Dissipative Potentials and the Motion of a Classical Charge

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(Received January 25, 1962; revised manuscript received March 29, 1962)

The equation of motion of a charge is obtained as a second-order integro-differential equation, directly from Maxwell's field equations and Lorentz's force equation without the use of the Dirac equation of motion. The validity of the field equations is assumed everywhere, the position of the charge being treated as a point singularity. The force on the charge at the field point is given by the expression for the Lorentz force produced by the fields of a source charge in the limiting case where the field charge is identified with the source. The only fields which need to be considered are the retarded solutions of the field equations, in agreement with causality. In order to obtain the equation of motion, it is necessary to formulate the potentials of the field of a point source in dissipative form. In this form, the potentials satisfy the Lorentz gauge condition identically. They are Fourier integrals which contain $\delta_\pm$ functions in the Fourier coefficients. When the equation of motion is applied to the examples of relativistic motion of a free charge and to the nonrelativistic simple harmonic motion of a charge, it is found that the principal-part terms of the $\delta_\pm$ functions provide the electromagnetic kinetic reaction, while the δ function terms provide the dissipative effects. The only divergence stems from the principal-part integrals, and is the Coulomb self-energy of the charge. It can be removed by renormalization of mass. The results are consistent with the Abraham theory that the mass of an electron (and positron) is wholly electromagnetic.

INTRODUCTION

IN a recent paper,[1] Rohrlich has shown that by starting with Dirac's equation for the classical motion of a charge and imposing an additional condition which, in effect, eliminates the nonphysical solutions, an integro-differential equation of second order may be obtained. The motion of a charge obeying such an equation is determined, as in Newtonian dynamics, once the initial position and velocity are specified. In this paper, we obtain an integro-differential equation of second order for motion of a charge, directly from Maxwell's field equations and Lorentz's force equation, without the use of Dirac's equation or subsidiary conditions. As does Dirac,[2] we assume the validity of the field equations everywhere, treating the position of the charge as a point singularity. We calculate the force on a charge at a field point, using the expression for the Lorentz force produced by the fields of a source charge in the limiting case where the field charge is identified with the source. The only fields which need be considered are the retarded solutions of the field equations, in agreement with causality.

In order to formulate the integro-differential equation, we start in Sec. 1 with the potentials of the field of a point source expressed as a Fourier integral. These are the dissipative potentials. They are derived in Appendix I. In Sec. 2, we derive the integro-differential equation of motion, together with the formulas for momentum and energy of a charge in its self-field. In Sec. 3, we apply these equations to the example of unaccelerated motion, and in Sec. 4, to the case of simple harmonic motion. In Sec. 5, we discuss the results obtained. In Appendix II are given details of the evaluation of some integrals for the simple harmonic oscillator example.

1. DISSIPATIVE POTENTIALS

It is shown in Appendix I that the potentials at the field point $(\mathbf{x},t)$ produced by a point charge e moving along a trajectory $\mathbf{x}(t')$ with velocity $\mathbf{v}(t')$ can be put in the dissipative form

$$A_\sigma(\mathbf{x},t) = \mp(ei/8\pi^3)\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)] \times\int dt'\, v_\sigma(t')\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t')]-\omega(t-t')\}, \quad (1.1)$$

where

$$v_\sigma(t)=[d\mathbf{x}(t)/dt,1], \quad \sigma=1,\cdots 4, \quad (1.2)$$

and the upper sign indicates the advanced, the lower sign, the retarded potential. The speed of light is taken as unity. The $\delta_\mp$ functions are defined as

$$\delta_\mp(x)=\pi\delta(x)\mp iP(1/x), \quad (1.3)$$

so that separation of δ-function and principal-part terms gives

$$\mp[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)] = \mp\pi[\delta(\omega-k)-\delta(\omega+k)]+2ikP(\omega^2-k^2)^{-1}. \quad (1.4)$$

