Analysis of Lattice and Ionized Impurity Scattering in p-Type Germanium*†

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The scattering of holes in p-type germanium by acoustical and optical phonons, and by ionized impurities has been studied over a wide range of temperature from 7 to 300°K, and for impurity concentrations from 10^{13} to 10^{17} cm^{-3}. The mobility was analyzed on the basis of the following approximations: (1) The light- and heavy-hole valence bands are parabolic with spherical constant-energy surfaces and effective-mass ratios of 0.043 and 0.35; (2) the relaxation time for lattice scattering is the same for the heavy and light holes; and (3) the ionized impurity scattering is describable by the Brooks-Herring formula with the scattering limited to intraband transitions. The following results were obtained: (1) The expectation was substantiated that the lattice scattering at $T \leq 70$°K can be attributed to acoustical phonon interactions; the mobility $\mu_{ac}=3.37\times10^{7}\, T^{-3/2}$ cm^2/V-sec was obtained by treating it as the sole adjustable parameter in the range 10 to 70°K; (2) the stronger $T^{-2.3}$ dependence of mobility in the range 120 to 300°K can be attributed to the advent of optical mode scattering if the ratio of optical to acoustical mode coupling constants $(\mathcal{E}_{op}/\mathcal{E}_{ac})^2$ is chosen to be 3.8; (3) the Brooks-Herring formula describes the ionized impurity scattering very well from 30 to 300°K for impurity concentrations $N_I \leq 10^{15}$ cm^{-3}, but *overestimates* the mobility for higher N_I or lower T. The eventual failure of the analysis may be attributed to the neglect of carrier-carrier and interband ionized impurity scattering.

1. INTRODUCTION

THIS work is primarily concerned with the analysis of the scattering mechanisms in p-type germanium from the temperature dependence of the Hall mobility. The existing information on this subject is surprisingly incomplete. Only two aspects of the hole mobility can be said to have been investigated in any detail. One, the strong dependence of the Hall mobility and the magnetoresistance on magnetic field strength, has been studied[1–4] intensively, primarily in terms of the peculiarities of the valence-band structure of germanium as determined by cyclotron resonance experiments. However, these studies have not particularly contributed to the elucidation of the scattering mechanisms. In fact, the complete analysis of the field dependence of the Hall mobility awaits the clarification of the strength and energy dependence on the various scattering mechanisms for holes. The second aspect is the temperature dependence of the mobility in the case of pure lattice scattering, which has been shown by several investigators[5–7] to be given by the empirical relationship $\mu \propto T^{-2.3}$ in the range 120 to 300°K. The deviation from a $T^{-3/2}$ dependence indicated that a simple acoustical mode scattering mechanism is inadequate to describe the mobility.

Ehrenreich and Overhauser,[8] and Harrison,[9] showed that a combination of optical and acoustical mode scattering could account for the strong temperature dependence. However, their calculations were sensitive to the value chosen for the optical phonon energy which was not known at that time. Once this value was established by neutron scattering experiments,[10] Conwell[11] was able to fit the empirical temperature dependence, in a simplified treatment, with *two* adjustable parameters—the magnitudes of the coupling constants for acoustical and optical mode scattering. The description of the lattice scattering in this manner is not unique, but several aspects of it are subject to verification. In particular, the optical mode scattering should become negligible below about 70°K.

One purpose of the present work was to investigate the low temperature region to determine if the existence of a $T^{-3/2}$ dependence of lattice mobility, characteristic of acoustical mode scattering, could be established. At low temperature, the lattice mobility is so high that some influence of ionized impurity scattering even in our purest samples had to be taken into account. Although Brooks[12] has discussed how to treat impurity scattering in p-type germanium, no comprehensive analysis of experimental data and test of the theory has been presented in the literature for this material.

In the present work, the magnitudes of the acoustical and optical phonon scattering parameters were determined, by making each the *sole* adjustable parameter in appropriate temperature ranges. With the use of these parameters, it was possible to determine the range of temperature and impurity concentration over which the Brooks-Herring treatment of ionized impurity scattering gave reasonable agreement with experiment. In the calculation of mobility, many simplifying assumptions were made. These are specified in the next section which briefly discusses the theory.

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† Part of this work was presented at the *International Conference on Semiconductor Physics, Prague, 1960* (Czechoslovakian Academy of Physics, Prague, 1961).
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[1] R. K. Willardson, T. C. Harman, and A. C. Beer, Phys. Rev. **96**, 1512 (1954).
[2] A. C. Beer and R. K. Willardson, Phys. Rev. **110**, 1286 (1958).
[3] A. C. Beer, J. Phys. Chem. Solids **8**, 507 (1959).
[4] C. Goldberg, E. N. Adams, and R. E. Davis, Phys. Rev. **105**, 865 (1957).
[5] W. C. Dunlap, Phys. Rev. **79**, 286 (1950).
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[7] F. J. Morin, Phys. Rev. **93**, 62 (1954).
[8] H. Ehrenreich and A. W. Overhauser, Phys. Rev. **104**, 331, 649 (1956).
[9] W. Harrison, Phys. Rev. **104**, 1281 (1956).
[10] B. N. Brockhouse and P. K. Iyengar, Phys. Rev. **111**, 747 (1958).
[11] E. Conwell, Sylvania Technologist **12**, No. 2 (1959).
[12] H. Brooks, *Advances in Electronics and Electron Physics* (Academic Press Inc., New York, 1955), Vol. 7.

2. THEORY

2.1 General Band Structure Approximations

For the analysis of the transport properties, the valence-band structure is treated in an approximate fashion. The known warping of the bands, which is especially pronounced for the heavy hole band, is neglected and the approximation of spherical constant energy surfaces and parabolic bands is made for both carriers. The effective mass ratios of the heavy and light holes m_h and m_l are taken as 0.35 and 0.043, respectively. These values represent an approximate average over the various directions within the bands as determined by cyclotron resonance measurements.[13] The ratio of heavy- to light-hole concentrations p_h/p_l is then given by $(m_h/m_l)^{3/2}=23.2$. These quantities were assumed to be independent of temperature, in the absence of any evidence to the contrary at the time this work was done.[14]

The total hole mobility is obtained by weighting the mobilities of the heavy and light holes μ_h and μ_l according to their relative populations:

$$\begin{aligned}\mu&=(\mu_l p_l+\mu_h p_h)/(p_l+p_h).\\ \mu&=(\mu_l+23.2\mu_h)/24.2.\end{aligned}\tag{1}$$

The mobility for each band must be calculated separately in terms of the various scattering mechanisms before the individual contributions are combined according to Eq. (1). The assumptions made in treating the various scattering mechanisms and the specific details of the calculations are outlined in the following sections.

