians giving rise to first-order equations of motion are available. These are[11]

$$\mathfrak{L}_1=-\tfrac{1}{4}[\{b_\nu;\partial_\mu f^{\mu\nu}\}-\{\partial_\mu b_\nu;f^{\mu\nu}\}]+\tfrac{1}{4}[f_{\mu\nu}\cdot f^{\mu\nu}+4gb_\nu\cdot(f^{\mu\nu}\times b_\mu)], \quad (3.11a)$$

[10] J. Schwinger, Proc. Nat. Acad. Sci. 44, 223, 617 (1958) and reference 6.

[11] The ambiguity arises, of course, from the cubic term in $\mathfrak{L}$. In general, one may form three independent Hermitian orderings from a product of three operators. Since the cubic structure here is bilinear in b_μ, two of the forms are identical.

and

$$\mathcal{L}_2=-\tfrac{1}{4}[\{b_\nu;\partial_\mu f^{\mu\nu}\}-\{\partial_\mu b_\nu;f^{\mu\nu}\}] +\tfrac{1}{4}[f_{\mu\nu}\cdot f^{\mu\nu}+2g\{f^{\mu\nu};(b_\mu\times b_\nu)\}]. \quad (3.11b)$$

The "dot" and "cross" refer to isotopic indices, e.g.,

$$\{b_\mu;b_\nu\}\equiv\{b_\mu{}^a,b_\nu{}^a\} \quad (3.12)$$

and

$$\{b_\mu,\times b_\nu\}_c\equiv\{b_\mu{}^a,b_\nu{}^b\}\epsilon_{abc}. \quad (3.13)$$

On *a priori* grounds, $\mathcal{L}_1$ and $\mathcal{L}_2$ are equally acceptable and hence, in principle, two different quantum Y-M systems may exist. We shall see below, however, that both $\mathcal{L}_1$ and $\mathcal{L}_2$ yield essentially the same theory (though $\mathcal{L}_1$ is a slightly better defined operator).

Both Lagrangians are in the standard form of Eq. (3.3) where χ is the thirty-component column symbol, $\chi=(b_\mu{}^a,f_{\mu\nu}{}^a)$. If one were to explicitly introduce a matrix notation, one would find that the A^μ matrices are antisymmetric. Consequently, the variations $\bar{\delta}b_\mu$ and $\bar{\delta}f_{\mu\nu}$ are c numbers for this Bose system. Performing the variations (remembering that b_μ and $f_{\mu\nu}$ are treated as independent variables), both Lagrangians give rise to the same Lagrange field equations

$$\partial_\nu f^{\mu\nu}=j^\mu\equiv g\{f^{\mu\nu},\times b_\nu\}, \quad (3.14a)$$

$$f_{\mu\nu}=b_{\mu,\nu}-b_{\nu,\mu}-g\{b_\mu,\times b_\nu\}, \quad (3.14b)$$

and the same generator G_χ (with $W^\mu=0$):

$$G_\chi=-\frac{1}{2}\int d\sigma_\nu[\bar{\delta}b_\mu\cdot f^{\mu\nu}-\bar{\delta}f^{\mu\nu}\cdot b_\mu]. \quad (3.15)$$

Here, j^μ is the isotopic current vector of the Y-M field itself, the two Lagrangians are distinguished by the fact that they give rise to different stress tensors and hence *a priori* different Hamiltonians. Since, to within an irrelevant divergence, T^{00} is the negative of $\mathcal{L}$ when the explicit time derivative terms in $\mathcal{L}$ have been deleted, one may easily find for the two Hamiltonians:

$$P^0{}_1=\frac{1}{4}\int d^3r(2f_{0k}\cdot f_{0k}-f_{jk}\cdot f_{jk}-2\{b_{k,j};f_{jk}\} -4gb_k\cdot(f_{jk}\times b_j)-2g[b_0;[f_{0k},\times b_k]]) \quad (3.16)$$

$$P^0{}_2=\frac{1}{4}\int d^3r(2f_{0k}\cdot f_{0k}+f_{jk}\cdot f_{jk} -2g[b_k;[b_0,\times f_{0k}]]). \quad (3.17)$$

In arriving at (3.16) and (3.17), some use of the Lagrange equations (3.14) was made to simplify the expressions.