We may readily verify that (1.1) is a solution of the D'Alembert equation

$$(\partial^2/\partial\mathbf{x}^2-\partial^2/\partial t^2)A_\sigma(\mathbf{x},t)=-4\pi e v_\sigma(t)\delta[\mathbf{x}-\mathbf{x}(t)]. \quad (1.5)$$

The substitution $t'=t-\tau$ in (1.1) gives

$$A_\sigma(\mathbf{x},t) = \mp(ei/8\pi^3)\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)] \times\int d\tau\, v_\sigma(t-\tau)\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t-\tau)]-\omega\tau\}. \quad (1.6)$$

In this form, the potentials satisfy identically the Lorentz gauge condition

$$(\partial/\partial\mathbf{x})\cdot\mathbf{A}+\partial\phi/\partial t=0, \quad A_\sigma=(\mathbf{A},\phi). \quad (1.7)$$

[1] F. Rohrlich, Ann. Phys. (New York) **13**, 93 (1961).
[2] P. A. M. Dirac, Proc. Roy. Soc (London) **A167**, 148 (1938).

The time integrations over t' and τ in (1.1) and (1.7) extend over the whole trajectory of the source charge. However, the $\delta_{\mp}$ functions are Fourier transforms of the Heaviside function θ according to the equation

$$\int d\omega\, \delta_{\mp}(\omega-k)\exp(-i\omega\tau)=2\pi\exp(-ik\tau)\theta(\mp\tau), \quad (1.8)$$

so that, performing the ω integration in (1.6), we obtain

$$A_\sigma(\mathbf{x},t)=\mp(e/2\pi^2)\int d\tau\,\theta(\mp\tau)v_\sigma(t-\tau)\int\int\int d\mathbf{k} \times k^{-1}\sin\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t-\tau)]+k\tau\}. \quad (1.9)$$

In this form the retarded potential (lower sign) conforms to the requirements of a causality principle according to which the potential at the field point at time t is determined by the trajectory of the source charge at times $t'=t-\tau\leq t$. This is the meaning of causality as used in this paper. In the particular case that the field point $\mathbf{x}$ is identified with $\mathbf{x}(t)$, the position of the source at time t, causality requires that the self-field be described by the retarded potential $A_\sigma[\mathbf{x}(t),t]$.

Performing the integrations over $\mathbf{k}$ and τ in (1.9),[3] we obtain

$$A_\sigma(\mathbf{x},t)=ev_\sigma(t')/[r\pm\mathbf{r}\cdot\mathbf{v}(t')], \text{ for } t'=t\pm r, \quad (1.10)$$

where $\mathbf{r}=\mathbf{x}-\mathbf{x}(t')$. These are just the Liénard-Wiechert potentials of a point source.

Note that half the sum of the retarded and advanced potentials contains the factor

$$\tfrac{1}{2}\{[\delta_+(\omega-k)-\delta_+(\omega+k)]-[\delta_-(\omega-k)-\delta_-(\omega+k)]\} =2ikP(\omega^2-k^2)^{-1}, \quad (1.11)$$

while half the difference contains

$$\tfrac{1}{2}\{[\delta_+(\omega-k)-\delta_+(\omega+k)]+[\delta_-(\omega-k)-\delta_-(\omega+k)]\} =\pi[\delta(\omega-k)-\delta(\omega+k)]. \quad (1.12)$$

According to (1.4) the first combination is equivalent to the principal-part term of the retarded potential alone, while the second is the δ-function term of the retarded potential alone. The principal-part terms in the retarded and advanced potentials are identical; the delta-function terms have opposite sign.