2.2 Lattice Scattering

It is assumed that below 70°K, the lattice scattering is by single phonons of the acoustical modes only, and that the lattice mobility is proportional to $T^{-3/2}$. The corresponding relaxation time τ_{ac} is assumed to be isotropic and given by

$$\tau_{ac}=AT^{-1}\epsilon^{-1/2},\tag{2}$$

where T is the lattice temperature, ϵ is the carrier energy, and A is a constant containing the deformation potential $\mathcal{E}_{ac}$ and is to be empirically determined by fitting the experimental mobility data for $T<70$°K. It is the sole adjustable parameter in this range.

Actually, the theory of lattice scattering for holes is quite complicated, involving warped bands and both intra and interband scattering.[8,15] The magnitude and angular dependence of the scattering matrix elements are functions of both the band in which the scattered carrier originates and of the band into which it is scattered, and possibly also of the initial direction of motion of the carrier.[8] Nevertheless, the theoretical studies[8] indicate that a $T^{-3/2}$ dependence for the mobility is still to be expected for scattering by acoustical modes.[15a] It should be noted, however, that even for a simple band structure, there is a low-temperature limit to the validity of a $T^{-3/2}$ mobility dependence. This occurs when the carriers interact with phonons whose energy is not much less than kT, and for which, therefore, equipartition of energy is no longer valid. Then the mobility should increase faster than $T^{-3/2}$ as the temperature decreases. This situation has been discussed by Dorn[16] for a single band, and by Conwell and Brown[17] for doubly degenerate bands, and may become applicable at $T<20$°K. This complication is neglected in the present analysis, which is mainly concerned with mobilities at $T\geq 20$°K.

Above 70°K, optical mode scattering is assumed to become important. This is treated in a simplified manner. Following Conwell,[18] the reciprocal relaxation time for scattering by the optical modes, which includes both phonon emission and absorption processes, can be given by

$$\frac{1}{\tau_{op}}=\frac{1}{\tau_{ac}}\frac{B}{A}\frac{\theta}{T}\left\{n\left(1+\frac{k\theta}{\epsilon}\right)^{1/2}+(n+1)\left(1-\frac{k\theta}{\epsilon}\right)^{1/2}\right\}.\tag{3}$$

where $k\theta$ is the optical phonon energy and $n=(e^{\theta/T}-1)^{-1}$. In using Eq. (3), $k\theta$ is taken to be a constant equal to 0.037 eV, the value of the optical phonon energy at zero wave number as determined from neutron scattering experiments.[10] The ratio (B/A) is equal to $\frac{1}{2}(\mathcal{E}_{op}^2/\mathcal{E}_{ac}^2)$, where $\mathcal{E}_{op}$ and $\mathcal{E}_{ac}$ are the deformation potentials for optical and acoustical mode scattering, respectively. In applying this expression at $T>70$°K, it is assumed that $1/\tau_{ac}$ can be extrapolated from its value determined at lower temperatures. Then (B/A) is empirically determined by fitting the experimental $T^{-2.3}$ dependence of mobility in the temperature range $120\text{°K}<T<300\text{°K}$. It is the *sole* adjustable parameter in this range.

It remains to specify the relative contributions of the light and heavy holes to the mobility. The commonly

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[13] G. Dresselhaus, A. F. Kip, and C. Kittel, Phys. Rev. **98**, 382 (1955); B. Lax and J. G. Mavroides, *ibid.* **100**, 1653 (1955); R. Dexter, H. J. Zeiger, and B. Lax, *ibid.* **104**, 641 (1956).

[14] Very recent cyclotron resonance measurements by D. M. S. Bagguley and R. A. Stradling, Proc. Phys. Soc. (London) **78**, 1078 (1961), indicate that the light-hole mass increases by approximately 16% between 10 and 100°K. However, such an increase in m_l can have only very slight influence on the total mobility, in fact *increasing* it by about 2%.

[15] G. L. Bir and G. E. Pikus, Soviet Phys.—Solid State **2**, 2039 (1961).

[15a] *Note added in proof.* This has been contradicted in a recent study by G. E. Tauber, J. Phys. Chem. Solids **23**, 7 (1962). Using a variational principle, generalized to take into account interband scattering and general energy surfaces, Tauber finds a $T^{-2.5}$ temperature dependence for mobility for acoustical scattering. However, this result is not consistent with our experimental results at $T<70$°K.

[16] D. Dorn, Z. Naturforsch. **12a**, 739 (1959).

[17] E. M. Conwell and A. L. Brown, J. Phys. Chem. Solids **15**, 208 (1960).

[18] E. Conwell, J. Phys. Chem. Solids **8**, 236 (1959).

made assumption is retained that both holes have the same lattice scattering relaxation time.[4,12] This assumption implies that both heavy and light holes are predominantly scattered into the heavy-hole band, where the density of states is much greater. Thus, the lattice relaxation time for the light holes is mainly determined by interband transitions. The basis for the assumption is essentially empirical. It is consistent with the magnitude of the magnetoresistance effects in p-type germanium, and with the fact that the cyclotron resonance lines at low temperature are nearly the same for heavy and light holes.[13] As already mentioned, the theoretical calculations of Ehrenreich and Overhauser[8] show that the situation may actually be much more complex. However, the assumption of equal relaxation times has the advantage of simplicity and some empirical justification; within the framework of the present method of analysis, there is no basis for making a better choice.[19]

The accuracy of the last assumption is completely unimportant in the empirical analysis of pure lattice scattering as long as the temperature dependence of τ is the same for both holes, and the ratio of the τ's for heavy and light holes is the same for optical as for acoustical mode scattering. It does matter, however, in combining lattice with impurity scattering (where the τ's are different for the two holes), and is critically important for the analysis of the magnetic field dependence of the magnetoresistance and Hall effect.

2.3 Impurity Scattering

The Brooks-Herring[12,20] formula was utilized to calculate the relaxation time τ_I for ionized impurity scattering. As is well known, this formula, based on a Born approximation calculation of scattering by a screened Coulomb potential, has a limited range of validity.[20,21] However, calculations based on the Born approximation may give quite good results even when the general criteria for its validity are violated.[22] It has been shown by Blatt,[20] using partial wave analysis, that the Brooks-Herring formula eventually overestimates the impurity scattering and gives excessively low values of mobility, when applied at too low temperatures and/or too high impurity concentrations. However, for impurity concentration $N_I=10^{16}/\text{cm}^3$, Blatt found the Brooks-Herring formula to be in good agreement with the partial wave analysis down to temperatures of about 50°K, at least for repulsive scattering centers. While we cannot simply scale these results to purer samples, it can be expected that for relatively pure samples, the Brooks-Herring formula should be useful to much lower temperatures.