Not all of the Lagrange equations are dynamical, i.e., involve time derivatives and hence not all give information as to how the system continues off an initial Cauchy surface. The constraint equations (those *independent* of time derivatives) instead give interrelations between the field variables at a fixed time; they allow one to eliminate redundant variables in favor of a minimum set of independent dynamical variables. The constraint equations of (3.14b) arise when one chooses $\mu=i$ and $\nu=j$. They allow one to eliminate f_{ij} in favor of b_i (and its *spatial* derivatives) in a fashion analogous to the relation $\mathbf{B}=\nabla\times\mathbf{A}$ in electrodynamics.[12] The constraint equations from Eq. (3.14a) arise when $\mu=0$:

$$\nabla\cdot\mathbf{E}-g\{E^i,\times b_i\}=0. \quad (3.18)$$

Here, $\mathbf{E}_a=(f_a{}^{01},f_a{}^{02},f_a{}^{03})$ is a spatial (and isotopic) vector. As in the analogous electromagnetic relation $(\nabla\cdot\mathbf{E}-\rho=0)$, Eq. (3.17) may be viewed as a differential equation to eliminate one of the (spatial) components of $\mathbf{E}_a$. The differential constraint (3.18) arises due to the gauge invariance in the theory. Their presence produces the following complication: Eq. (3.18) must hold for all time and hence one is free to take its time derivative. But on explicitly doing this, the time derivatives of $\mathbf{E}$ (and of $\mathbf{b}$) may be eliminated by the dynamical equations of (3.14). This gives rise to a new (nondynamical) equation[13] which must clearly be consistent with the previous ones, i.e., the constraint equations must be consistent with the dynamical ones. The calculation is most simply performed by taking the four-divergence of Eq. (3.14a) since this relation reads

$$\partial_0(\partial_i f^{0i}-j^0)=-\partial_i(\partial_\nu f^{i\nu}-j^i). \quad (3.19)$$

Equation (3.19) says that if the dynamical equations hold at a fixed time (i.e., the right-hand side vanishes) then the time derivative of the constraint [i.e., left-hand side of Eq. (3.18)] will vanish (which is just the consistency requirement discussed above). From the antisymmetry of $f^{\mu\nu}$, Eq. (3.19) reduces to the continuity equation for j^μ:

$$0=\partial_\mu j^\mu=g\{\partial_\mu f^{\mu\nu},\times b_\nu\}+g\{f^{\mu\nu},\times\partial_\mu b_\nu\}. \quad (3.20)$$

For the classical theory, Eq. (3.20) is an identity as a consequence of Eqs. (3.14). This is not *a priori* the case in the quantum theory. One finds upon eliminating derivatives by Eqs. (3.14), that

$$[f^{\mu\nu}{}_a,[b_\mu;b_\nu]]+[b^\mu{}_a,[b^\nu;f_{\mu\nu}]] +[b^\nu{}_a,[f_{\mu\nu};b^\mu]]=0. \quad (3.21)$$

Thus, the consistency between the constraint and dynamical equations in the quantum theory implies

[12] The existence of these so-called algebraic constraints is due to the requirement of Lorentz covariance. Thus, as will be seen below, in a particular Lorentz frame, two of the components of f^{0k} are independent dynamical variables needed to specify the state of the system on an initial Cauchy surface. In other Lorentz frames clearly the other components of the Lorentz tensor $f^{\mu\nu}$ will enter (though they won't be independent). Thus, constraints of this type also appear in the massive vector meson theory (which possesses no gauge invariance).

[13] Note that this phenomenon does not arise in the simpler constraints of Eq. (3.14b) since the time derivatives of f_{ij} appear nowhere in the dynamical equations, and hence no new relations arise from taking time derivatives.

conditions on the commutation relations.[14] Equations (3.21) thus constitute an additional check on the consistency of the quantization.