At each moment of time, the potentials $A_\sigma(\mathbf{x},t)$ have a singularity at the point $\mathbf{x}=\mathbf{x}(t)$, the position of the source charge. If we want the value of $A_\sigma[\mathbf{x}(t),t]$, the potential of the source charge at the position of the charge itself, we cannot simply set $\mathbf{x}=\mathbf{x}(t)$ inside the Fourier integrals in (1.1) which diverge (the principal-part integrals). In all Fourier integrals for field quantities, the value at the singular point must be obtained as the limit of the Fourier integral as $\mathbf{x}\rightarrow\mathbf{x}(t)$ after integration. This is the explanation of errors in earlier work.[4,5] Accordingly, at the singular point, (1.1) gives the self-field as

$$\begin{aligned}A_\sigma[\mathbf{x}(t),t] &=\mp(ei/8\pi^3)\lim_{\alpha\to0}\int\cdots\int d\mathbf{k}d\omega \\ &\times k^{-1}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})[\delta_{\mp}(\omega-k)-\delta_{\mp}(\omega+k)] \\ &\times\int dt'\,v_\sigma(t')\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}.\end{aligned} \quad (1.13)$$

2. EQUATIONS OF MOTION

In order to obtain a description of the motion of a particle in its self-field, we consider the motion of a second particle of charge e in the field of the source particle, in the limit as the velocity of the second particle and its position coincide with that of the source. A second charge e of mass μ in the field A_σ of the first has momentum $\mathbf{p}$ and energy H, in the same gauge as that in which the potentials are expressed (Lorentz gauge in this discussion).

$$\mathbf{p}=\mu\mathbf{v}/(1-v^2)^{\frac{1}{2}}+e\mathbf{A}(\mathbf{x},t), \quad (2.1)$$

$$H=\mu/(1-v^2)^{\frac{1}{2}}+e\phi(\mathbf{x},t)+V(\mathbf{x}), \quad (2.2)$$

where $A_\sigma(\mathbf{x},t)=[\mathbf{A}(\mathbf{x},t),\phi(\mathbf{x},t)]$ is the potential four-vector for which the first charge is the source, $\mathbf{v}$ is the velocity of the second charge at the field point $(\mathbf{x},t)$ of the first, and $V(\mathbf{x})$ is the potential of an external force. The equation of motion of the second charge is

$$\mu(d/dt)[\mathbf{v}/(1-v^2)^{\frac{1}{2}}] =e[\mathbf{E}(\mathbf{x},t)+\mathbf{v}\times\mathbf{B}(\mathbf{x},t)]+\mathbf{F}_{\text{ext}}, \quad (2.3)$$

where the external force $\mathbf{F}_{\text{ext}}=-\partial V/\partial\mathbf{x}$, and

$$\mathbf{E}(\mathbf{x},t)=\mp(e/8\pi^3)\int\cdots\int d\mathbf{k}d\omega\,k^{-1}[\delta_{\mp}(\omega-k)-\delta_{\mp}(\omega+k)]\int dt'[\mathbf{k}-\omega\mathbf{v}(t')]\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t')]-\omega(t-t')\}, \quad (2.4)$$

$$\mathbf{B}(\mathbf{x},t)=\mp(e/8\pi^3)\int\cdots\int d\mathbf{k}d\omega\,k^{-1}[\delta_{\mp}(\omega-k)-\delta_{\mp}(\omega+k)]\int dt'[\mathbf{v}(t')\times\mathbf{k}]\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t')]-\omega(t-t')\}. \quad (2.5)$$

We now identify the field charge with the source charge by letting $\mathbf{v}=\mathbf{v}(t)$ and $\mathbf{x}\rightarrow\mathbf{x}(t)$. We obtain from (2.1)

[3] D. Iwanenko and A. Sokolow, *Klassische Feldtheorie* (Akademie-Verlag, Berlin, 1953).
[4] I. Prigogine and B. Leaf, Physica **25**, 1067 (1959).
[5] I. Prigogine and B. Leaf, Bull. Acad. Roy. Belg. **46**, 915 (1960).