The application of the Brooks-Herring formula to light and heavy holes was restricted to the case of *intraband* scattering, as suggested by Brooks.[12] Then, the relaxation time τ_I is given by[23]

$$\frac{1}{\tau_I{}^i}=\frac{N_I\pi q^4}{K^2(2m_i)^{1/2}\epsilon^{3/2}}\left[\ln(b_i+1)-\frac{b_i}{b_i+1}\right]. \tag{4}$$

where the "i" identifies the particular type of hole, and

$$b_i=\frac{2Km_ikT\epsilon}{\pi\hbar^2q^2p'},\quad p'=p+(p+N_D)\left(1-\frac{p+N_D}{N_A}\right),$$
$$N_I=p+2N_D.$$

N_I is equal to the total concentration of ionized centers; N_D and N_A are the concentrations of ionized donor and acceptor centers; p is the total hole concentration, and K is the dielectric constant.

It is worthwhile to discuss briefly the restriction of the analysis to intraband scattering. If the screening radius r_0 used to describe the screened Coulomb potential is large (i.e., for relatively small impurity concentrations), the scattering probability[24] greatly favors those events which involve small changes in momentum. Then intraband scattering is much more probable than interband scattering which requires relatively large momentum changes. However, when the impurity concentration becomes large and the screening radius small, the scattering probability tends to become relatively independent of the magnitude of the momentum change and interband transitions may become important. From the form of the scattering probability, it can be seen that the condition required to validate the intraband scattering model for ionized impurity scattering is that $(kr_0)^2\gg1$,[25] where k is the carrier wave vector. However, this is also the condition for which the Born approximation (and hence the Brooks-Herring formula) happens to work fairly well.[26]

[19] D. M. S. Bagguley, R. A. Stradling, and J. S. Whiting, Proc. Roy. Soc. (London) **A262**, 340 (1961). This paper, published after the completion of our work, gives results of an extensive cyclotron resonance investigation, including analytical expressions for the reciprocal collision time for several p-type Ge samples. For the purest sample, in the range 2 to 120°K, they give the following results: light hole, $\langle\tau\rangle^{-1}=(0.24\times10^{11}+1.85\times10^8T^{3/2})$ sec^{-1}; heavy hole, $\langle\tau\rangle^{-1}=(0.15\times10^{11}+1.85\times10^8T^{3/2})$sec^{-1}; where the $\langle\tau\rangle$ obtained from the shape of the resonance lines represent a different average over the Boltzmann distribution than does the conductivity relaxation time. It is not clear to what extent $\langle\tau\rangle$ is influenced by other factors than pure lattice scattering and, in particular, to what extent the difference in the constant term between the two types of holes represents impurity scattering effects. Therefore, this new work furnishes no grounds for altering the simple assumption with regard to the equality of the relaxation times.

[20] F. J. Blatt, in *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1957), Vol. 4.

[21] N. Sclar, Phys. Rev. **104**, 1548 (1956).

[22] See, e.g., D. Bohm, *Quantum Mechanics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1951), Chap. 21.

[23] The collision rates published by Brooks (reference 12) and Blatt (reference 20) are a factor of 2 too small. However, their expressions for μ_I are correct. See also R. B. Dingle, Phil. Mag. **46**, 837 (1955).

[24] See e.g., Bohm, reference 22, Chap. 21, p. 537, Eq. 34(a).

[25] This is also the condition $b_i\gg1$ in Eq. (4).

[26] This is actually the condition for the Born approximation cross section to approach the classical Rutherford cross section. Bohm (reference 22) discusses why this apparent contradiction may lead to good results.

Therefore, as long as we restrict the description of the impurity scattering to the Brooks-Herring formula, it is not worthwhile to consider interband scattering. However, this restriction may limit the range of applicability of the analysis.

To test empirically the range of utility of the Brooks-Herring formula, it was applied down to 7°K for a relatively pure sample ($N_I \simeq 10^{13}/\text{cm}^3$), to 20°K for several less pure samples ($N_I < 2\times10^{15}/\text{cm}^3$), and at 77 and 300°K for samples with $N_I \leq 10^{17}/\text{cm}^3$.

Neutral impurity scattering has not been included in the analysis because for most of the samples investigated it should be quite small compared to lattice and ionized impurity scattering. Also, carrier-carrier scattering has been completely neglected in this work. (See, however, the discussion in Sec. 6.2.)

2.4 Calculation of Mobility

Because the τ_I's for light and heavy holes are different, the collision frequencies for lattice and ionized impurity scattering have to be added for each band separately, and averaged over the Maxwell-Boltzmann distribution function in the usual manner.[26a] The general expression for the mobility in a particular band is then given by

$$\mu_i = \frac{e}{m_i}\frac{\langle v^2\tau^i\rangle}{\langle v^2\rangle} = \frac{e}{m_i}\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{x^{3/2}e^{-x}dx}{(1/\tau_{ac}+1/\tau_{op}+1/\tau_I{}^i)}. \quad (5)$$

where $x=\epsilon/kT$.

At temperatures below 77°K, where optical mode scattering is unimportant, Eq. (5) was not evaluated directly; instead the mobilities for lattice and ionized impurity scattering were separately calculated for each band and then combined according to a prescription of Conwell[27] which involves a small approximation in the treatment of the log term in Eq. (4). In all other cases, the integration of Eq. (5) was carried out graphically with no approximations. Finally, the total hole mobility for both bands was obtained using Eq. (1).

When the impurity scattering can be neglected, the total hole mobility is given by

$$\mu_{ac+op} = \mu_{ac}\int_0^\infty \frac{xe^{-x}dx}{1+C[(1+\theta/Tx)^{1/2}+e^{\theta/T}(1-\theta/Tx)^{1/2}]}, \quad (6)$$

where $C=(B/A)(\theta/T)/(e^{\theta/T}-1)$, and μ_{ac} is the total mobility for acoustical mode scattering. Since the value of μ_{ac} is obtained from the low-temperature analysis, the ratio (B/A) is the only adjustable parameter.

3. EXPERIMENTAL TECHNIQUES

Hall and resistivity measurements were made with standard dc potentiometric techniques and circuitry. Etched samples, rectangular in shape, with ohmic soldered end contacts were used. Typical finished dimensions of such samples were 1 mm×3 mm×14 mm. The resistivity and electrical uniformity of each sample was determined at 77°K with the aid of a movable point probe while the sample was immersed in liquid nitrogen. This technique was employed to accurately determine the electrical length between small soldered probes, which were used for resistivity measurements at other temperatures. Resistivities of uniform material, measured by this procedure, were found to be reproducible to within 1%.