IV. QUANTIZATION IN THE RADIATION GAUGE

In order to verify whether the consistency requirements discussed in the previous section are indeed satisfied, one needs the explicit form of the canonical commutation relations. In principle, these are to be obtained from Eq. (3.10) with G_X given explicitly by Eq. (3.15). The possibility of performing gauge transformations, however, complicates the analysis, since then some quantities can be changed arbitrarily and thus do not represent dynamical variables. In a linear theory, such as the free Maxwell field, the gauge-variant quantities automatically cancel out of G_X (and other relevant parts of the theory), leaving only the independent dynamical variables. However, this is not the case for the Y-M field, which, due to the appearance of the self-interaction structures, has a form more akin to coupled electrodynamics. As in that case, well-defined commutation relations are obtained only after a gauge condition has been imposed.

From general considerations, a gauge condition involves specifying one condition on the field $\mathbf{b}$ (or more precisely three when the isotopic index is not suppressed). Thus, one component of $\mathbf{b}$ is specified throughout space-time. The equation of the time derivative of that component [obtained from Eq. (3.14b)] then may be used to determine b_0. One is thus left with two independent components of $\mathbf{b}$ and (as was seen in the previous section) two independent components of $\mathbf{E}$. These four variables (or including the isotopic labels, 12 variables) are to be arranged into two conjugate pairs for the purposes of quantization. In this section, quantization using the radiation gauge will be considered. A complete set of commutation relations cannot be obtained here since the constraint equations cannot be solved in closed form (to explicitly give the dependent variables in terms of the canonical ones). In the next section, a gauge will be exhibited allowing a rigorous solution of the constraint equations.

In the radiation gauge, it is convenient to divide spatial vectors into their transverse and longitudinal parts, $\mathbf{V}=\mathbf{V}^T+\mathbf{V}^L$; here the transverse vector $\mathbf{V}^T$ obeys $\nabla\cdot\mathbf{V}^T\equiv 0$ and the longitudinal part, $\mathbf{V}^L$, is the gradient of a scalar (and so $\nabla\times\mathbf{V}^L=0$). Thus, the gauge condition

$$\nabla\cdot\mathbf{b}_a=0, \tag{4.1}$$

implies that $\mathbf{b}$ is a pure transverse vector: $\mathbf{b}_a{}^L=0$. Choosing the surface σ to be perpendicular to the time axis, the generator G_X of Eq. (3.15) becomes

$$G_X=\frac{1}{2}\int d^3r[(-\mathbf{E}^T)\cdot\delta_0\mathbf{b}^T-\mathbf{b}^T\cdot\delta_0(-\mathbf{E}^T)]. \tag{4.2}$$

Decomposition of Eqs. (3.14) into transverse and longitudinal parts leads to the dynamical equations

$$b_k{}^T{}_{,0}=E_k{}^T-g\{b_0,\times b_k{}^T\}^T, \tag{4.3a}$$

$$-E_k{}^T{}_{,0}=f_{jk,j}+g\{b_0,\times E_k\}^T-g\{b_j,\times f_{jk}\}^T, \tag{4.3b}$$

the constraint equations,

$$f_{ij}=b_i{}^T{}_{,j}-b_j{}^T{}_{,i}-g\{b_i{}^T,\times b_j{}^T\} \tag{4.4a}$$

$$E_k{}^L{}_{,k}=g\{E_k{}^L,\times b_k{}^T\}+g\{E_k{}^T,\times b_k{}^T\}, \tag{4.4b}$$

the condition that determines b_0 (now that the gauge is fixed),

$$0=b_k{}^L{}_{,0}=b_{0,k}+E_k{}^L-g\{b_0,\times b_k{}^T\}^L, \tag{4.5}$$

and the equation for the time derivative of $\mathbf{E}^L$

$$E_k{}^L{}_{,0}=-g\{b_0,\times E_k\}^L+g\{b_j,\times f_{jk}\}^L. \tag{4.6}$$

If one writes $\mathbf{E}^L=\nabla\phi$, one sees that Eq. (4.4b) should, in principle, determine ϕ (and hence $\mathbf{E}^L$) while Eq. (4.5) can then be used to determine b_0. That Eq. (4.6) is consistent with these solutions [when time derivatives are eliminated by Eqs. (4.3)] is, of course, condition (3.21) in this gauge. Further, $\mathbf{b}^T$ and $-\mathbf{E}^T$ are canonically conjugate, as can be seen from the form[15] of G_X. In fact Eqs. (3.10) read explicitly