and (2.2)

$$\mathbf{p}=\mu\mathbf{v}(t)/[1-v^2(t)]^{\frac{1}{2}}+e\mathbf{A}[\mathbf{x}(t),t], \tag{2.6}$$

$$H=\mu/[1-v^2(t)]^{\frac{1}{2}}+e\phi[\mathbf{x}(t),t]+V[\mathbf{x}(t)], \tag{2.7}$$

with $A_\sigma[\mathbf{x}(t),t]$ given by (1.13). The equation of motion (2.3) becomes

$$\mu(d/dt)\{\mathbf{v}(t)/[1-v^2(t)]^{\frac{1}{2}}\}=\mathbf{F}[\mathbf{x}(t),t]+\mathbf{F}_{\text{ext}}[\mathbf{x}(t)], \tag{2.8}$$

where $\mathbf{F}[\mathbf{x}(t),t]$ is the Lorentz self-force,

$$\mathbf{F}[\mathbf{x}(t),t]=\mp(e^2/8\pi^3)\lim_{\alpha\to 0}\int\cdots\int d\mathbf{k}d\omega\, k^{-1}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)]$$
$$\times\int dt'\{\mathbf{k}[1-\mathbf{v}(t)\cdot\mathbf{v}(t')]+\mathbf{v}(t')[\mathbf{k}\cdot\mathbf{v}(t)-\omega]\}\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}. \tag{2.9}$$

Multiplication by $\mathbf{v}(t)$ gives the energy equation

$$(d/dt)\{\mu/[1-v^2(t)]^{\frac{1}{2}}\}-\mathbf{v}(t)\cdot\mathbf{F}_{\text{ext}}=e\mathbf{v}(t)\cdot\mathbf{E}[\mathbf{x}(t),t]. \tag{2.10}$$

Equation (2.8) is a second-order integro-differential equation. For a given external force, the value of $\mathbf{x}(t)$ is determined once the initial values of $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are specified. If, instead of prescribing $\mathbf{F}_{\text{ext}}$ and solving for $\mathbf{x}(t)$, we assume a particular motion of the particle by prescribing $\mathbf{x}(t)$, then $\mathbf{F}_{\text{ext}}$ may be obtained from the equation; but it may turn out that this force is different from the one required for a neutral particle executing the prescribed motion $\mathbf{x}(t)$. It may even occur that $\mathbf{F}_{\text{ext}}$ in (2.8) is not derivable from a potential.

With the use of (1.4) we may separate the self-force $\mathbf{F}[\mathbf{x}(t),t]$ into two parts arising from the principal-part and δ-function terms, respectively.

$$\mathbf{F}[\mathbf{x}(t),t]=\mathbf{F}^K[\mathbf{x}(t),t]+\mathbf{F}^D[\mathbf{x}(t),t], \tag{2.11}$$

$$\mathbf{F}^K[\mathbf{x}(t),t]=(e^2i/4\pi^3)\lim_{\alpha\to 0}\int\cdots\int d\mathbf{k}d\omega\,\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})(\omega^2-k^2)^{-1}\int dt'$$
$$\times\{\mathbf{k}[1-\mathbf{v}(t)\cdot\mathbf{v}(t')]+\mathbf{v}(t')[\mathbf{k}\cdot\mathbf{v}(t)-\omega]\}\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}, \tag{2.12}$$

$$\mathbf{F}^D[\mathbf{x}(t),t]=\mp(e^2/8\pi^2)\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta(\omega-k)-\delta(\omega+k)]\int dt'$$
$$\times\{\mathbf{k}[1-\mathbf{v}(t)\cdot\mathbf{v}(t')]+\mathbf{v}(t')[\mathbf{k}\cdot\mathbf{v}(t)-\omega]\}\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}. \tag{2.13}$$

Similarly, in (2.6) and (2.7) we separate the electromagnetic momentum and energy, letting

$$A_\sigma[\mathbf{x}(t),t]=A_\sigma^K[\mathbf{x}(t),t]+A_\sigma^D[\mathbf{x}(t),t], \tag{2.14}$$

where, according to (1.13),

$$A_\sigma^K[\mathbf{x}(t),t]=-(e/4\pi^3)\lim_{\alpha\to 0}\int\cdots\int d\mathbf{k}d\omega\,\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})(\omega^2-k^2)^{-1}$$
$$\times\int dt'\, v_\sigma(t')\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}, \tag{2.15}$$

$$A_\sigma^D[\mathbf{x}(t),t]=\mp(ei/8\pi^2)\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta(\omega-k)-\delta(\omega+k)]\int dt'\, v_\sigma(t')\exp i\{\mathbf{k}\cdot[\mathbf{x}(t)-\mathbf{x}(t')]-\omega(t-t')\}. \tag{2.16}$$

As we shall see in the following examples, the kinetic parts, $\mathbf{F}^K$, A_σ^K, arising from the principal-part integrals, diverge. The dissipative parts, $\mathbf{F}^D$, A_σ^D, arising from the δ-function integrals, are finite; accordingly, the limit $\boldsymbol{\alpha}\to 0$ has been taken inside these integrals.