All Hall coefficients were measured with pressure contact Hall probes, consisting of sharply pointed 5-mil tungsten wires. The use of such small probes eliminated the errors due to the shorting effect of larger soldered probes on the sample current in the probe region.[5,28,29] The reproducibility of Hall coefficient measurements was within 1% at different positions on any sample of uniform resistivity.

Measurements of Hall coefficient and resistivity vs temperature were made in conventional gas-exchange cryostats. Liquid helium coolant was used for measurements below 15°K, liquid hydrogen for the range 13 to 77°K, and liquid nitrogen for the range 77 to 300°K.

4. DETERMINATION OF LATTICE SCATTERING PARAMETERS

To determine the temperature dependence of mobility due to lattice scattering below 70°K, an extremely pure sample is required to minimize the impurity scattering contribution. The purest sample available to us (Ge 112A) had a carrier concentration $(N_A-N_D)=1.25\times10^{13}/\text{cm}^3$. This sample was thoroughly investigated in the range 7 to 300°K. Above 100°K, impurity scattering was negligible. Below 100°K, corrections for ionized impurity scattering were necessary, but the lattice scattering was dominant over most of the measured temperature range.

4.1 Experimental Data

Figure 1 shows the experimental data for sample Ge 112A. The resistivity was determined between 7 and 300°K, while the Hall coefficient was measured only at $T\leq77$°K at a magnetic field strength of 7000 G. To determine the conductivity mobility and carrier concentration from the data, it is necessary to know the ratio of Hall to conductivity mobility, μ_H/μ. Since

[26a] *Note added in proof.* It should be noted that the usual assumption that collision frequencies for phonon and impurity scattering processes are independent of each other and therefore additive has been criticized recently by H. Reiss and A. I. Anderman [Phys. Rev. **122**, 1135 (1961)], and H. L. Frisch and J. L. Lebowitz [Phys. Rev. **123**, 1542 (1961)]. They present a reformulation of the problem for some special cases.

[27] E. Conwell, Proc. Inst. Radio Engrs. **40**, 1327 (1952).

[28] W. C. Dunlap, *An Introduction to Semiconductors* (John Wiley & Sons, Inc., New York, 1957).

[29] M. Glicksman, J. Phys. Chem. Solids **8**, 511 (1959).

the Hall mobility is very large, $\mu_H > 40\,000$ cm²/V-sec at temperatures below 77°K, it is expected that the strong field approximation is applicable at $H = 7000$ G, in which case the ratio $\mu_H/\mu = 1$.[2,30] In support of this conclusion, the Hall mobility was found to be independent of H in the range 1 to 7 kG at 20°K, and varied by only 5% in this range at 77°K. Also, a value of $\mu_H/\mu = 1$, and, therefore, $p = 1/qR_H$, is consistent with the constancy of the Hall data above 35°K where exhaustion should prevail for such a pure sample.

At temperatures above 77°K, the mobility decreases and it may no longer be valid to take $\mu_H/\mu = 1$. In this range, the temperature dependence of mobility is obtained directly from the conductivity curve in the range where the hole concentration should be constant, i.e., below the onset of intrinsic generation at $T \geq 260$°K. For $T > 260$°K, the mobility is extrapolated to 300°K to give a value of 2060 cm²/V-sec, some 5% higher than the highest drift mobilities which have been reported on less pure samples.[6]

In the range 100 to 300°K, μ can be represented by $1.02 \times 10^9 T^{-2.3}$ cm²/V-sec in good agreement with previous determinations.[6,7] At lower temperatures, the slope of the mobility data decreases drastically, falling below that of a $T^{-3/2}$ dependence as represented by the labeled solid line in Fig. 1. However, it becomes necessary here to correct for impurity scattering.

In order to make theoretical corrections for ionized impurity scattering, the compensating donor concentration N_D was determined by applying the dissociation equation to the Hall data in the range of carrier freeze-out at low temperature where $p \ll N_D$. Then,

$$p = [(N_A - N_D)/gN_D]N_v \exp(-E_I/kT). \quad (7)$$

In the analysis, the degeneracy g of the ground state of the low-lying acceptor levels was taken equal to 4.[31] The total density of states in the two valence bands is N_v, equal to $1.05 \times 10^{15} T^{3/2}$/cm³ on the basis of the approximations and constants given in Sec. 2.1. $N_A - N_D$ was set equal to 1.25×10^{13}/cm³, the value of $1/qR_H$ at 77°K at $H = 7000$ G. A value of 1.9×10^{12}/cm³ was obtained for N_D, and 0.0106 eV for E_I. The latter value is about midway between the acceptor activation energies of the three elements with the largest segregation coefficients in germanium (Al, 0.0102 eV; B, 0.0104 eV; and Ga, 0.0108 eV).[32] This is reasonable, since this material was not intentionally doped.

4.2 Acoustical Mode Scattering Parameter

Figure 2 shows a comparison of the calculated and experimental mobility temperature curves, in the range 7°K < T < 70°K where only acoustical mode and ionized impurity scattering are taken into account as discussed in Sec. 2.4. The solid curve represents the calculated mobility. The lattice mobility chosen in the calculation to give the best fit to the data is shown by the dashed

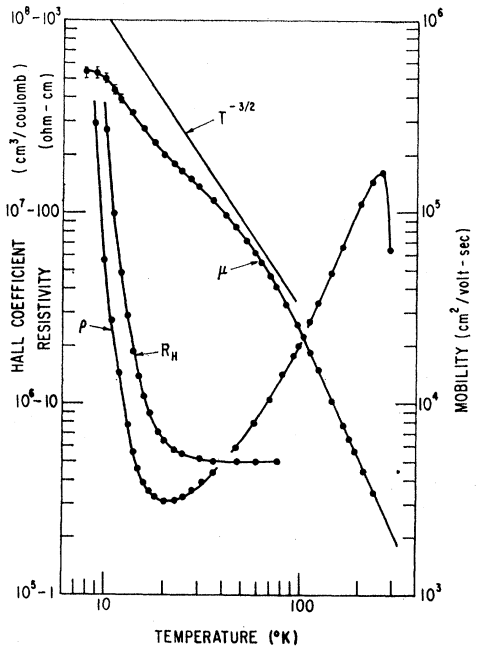

FIG. 1. Experimental Hall, resistivity, and mobility data for sample Ge 112A.