$$[\mathbf{b}^T(\mathbf{r}',t),G_X]=\tfrac{1}{2}i\delta_0\mathbf{b}^T(\mathbf{r}',t),$$
$$[\mathbf{E}^T(\mathbf{r}',t),G_X]=\tfrac{1}{2}i\delta_0\mathbf{E}^T(\mathbf{r}',t),$$

which lead directly to

$$[b_j{}^{aT}(\mathbf{r},t),b_k{}^{bT}(\mathbf{r}',t)]=0=[E_j{}^{aT}(\mathbf{r},t),E_k{}^{bT}(\mathbf{r}',t)], \tag{4.7a}$$

$$[b_j{}^{aT}(\mathbf{r},t),E_k{}^{bT}(\mathbf{r}',t)]=-i\delta_{ab}\delta_{jk}{}^T(\mathbf{r}-\mathbf{r}')$$
$$=-i\delta_{ab}\left(\delta_{jk}\delta^3(\mathbf{r}-\mathbf{r}')+\partial_j\partial_k\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right). \tag{4.7b}$$

The quantity $\delta_{jk}{}^T(\mathbf{r})$ is the usual transverse δ function appearing in electrodynamics. In fact, relations (4.7) are identical to those of three independent electromagnetic fields.

To obtain a complete set of commutation relations, which are needed to verify the consistency conditions of the previous section, one must express the constraint variables in terms of the canonical ones. Equation (4.4a) automatically gives f_{ij} in terms of $b_k{}^T$. Also, one must solve Eqs. (4.4b) and (4.5) for $\mathbf{E}^L$ and b_0. Here the situation differs from even coupled electrodynamics

[14] A similar phenomenon arises in coupled electrodynamics. For classical point particles, the continuity equation, $\partial_\mu j^\mu=0$, is an identity. For classical fields it follows as a consequence of the field equations. But in quantum electrodynamics, it is a consequence of both the field equations and the canonical commutation relations (a relation analogous to Eq. (3.21) also arising).

[15] In general, if G_X can be put in the form

$$\tfrac{1}{2}\int d^3r\ \Sigma(\pi_A\delta_0\phi_A-\phi_A\delta_0\pi_A)$$

then π_A is canonically conjugate to ϕ_A, since such a G_X arises from varying an $\mathcal{L}$ of the form $\frac{1}{2}\ \Sigma(\pi_A\partial_0\phi_A-\phi_A\partial_0\pi_A)-\mathcal{H}(\pi_A,\phi_A)$, where $\mathcal{H}$ is the Hamiltonian density.

in that a closed form solution for these variables is not available.[16] To obtain some (albeit unrigorous) information, we resort now to a perturbation solution of these equations. To first order in the coupling constant, one easily finds

$$b_0{}^{(1)}=-g\int d^3r'\frac{\{b_k{}^T(\mathbf{r}'),\times E_k{}^T(\mathbf{r}')\}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{4.8}$$

and

$$\mathbf{E}^{L(1)}=\boldsymbol{\nabla} b_0{}^{(1)}. \tag{4.9}$$

Equations (4.8) and (4.9) are adequate for evaluating terms up to order g^2 in the Hamiltonians of Eqs. (3.16) and (3.17). Thus, the double commutator term of Eq. (3.16) may be approximated by

$$2g[b_0{}^{(1)};[E_k{}^T,\times b_k{}^T]].$$

A direct evaluation using the canonical commutation relations shows that this term vanishes. A similar analysis for the corresponding term in Eq. (3.17) gives

$$2g[b_k{}^T;[b_0{}^{(1)},\times E_k{}^T]]=4g^2\int d^3r'\frac{[\delta_{jk}{}^T(\mathbf{r}-\mathbf{r}')]^2(\epsilon_{abc})^2}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{4.10}$$

While expression (4.10) is divergent, it is a c number, and hence will not presumably affect the Heisenberg equations of motion. Thus, if one discards this infinite c number, both Hamiltonians reduce to the same value (to order g^2):

$$P^0=P^0{}_1=P^0{}_2=\int d^3r(\tfrac{1}{2}\mathbf{E}^T\cdot\mathbf{E}^T+\tfrac{1}{4}f_{jk}\cdot f_{jk}+\tfrac{1}{2}\mathbf{E}^{L(1)}\cdot\mathbf{E}^{L(1)}) \tag{4.11}$$

where f_{ij} and $\mathbf{E}^L$ are to be expressed in terms of canonical variables by Eqs. (4.4a), (4.8), and (4.9). One may now examine the Heisenberg equations of motion for the independent variables $\mathbf{b}^T$ and $\mathbf{E}^T$:

$$i\mathbf{b}^T{}_{,0}=[\mathbf{b}^T,P^0], \tag{4.12a}$$

$$i\mathbf{E}^T{}_{,0}=[\mathbf{E}^T,P^0]. \tag{4.12b}$$

A straightforward calculation of the right-hand side, using the canonical commutation relations (4.7) shows that Eqs. (4.12) are identical to Eqs. (4.3), to the required order. Note also that to order g^2, the Hamiltonian of Eq. (4.11) is clearly positive definite.

We consider next the second consistency check of Sec. III. To the desired order, Eq. (3.21) reduces to

$$\begin{aligned}[E_k{}^{aT},[b_0{}^{(1)};b_k{}^T]]&+[b_k{}^{aT},[E_k{}^T;b_0{}^{(1)}]]+[b_0{}^{a(1)},[b_k{}^T;E_k{}^T]]\\&+\tfrac{1}{2}[f_{jk}{}^a,[b_j{}^T;b_k{}^T]]+\tfrac{1}{2}[b_k{}^{aT},[f_{jk};b_j{}^T]]\\&+\tfrac{1}{2}[b_j{}^T,[b_k{}^T;f_{jk}]]=0.\end{aligned} \tag{4.13}$$

The last three terms vanish, since by Eqs. (4.4a) and (4.7a), the inner commutators are zero. The inner commutator of the third term is a c number, albeit infinite, and hence the commutator with $b_0{}^{(1)}$ presumably vanishes.[17] Applying the commutation relations to the remaining terms, one obtains:

$$[E_k{}^{aT},[b_0{}^{(1)};b_k{}^T]]=[b_k{}^{aT},[E_k{}^T;b_0{}^{(1)}]]=2g\epsilon_{acc}\int d^3r'\frac{[\delta_{jk}{}^T(\mathbf{r}-\mathbf{r}')]^2}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{4.14}$$

This expression presumably vanishes, because the Levi-Civita ϵ_{abc} symbol is totally antisymmetric. This interpretation of the indeterminate structure assumes, basically, that the δ function of Eq. (4.7b) is to be viewed as the limit of a spread out (c number) function, and that this limit is to be taken at the end. Thus, to the first nonvanishing order, the two consistency requirements in the radiation gauge are satisfied.

V. QUANTIZATION IN THE GAUGE $b_3{}^a=0$

The radiation gauge possesses the drawback that a closed form solution of the constraint equations (for the desired variables) is not obtainable. It is thus not clear whether meaningful operator relationships exist. In this section, we examine an alternate gauge condition[18]:

$$b_3{}^a=0. \tag{5.1}$$

While this gauge is not as esthetically pleasing as the usual radiation gauge, it *does* have the extra advantage of allowing rigorous solutions of the constraint equations in terms of reasonably well-defined operators.