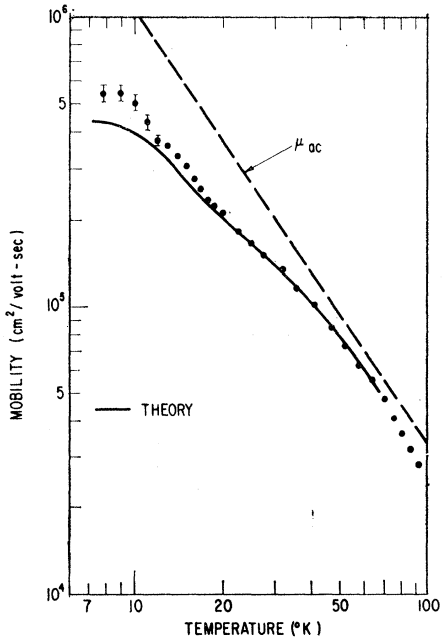

FIG. 2. Comparison of theory and experiment for sample Ge 112A, for $T < 70$°K. Theoretical curve includes acoustical mode lattice scattering and ionized impurity scattering.

[30] J. A. Swanson, Phys. Rev. **99**, 1799 (1955).

[31] W. Kohn, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1957), Vol. 5, p. 303.

[32] T. H. Geballe and F. J. Morin, Phys. Rev. **95**, 1085 (1954).

TABLE I. Summary of parameters.

Used
$m_l=0.043m_e$
$m_h=0.35m_e$
$p_h/p_l=23.2$
$N_v=1.05\times10^{15}T^{3/2}/\text{cm}^3$
Derived
$\mu_{ac}=3.37\times10^7T^{-3/2}$ cm²/V-sec
$(\mu_h)_{ac}=2.60\times10^7T^{-3/2}$ cm²/V-sec
$(\mu_l)_{ac}=2.12\times10^8T^{-3/2}$ cm²/V-sec
$\tau_{ac}=6.92\times10^{-9}T^{-1}\epsilon^{-1/2}$ sec
$\mu_{ac+op}=1.02\times10^9T^{-2.3}$ cm²/V-sec
$(\mathcal{E}_{op}^2/\mathcal{E}_{ac}^2)=3.8$

line labeled μ_{ac} and is expressed by

$$\mu_{ac}=3.37\times10^7T^{-3/2}\ \text{cm}^2/\text{V-sec}. \quad (8)$$

The corresponding heavy- and light-hole components of this mobility, and the value of the relaxation time are summarized in Table I.

The fit of the theory to the experimental data is very good in the temperature range $20°\text{K}\leq T\leq70°\text{K}$. The only appreciable discrepancies appear at low temperatures where the calculated mobility becomes about 20% too low at $T<10°$K. There is some uncertainty in the experimental values below 10°K, as indicated by the vertical lines through the data points. This is due to the fact that the resistivity and Hall coefficient are changing very rapidly with temperature in this range, and the two were not determined at exactly the same temperatures. As for the calculated mobility, any error in determining N_D is strongly reflected in the magnitude of μ when most of the free carriers are frozen out at low temperature. Also, it is expected, as mentioned before, that the Brooks-Herring formula overestimates the impurity scattering at low temperature, yielding too low values for the mobility. Finally, it should be noted that as the temperature decreases the lattice mobility can increase faster than $T^{-3/2}$ when the equipartition assumption eventually fails at low temperature. However, in view of the various simplifying assumptions, and the neglect of many details of the band structure, it might be more appropriate to wonder why the discrepancies here are so small rather than to try to account for them.

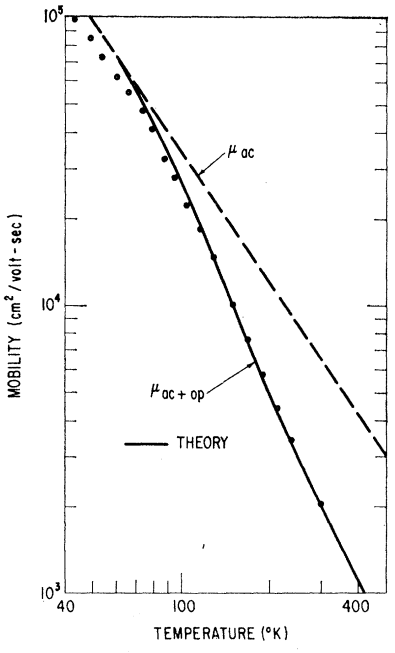

FIG. 3. Comparison of theory and experiment for sample Ge 112A, at $T>100°$K. Theoretical curve includes acoustical and optical mode lattice scattering. Dashed line shows extrapolated mobility for pure acoustical scattering.

4.3 Optical Mode Lattice Scattering

With the assumption that μ_{ac} as given by Eq. (8) can be extrapolated to higher temperature, and that the excess scattering above 70°K can be attributed to optical mode interactions, the magnitude of the latter can be obtained by fitting the combined optical and acoustical mode scattering to the experimental $T^{-2.3}$ behavior. Using Eq. (6), a best fit to the experimental data above 100°K, where ionized impurity scattering is negligible, can be obtained if $(\mathcal{E}_{op}/\mathcal{E}_{ac})^2$ is taken equal to 3.8 (i.e., $B/A=1.9$). The result of this calculation is shown in Fig. 3 by the solid line labeled μ_{ac+op}, which matches the data quite well. At $T<100°$K, the theoretical curve lies somewhat above the data points, probably because the small contribution of ionized impurity scattering has not been included in the calculation. Also, a slightly decreasing slope for the calculated curve may be noted above 250°K. This may be attributed to the fact that the equipartition condition for optical phonons is gradually approached with increasing temperature. Eventually, at $T>\theta$, Eq. (3) should again[33] give a mobility proportional to $T^{-3/2}$. We have not looked for such a decrease in temperature dependence of mobility, but the extension of mobility measurements to high temperatures ($\geq500°$K) might serve as an additional check of the optical phonon scattering hypothesis.

5. DETERMINATION OF THE RANGE OF UTILITY OF THE IMPURITY SCATTERING THEORY

This section will be mainly concerned with the comparison of theoretical and experimental mobilities in samples with various concentrations of impurities.

In analyzing the more impure samples, we are again faced with the problem of determining the ratio μ_H/μ. The following approach was used: (1) For samples with $\mu_H>20\,000$ cm²/V-sec, μ_H/μ was taken equal to 1 for $H\geq3500$ G. This is a reasonable approximation

[33] See C. Herring, Bell System Tech. J. **34**, 237 (1955), Fig. 4 for an analogous situation involving intervalley scattering in n-type Ge.

according to the theoretical analysis of Beer and Willardson.[2] From their work, μ_H/μ exceeds 1 by no more than 5% at 77°K, for $N_I \leq 10^{15}/\text{cm}^3$ and $H > 3000$ G. It is further justified by the fact that the Hall coefficient changes very little between 3550 and 8000 G, and by the temperature independence of the Hall coefficient in that portion of the exhaustion range over which $\mu_H \geq 20\,000$ cm²/V-sec. (2) For more impure samples, and still lower mobilities, appreciable conversion factors are necessary. Estimates of μ_H/μ at 77°K were made using the relation

$$\mu_{77°K} = \mu_{300°K}(\rho_{300°K}/\rho_{77°K})(p_{300°K}/p_{77°K}), \qquad (9)$$

where the resistivity ratio is measured, $\mu_{300°K}$ is assumed to be correctly given by Prince's drift mobility measurements[6] for holes, and $p_{300°K}/p_{77°K}$ is calculated from the dissociation equation (neglecting compensation and changes in impurity activation energy with concentration). Such estimates confirm that μ_H/μ is nearly equal to 1 for the purer samples; it is appreciably larger for the more impure samples. For the latter samples, the accuracy of the conversion factor is fortunately not critical in the analysis of the experiment data, as we shall see from the discussion of Fig. 6 in Sec. 5.2.

TABLE II. Summary of properties of samples at 77 and 300°K.

Sample	$\rho_{77°K}$	$(\mu_H)_{77°K}$	$\rho_{300°K}$	$\mu_{300°K}$[a]	$(N_A - N_D)$
112A	12.2	41 200	(Intrinsic)	2050	1.25×10^{13}
19B	1.35	30 300[b]	20.2	1900	1.52×10^{14}
214A	1.05	34 000	16.6	1940	1.81×10^{14}
8A	0.79	29 900[b]	12.7	1800	2.65×10^{14}
128	0.28	22 600	3.2	1980	9.75×10^{14}
21C	0.215	19 500[b]	2.32	1800	1.51×10^{15}
106C	0.161	18 300	1.55	1800	2.23×10^{15}

[a] Derived as described in text.
[b] At $H = 3550$ G, the others at 8000 G.

5.1 Temperature Dependence of Mobility Below 77°K for a Range of Impurity Concentrations

Comparison of the theoretical and experimental variation of mobility with temperature in the range 15 to 77°K for several samples (see Table II) with $N_I < 2\times10^{15}/\text{cm}^3$ is shown in Fig. 4. In addition, some published but unanalyzed Hall mobility data by Tyler *et al.*[34] were incorporated in this study. Mobility data for some of their purer samples are compared with theoretical curves in Fig. 5. In both figures, the pure lattice mobility, μ_{ac} determined in Sec. 4.2, is shown by the dashed line. In the calculation of mobility, compensation was neglected. This is reasonable for samples all intentionally doped with gallium, at least at $T > 20$°K where freezeout is not yet very strong.

[34] W. W. Tyler, H. Woodbury and T. J. Soltys, in review by R. Newman and W. W. Tyler, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1959), Vol. 8, p. 68.

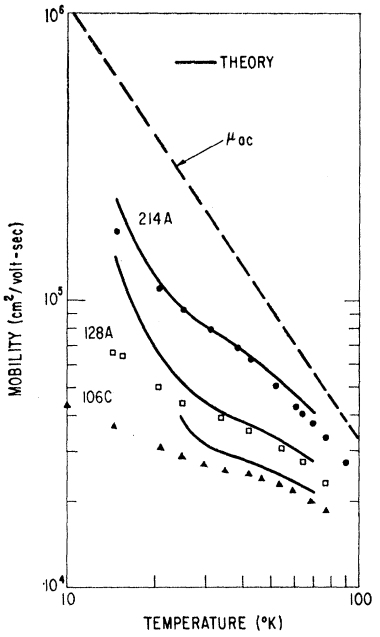

FIG. 4. Comparison of theory and experiment for progressively more impure samples. See Table II for sample data.

In Fig. 4, the theory fits the data of sample 214A very well down to 20°K, below which carrier freezeout becomes appreciable and knowledge of the compensation becomes very important for the mobility calculations. From the more impure samples 128A and 106C it can be seen that the fit to the experimental data becomes progressively worse, especially below 30°K, where the calculated mobilities rise rapidly above the

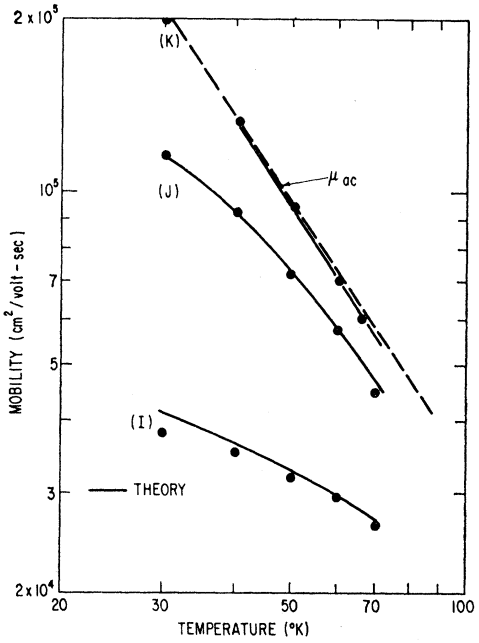

FIG. 5. Comparison of theory and experiment for one very pure sample (K) and two more impure samples (J and K). Data are those of Tyler *et al.* (reference 34).

measured mobilities. Whereas, small amounts of compensating impurities could be expected to explain the discrepancies at 15°K for sample 214A, the amount of compensation required to fit the measured mobilities for the more impure samples (128A and 106C) below 30°K seems unreasonably high, $\approx 10\%$. Actually, the concentration of minority impurities in 106C as deduced from the low-temperature Hall data is less than 1% of the total gallium concentration.

In Fig. 5 the data of Tyler *et al.* extend down to only 30°K as shown. Since these measurements were made with $H=6000$ to 8000 G, μ can be taken equal to μ_H, as before. For one very pure sample (K) represented by the highest data points in Fig. 5, freezeout on a deep-lying energy level apparently occurred, but the total impurity concentration was so small that it contributed negligibly to the carrier scattering, and the experimental data lies parallel to and almost coincides with the assumed pure lattice scattering curve. The calculated mobility curves match the data extremely well for the two purer samples, J and K; the mismatch for sample I, is comparable to that for our sample (128A) over the equivalent temperature range.

5.2 Dependence of Impurity Scattering on N_I at 77°K

In order to define the limitations of the present treatment of the ionized impurity scattering more exactly, theory and experiments are compared over an extended range of impurity concentrations at 77°K where the lattice scattering is predominately of a simple kind (i.e., acoustical mode scattering≫optical mode scattering).