In this section, we shall adopt the convention that spatial indices i, j run over only 1, 2 and explicitly separate out components along the 3 direction. In terms of this notation, the dynamical equations read

$$b_{i,0}=b_{0,i}+E^i-g\{b_0,\times b_i\}, \tag{5.2a}$$

$$-E^i{}_{,0}=f_{ji,j}+f_{3i,3}-g\{E^i,\times b_0\}+g\{b_j,\times f_{ji}\}. \tag{5.2b}$$

The constraint equations are

$$f_{ij}=b_{i,j}-b_{j,i}-g\{b_i,\times b_j\}, \tag{5.3a}$$

$$f_{i3}=b_{i,3}=-f_{3i}, \tag{5.3b}$$

$$E^3{}_{,3}=-E^i{}_{,i}+g\{E^i,\times b_i\}, \tag{5.3c}$$

while the condition determining b_0 (arising from the requirement that $b_{3,0}=0$) is simply

$$-E^3=b_{0,3}. \tag{5.4}$$

The final field equation is

$$E^3{}_{,0}=f_{3j,j}+g\{E^3,\times b_0\}+g\{b_j,\times f_{3j}\}. \tag{5.5}$$

[16] J. Schwinger [Phys. Rev. **125**, 1043 (1962); **127**, 324 (1962)] has recently obtained a formal solution to these equations.

[17] In explicitly verifying that $\partial_\mu j^\mu$ vanishes for the analogous electromagnetic case (see reference 13), one must similarly assume that the commutator of A_0 with an infinite c number vanishes.

[18] This gauge was first considered for the Maxwell field (in another connection) by J. Anderson.

The condition that Eq. (5.5) be consistent with the other field equations leads to Eq. (3.21) in this gauge. The generating function G_X now has the form

$$G_X=\frac{1}{2}\int d^3r[\delta_0 b_i\cdot(-E^i)-b_i\cdot\delta_0(-E^i)] \tag{5.6}$$

which shows that b_i and $-E^i$ $(i=1,2)$ now are the canonically conjugate variables.[15] Condition (3.10) now gives rise to the following canonical commutation relations:

$$[b_i{}^a(\mathbf{r},t),b_j{}^b(\mathbf{r}',t)]=0=[E_i{}^a(\mathbf{r},t),E_j{}^b(\mathbf{r}',t)], \tag{5.7a}$$

$$[b_i{}^a(\mathbf{r},t),\ -E_j{}^b(\mathbf{r}',t)]=i\delta_{ab}\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}'). \tag{5.7b}$$

Equations (5.3) and (5.4) now allow one to eliminate the dependent variables f_{ij}, f_{i3}, E^3, and b_0 from the theory. Thus, Eqs. (5.3a) and (5.3b) explicitly express f_{ij} and f_{i3} in terms of the canonical coordinates b_i. The other equations have the solution

$$E^3(\mathbf{r},t)=\int_{-\infty}^{z} dz'[-E^i{}_{,i}(\mathbf{r}',t)+j^0(\mathbf{r}',t)] \tag{5.8}$$

and

$$b_0(\mathbf{r},t)=\int_{-\infty}^{z} dz'\int_{-\infty}^{z'} dz''[E^i{}_{,i}(\mathbf{r}'',t)-j^0(\mathbf{r}'',t)], \tag{5.9}$$

where $\mathbf{r}'$ means (x,y,z'). From Eq. (3.14a), one has that

$$j^0=g\{E^i,\times b_i\} \tag{5.10}$$

in this gauge, and hence depends only on the canonical variables (recalling that $i,j=1,2$ in this section). The remaining commutation relations can now be found from the canonical ones of Eqs. (5.7).

Having obtained a complete set of commutation relations, we consider next the Heisenberg equations of motion. The double commutation structure in Eq. (3.16) vanishes. The corresponding term in Eq. (3.17) becomes

$$2g[b_j;[b_0,\times E^j]]$$
$$=-48g(z-z)\theta(z-z)\delta^2(x-x)\delta^3(\mathbf{r}-\mathbf{r}), \tag{5.11}$$

where $\delta^2(x)$ means $\delta(x)\delta(y)$ and $\theta(z)$ is the step function. Though the right-hand side of Eq. (5.11) is indeterminate, it is a c number and presumably will not affect the Heisenberg equations. Thus, if one discards this structure, the two Hamiltonians become identical and take the form

$$P^0=P^0{}_1=P^0{}_2=\int d^3r[\tfrac{1}{2}E^i\cdot E^i+\tfrac{1}{2}E^3\cdot E^3$$
$$+\tfrac{1}{2}f_{i3}\cdot f_{i3}+\tfrac{1}{4}f_{ij}\cdot f_{ij}], \tag{5.12}$$