The results are shown in Fig. 6, where the solid line represents the calculated mobility plotted against N_I, the total concentration of ionized impurities. The circles represent our own Hall mobility data, obtained at $H=3550$ G, and *selected* to represent the highest mobilities observed at these concentrations, thus, eliminating compensated samples. The squares are for data taken from Tyler *et al.*[34] measured at $H\geq 6000$ G.

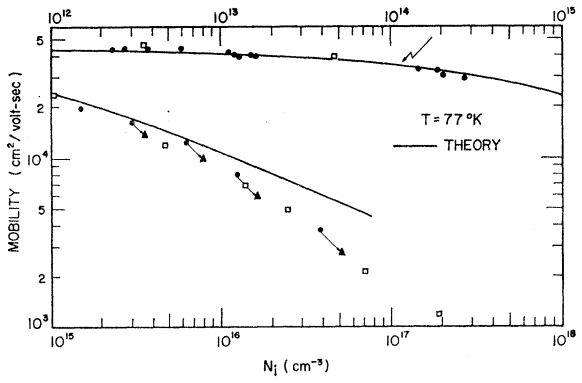

FIG. 6. Dependence of mobility on ionized impurity concentration at $T=77$°K. Circles are for μ_H data at $H=3550$ G; triangles represent converted Hall to conductivity mobility data (see text); squares are μ_H data of Tyler *et al.* (reference 34) at $H=6000$–8000 G.

The experimental data actually represent Hall mobility vs $1/qR_H$, and cannot be directly compared to the theoretical curve of conductivity mobility vs N_I without further analysis. However, according to the previous discussion of the relation between Hall and conductivity mobility, we take $\mu_H/\mu=1$ for the purer samples, for which $\mu_H\geq 20\,000$ cm²/V-sec or $1/qR_H \leq 2\times 10^{15}$/cc. Then for these samples, the experimental data can be taken to very nearly represent conductivity mobility vs the ionized impurity concentration, and, therefore, can be directly compared with the theory. For our more impure samples, μ_H/μ is estimated from Eq. (9). Application of the conversion factor $r=\mu_H/\mu$ in Fig. 6 translates the original Hall mobility data (represented by the circles) to the corrected data (represented by the triangles). An arrow marks the translation, which always carries the date point to lower mobility and higher N_I. Fortunately, the correction does not appreciably alter the shape of the experimental curve, since the corrected data lie approximately along the same line as the original points.

The theoretical calculation at 77°K can be greatly simplified by a small approximation which introduces negligible error. At 77°K, the experimental value for the lattice mobility, as determined from the measurements of the very pure samples, is 44 000 cm²/V-sec. Extrapolation of μ_{ac} to this temperature gives a value of 49 800 cm²/V-sec. Thus, the optical mode contribution to the lattice scattering is about 10% at 77°K. In the past, for example in the analyses of Hall and magnetoresistance measurements at 77°K,[1–4] the assumption has usually been made that all the lattice scattering is by acoustical modes and that the lattice relaxation time has a simple $\epsilon^{-1/2}$ dependence. This scheme can be used to calculate mobility vs impurity concentration, with the approximation that the acoustical mode lattice mobility be taken as 44 000 cm²/V-sec. It gives results which are identical to that for an exact calculation using Eq. (5) and including optical mode scattering.

In the calculation of the impurity scattering term, N_D was taken equal to zero; for $N_A\leq 10^{15}$ cm^{-3} the ratio $p/N_A=1.00$ and for $N_A>10^{15}$ cm^{-3}, p/N_A was estimated from the dissociation equation.

The theoretical curve (solid line) matches the experimental data extremely well for $N_I\leq 10^{15}$ cm^{-3}. However, the theory *overestimates* the mobility progressively more and more, with further increase in N_I. At $N_I=4\times 10^{16}$/cm^3, the theoretically calculated mobility is *too high* by over 50%. This pattern of the discrepancy between theory and experiment conforms to that already observed in Sec. 5.1 at lower temperatures.

5.3 Dependence of Impurity Scattering on N_I at 300°K

In order to test the impurity scattering formulation at a temperature where the energy dependence of the lattice scattering relaxation time differs from a simple $\epsilon^{-1/2}$ relation, calculations of mobility vs impurity concentration were also carried out at 300°K, where the predominant lattice scattering is by the optical modes. The mobility calculated according to Eq. (5) is shown by the solid line in Fig. 7.

Satisfactory comparison of the theory with experimental Hall data at 300°K is complicated by the fact that the conversion factors for Hall to conductivity mobility are not known at this temperature, nor is it known how large the magnetic field strength should be to attain the strong-field approximation. Consequently, the mobility data has been culled from several sources, and it is apparent that the scatter in the data is appreciable. The solid points represent our own data, obtained indirectly from measurements of the strong-field mobility at 77°K, and from the resistivities at 77 and 300°K,[35] using Eq. (9) in reverse. This relationship can only be safely applied for those samples for which μ_H(77°K) is sufficiently large ($\geq$20 000 cm²/V-sec) so that $\mu_H/\mu=1$ at 77°K. The solid squares represent Prince's drift mobility data[6] for holes in n-type Ge at 300°K. These data show a good deal of scatter; also ambipolar effects apparently already reduce the mobility at $N_I\leq 2\times 10^{14}$/cm³. The solid triangles represent recent Hall mobility data of Vinogradova *et al.*[36] obtained at $H=17\,000$ G. The fact that their data lie well above all the other data would indicate that their magnetic field strength is not yet sufficiently high to achieve the strong-field approximation. Our own data agree fairly well with the highest of the drift mobility values of Prince.

Comparison between theory and experiment is not too well defined here because of the scatter and uncertainty in the experimental data. It does seem, however, that the theory overestimates the mobility (as it did at 77 and 20°K) at the higher concentrations, but the discrepancies are not as large as at the lower temperatures. This may be due merely to the fact that the lattice mobility is already so small at 300°K that the impurity scattering does not have much effect on the total mobility.

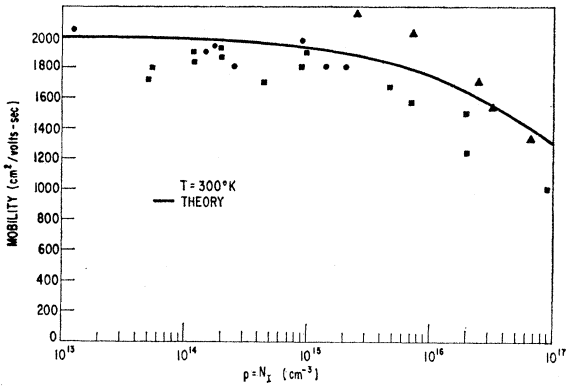

FIG. 7. Dependence of mobility on ionized impurity concentration at 300°K. Circles are for derived mobility data given in Table II; squares are Prince's drift mobility data (reference 6); triangles represent μ_H data of Vinogradova *et al.* (reference 36) at $H=17\,000$ G.