where f_{ij}, f_{i3}, and E^3 are given by Eqs. (5.3a), (5.3b), and (5.8), respectively. The validity of the Heisenberg equations can now be explicitly verified. Thus, using the commutation relations, one may verify that the equations,

$$ib_{i,0}=[b_i,P^0], \tag{5.13a}$$

$$iE^i{}_{,0}=[E^i,P^0], \tag{5.13b}$$

are identical to the Lagrange equations (5.2a) and (5.2b). Note also that Eq. (5.12) shows rigorously that P^0 is a positive definite operator.

Turning to the second consistency requirement, Eq. (3.21) becomes, in this gauge:

$$-[E_j{}^b,[b_j{}^a,b_0{}^a]]$$
$$-[b_j{}^b,[b_0{}^a,E_j{}^a]]-[b_0{}^b,[E_j{}^a,b_j{}^a]]$$
$$+\tfrac{1}{2}[f_{ij}{}^b,[b_i{}^a,b_j{}^a]]+\tfrac{1}{2}[b_i{}^b,[b_j{}^a,f_{ij}{}^a]]$$
$$+\tfrac{1}{2}[b_j{}^b,[f_{ij}{}^a,b_i{}^a]]=0. \tag{5.14}$$

Utilizing the same arguments as those presented for Eq. (4.13), one finds that the last four terms vanish by virtue of Eqs. (5.3a), and (5.7), and the two remaining terms become

$$[E_j{}^b,[b_j{}^a,b_0{}^a]]=3i(z-z)\theta(z-z)\{[E_j{}^b,\partial_j\delta^2(x-x)]$$
$$+2g\delta^3(\mathbf{r}-\mathbf{r})\delta^2(x-x)\epsilon_{aba}\} \tag{5.15}$$

and

$$[b_j{}^b,[b_0{}^a,E_j{}^a]]$$
$$=2g(z-z)\theta(z-z)\delta^2(x-x)\delta^3(\mathbf{r}-\mathbf{r})\epsilon_{aba}. \tag{5.16}$$

If one employs the previously mentioned interpretation of the indeterminate structures, both of these expressions presumably vanish, either because the commutator of an operator with a c number appears or because the Levi-Civita ϵ_{abc} symbol is totally antisymmetric.

VI. CONCLUSIONS

In the preceding sections, some of the purely quantum difficulties that arise in attempts to set up a consistent quantum Yang-Mills theory have been discussed. In particular, the consistency between the Heisenberg equations (explicitly evaluated using the commutation relations) and the Lagrange equations was examined, as was the question of consistency between the constraint equations and the dynamical ones. Both of these points arise basically from the gauge invariance of the theory. Thus, the second condition (which is a rephrasing of the fact that the continuity equation must be satisfied by the isotopic current vector due to the antisymmetry of $f_{\mu\nu}$) does not even occur for theories without a gauge group. Similarly, the consistency between the Heisenberg and Lagrange equations (which is satisfied as a tautology in the classical theory) is much less obvious in the quantum gauge theories than in nongauge theories, due to the complexity of the constraints. This is particularly true in a nonlinear theory where the ordering of factors becomes more complicated.

In order to verify that the above consistency conditions are satisfied, it is necessary to have a complete set of commutation relations. To obtain these, one must impose a gauge condition. Two such conditions were examined: the radiation gauge ($\nabla\cdot\mathbf{b}^a=0$) and the gauge $b_3{}^a=0$. The radiation gauge leads to canonical commutation relations similar to those of electrodynamics. It has, however, the drawback that the constraint equations cannot be solved rigorously. The above consistency requirements were shown, however, to be satisfied in lowest-order perturbation theory. The gauge where $b_3{}^a=0$ is more complex than the radiation gauge due to the loss of three-space rotational invariance. This gauge does have the advantage of affording a rigorous solution of the constraint equations and hence a complete verification of the consistency conditions.[19] It might also be mentioned that in this gauge, the constraint variable E^3 (and b_0) depend only linearly on the canonical variables and linearly on the isotopic current operator [see Eqs. (5.8) and (5.9)]. One might hope, then, that E^3 and b_0 are operators with well-defined matrix elements.