6. DISCUSSION AND CONCLUSIONS

6.1 Lattice Scattering

The expectation has been substantiated that at $T\leq 70$°K, the lattice mobility in p-type germanium can be represented by a simple $T^{-3/2}$ dependence, characteristic of acoustical mode scattering. The $T^{-2.3}$ variation of mobility at $T>100$°K can be attributed to the onset of additional scattering by optical modes. The magnitudes of the acoustical and optical mode scattering parameters were determined by treating them as adjustable parameters in the different temperature ranges on the basis of several simplifying assumptions. The various parameters, used and deduced, are summarized in Table I. We shall now consider briefly to what extent these parameters are consistent with some few available results from independent experiments. These pertain mainly to the acoustical mode scattering.

The results of hot-carrier measurements,[37] where the dependence of the mobility on electric field strength is determined, are in excellent accord with the parameters for acoustical mode scattering, but apparently not with those for optical mode scattering. However, the latter result may be a reflection of inadequacies in the hot-carrier theory.

The recent cyclotron resonance results of Bagguley *et al.*,[19] as mentioned before, show the existence of a

[35] For $\rho_{300°K}\geq 15$ Ω-cm, additional corrections were made for intrinsic contribution in the resistivity which served to slightly decrease $\mu_{300°K}$; see, e.g., R. Bray, Phys. Rev. **100**, 1047 (1955).

[36] M. N. Vinogradova, O. A. Golikova, B. P. Mitronin, and L. S. Stil'bans, *Proceedings of the International Conference on Semiconductor Physics, Prague, 1960* (Czechoslovakian Academy of Sciences, Prague, 1961), p. 85. Vinogradova *et al.* attempted an extremely simplified analysis of their mobility data in the range 100 to 450°K. They assumed $\mu_H/\mu=1$ at $H=17\,000$ G even for the most impure samples. The separation of the mobility contributions for the two types of holes was not made. Lattice and impurity scattering were combined by simple addition of reciprocal mobilities. The very simple empirical result was obtained that the mobility due to ionized impurity scattering is *independent* of temperature in the range 100 to 450°K, and that over the whole range the lattice mobility is proportional to $T^{-2.3}$. In view of the oversimplification of the analysis, it is difficult to ascribe any significance to this result.

Note added in proof. See also M. N. Vinogradova *et al.*, Soviet Phys.—Solid State **2**, 1298 (1960). This work has been reinterpreted and extended by O. A. Golikova, B. Ya. Moizhes, and L. S. Stil'bans, Soviet Phys.—Solid State **3**, 2259 (1962). In the latter work, the conclusions of Vinogradova *et al.*, are criticized and the effects of ionized impurity scattering are re-evaluated. Additional data relevant to our Figs. 6 and 7 are presented.

[37] R. Bray and D. M. Brown, *Proceedings of the International Conference on Semiconductor Physics, Prague 1960* (Czechoslovakian Academy of Sciences, Prague, 1961), p. 82.

$T^{-3/2}$ dependence for the average cyclotron resonance collision time $\tau_{c.r.}$ in the acoustic mode scattering range. The actual magnitude is in agreement with our average conductivity relaxation time τ_σ; thus

$$\tau_{c.r.}=\langle\tau\rangle=5.4\times10^{-9}T^{-3/2}\text{ sec},$$
$$\tau_\sigma=\langle v^2\tau_{ac}\rangle/\langle v^2\rangle=5.2\times10^{-9}T^{-3/2}\text{ sec}.$$

However, according to Bagguley *et al.*,[19] it is expected that $\tau_\sigma=\frac{2}{3}\tau_{c.r.}$ because of the different type of averaging that is involved.

Finally, Mochan *et al.*[38] have deduced somewhat indirectly from thermal-emf measurements that the part of the scattering due to longitudinal acoustical phonons can be described by a relaxation time proportional to $T^{-3/2}$ in the range 50 to 200°K.

We are not aware of any independent experimental evidence to confirm the strength of the optical mode scattering.

6.2 Impurity Scattering

The range of applicability of the Brooks-Herring formula, with the restriction to intraband scattering, was empirically explored. The fit of the theory to the experimental data was found to be quite good for $N_I\leq10^{15}$ cm^{-3} in the temperature range from 30 to 300°K. However, the results attested to the increasing inadequacy of the impurity scattering treatment at lower temperatures, and for higher impurity concentrations. Where the theory began to fail, it consistently *overestimated* the mobility. It remains to consider the sources of this discrepancy.

The inadequacy of the Brooks-Herring formula, as demonstrated theoretically by Blatt,[20] leads to overestimation of the strength of ionized impurity scattering for the more impure samples, which is in the wrong direction to account for the present observations. It would appear, therefore, that some of the sources of scattering neglected in the present analysis, i.e., carrier-carrier scattering, neutral impurity scattering, and interband ionized impurity scattering, must be important for $N_I>10^{15}$/cm^3. The incorporation of carrier-carrier scattering has been shown by Appel and Bray,[39] to make a marked improvement, extending the range of agreement of the theory to $N_I\approx10^{16}$/cm^3 at least at 77°K. For still higher N_I, it would appear that interband ionized impurity scattering may be the source of the additional scattering.[40] However, as we have already pointed out, theoretical treatment of this type of scattering is only worthwhile within the framework of a more general treatment of ionized impurity scattering itself. Neutral impurity scattering may also be important for the most impure samples, for which some freezeout must occur at $T\leq77$°K. However, it is not certain how to treat neutral impurity scattering in terms of inter and intraband scattering. Furthermore, it is difficult to estimate the number of neutral centers because it is not known how the impurity ionization energy varies with N_I, especially for $N_I>10^{16}$/cm^3.

In conclusion, it can be said that the relatively simple analysis of mobility in p-type germanium is quite successful, in fact, much more so than might have been expected. Where it fails, it is fairly clear what additional complexities are necessary in the theory.

ACKNOWLEDGMENTS

The authors are indebted to Miss Louise Roth for growing the germanium crystals used in this work and to Dr. Rajat Ray for assistance in the low temperature measurements.

[38] I. V. Mochan, Yu. N. Obragtov, and T. V. Smirnova, Soviet Phys.—Solid State **1**, 1239 (1960).

[39] J. Appel and R. Bray, following paper [Phys. Rev. **127**, 1603 (1962)].

[40] This was suggested to us by C. Herring.