For a nonlinear theory such as the Yang-Mills field, the imposition of a gauge condition is a nontrivial operation. This is due to the fact mentioned earlier (in Sec. II), that the theory is invariant, in general, under only c-number gauge transformations. Two different q-number related gauges, will, more likely than not, represent two physically different theories[20] (with different predictions for cross sections, etc.). As may easily be seen, the radiation gauge and $b_3{}^a=0$ gauge are indeed q-number related and so the theories of Secs. IV and V may have different physical content. Should this be the case, and should both gauges be Lorentz invariant, one would presumably have to resort to experiment to decide which gauge was correct. In electrodynamics, a valid gauge is the radiation gauge.[21] While the theoretical origin of this result is not clear, the radiation gauge *does* possess a preferred position in electrodynamics. It is the gauge in which the total vector potential A_μ equals the gauge-invariant part of A_μ (and hence is the gauge where the gauge-variant part of A_μ has been set to zero). Due to the more complicated nature of the Yang-Mills gauge transformation [Eq. (2.7)], neither of the gauges considered in this paper have this property. It would be of interest to discover the nature of the analogous preferred gauge for the Yang-Mills field.

[19] One of us (S.I.F.) has investigated in this gauge the case of the Yang-Mills field coupled to the nucleon field [Bull. Am. Phys. Soc. 7, 80 (1962)]. The consistency conditions can be completely verified here also.

[20] However, the gauge transformations associated with Lorentz transformations have q-number parameters and a valid theory must be invariant under these.

[21] It is not known whether or not, in a given Lorentz frame, there exist other experimentally correct gauges in electrodynamics that are q-number related to the radiation gauge.

PHYSICAL REVIEW VOLUME 127, NUMBER 5 SEPTEMBER 1, 1962

Suggested Quantum Numbers for Bosons of Mass $\approx 4m_\pi$†

E. M. Henley and B. A. Jacobsohn
Physics Department, University of Washington, Seattle, Washington
(Received April 30, 1962)

The decay modes of the ζ ($T=1$) and η ($T=0$) bosons of mass $\approx 4m_\pi$ are discussed, together with other experiments bearing on quantum number assignments for these mesons. Using an effective interaction which includes the influence of angular momentum barriers, we have made numerical estimates of branching ratios. Arguments based on these estimates lead to the most likely spin and parity assignments of $J^{PG}=0^{+\pm}$ for ζ and 0^{--} for η. Study of the reaction $\pi+\mathrm{He}\to\zeta+\mathrm{He}$ is proposed as a test of the 0^+ assignment.

RECENTLY, renewed evidence has been presented for an isospin 1 resonance at about 565 MeV $\approx 4m_\pi$.[1] We present arguments below in support of a suggestion that the quantum numbers for this resonance (ζ) are either $J^{PG}=0^{++}$ or 0^{+-}. We furthermore discuss the relationship of the neutral component ζ^0 to the η which has $T=0$ and decays into three pions, and reexamine the quantum numbers for the latter.

Since the η and the ζ^0 have about the same mass, all presently existing experimental discussions of the decay modes refer to some combination of the two particles. For brevity, in this paper we use the name X for those phenomena which refer to whatever mixture of η and ζ^0 has been measured.

The arguments which we have used to arrive at the above assignment for the ζ are based on the following pieces of experimental evidence: (1) The decay of the ζ^+ is primarily into $\pi^+\pi^0$ with a width less than 15 MeV.[1,2] (2) The branching ratio for the X produced in $K^-+p\to\Lambda+X$ is $\lesssim 1/20$ for $\pi^+\pi^-$ as well as for $\pi^+\pi^-\gamma$, as indicated by the absence of these modes in the data of Bastien *et al.*[3] (3) This same X has[3] $\Gamma(\text{neutrals})/\Gamma(\pi^+\pi^-\pi^0)\approx 3/1$. (4) The experiment of

† Supported in part by the U. S. Atomic Energy Commission.

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