3. MOTION OF A FREE CHARGE

Let $\mathbf{v}(t)=\mathbf{v}(t')=\mathbf{v}$, a constant, so that

$$\mathbf{x}(t)-\mathbf{x}(t')=\mathbf{v}(t-t'). \tag{3.1}$$

According to (2.13),

$$\mathbf{F}^D=\mp(e^2/8\pi^2)\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta(\omega-k)-\delta(\omega+k)][(1-v^2)\mathbf{k}+(\mathbf{k}\cdot\mathbf{v}-\omega)\mathbf{v}]\int dt'\exp[i(\mathbf{k}\cdot\mathbf{v}-\omega)(t-t')]$$

$$=\mp[e^2(1-v^2)/4\pi]\int\cdots\int d\mathbf{k}d\omega\, k^{-1}[\delta(k-\mathbf{k}\cdot\mathbf{v})-\delta(k+\mathbf{k}\cdot\mathbf{v})]=0. \tag{3.2}$$

The δ functions can never be satisfied, since $v<1$, so that $\mathbf{F}^D$ vanishes. According to (2.12)

$$\mathbf{F}^K=(e^2i/4\pi^3)\lim_{\alpha\to 0}\int\cdots\int d\mathbf{k}d\omega\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})(\omega^2-k^2)^{-1}[(1-v^2)\mathbf{k}+(\mathbf{k}\cdot\mathbf{v}-\omega)\mathbf{v}]\int dt'\exp[i(\mathbf{k}\cdot\mathbf{v}-\omega)(t-t')]$$

$$=-[ie^2(1-v^2)/2\pi^2]\lim_{\alpha\to 0}\int\int\int d\mathbf{k}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})\mathbf{k}/[k^2-(\mathbf{k}\cdot\mathbf{v})^2]$$

$$=-(1-v^2)\lim_{\alpha\to 0}(\partial/\partial\boldsymbol{\alpha})[(e^2/\alpha)/(1-v_\perp^2)^{\frac{1}{2}}]. \tag{3.3}$$

where $\mathbf{v}_\perp=\mathbf{v}\cdot(\mathbf{1}-\boldsymbol{\alpha\alpha}/\alpha^2)$. Similarly, from (2.15) and (2.16),

$$A_\sigma{}^D=0, \tag{3.4}$$

$$A_\sigma{}^K=\lim_{\alpha\to 0}(e^2/\alpha)v_\sigma/(1-v_\perp^2)^{\frac{1}{2}}. \tag{3.5}$$

Now[6]

$$A_\sigma(\mathbf{r},t)=(ev_\sigma/\alpha)/(1-v_\perp^2)^{\frac{1}{2}},$$

where $\boldsymbol{\alpha}=\mathbf{r}(t)-\mathbf{r}'(t)$ is the Lorentz gauge potential at a field point which is at a distance $\boldsymbol{\alpha}$ at time t from a point charge moving with constant velocity $\mathbf{v}$. The vector $\boldsymbol{\alpha}$ is the displacement from the source point $\mathbf{r}'(t)$ to the field point $\mathbf{r}(t)$ at the moment t when the signal arrives at $\mathbf{r}$. Let

$$\boldsymbol{\alpha}'=\mathbf{r}(t)-\mathbf{r}(t')$$

be the displacement from source to field point when the source position is measured at the time t' such that

$$\mp(t-t')=\alpha',$$

where again the upper sign refers to the advanced, the lower sign to the retarded, time. Since the source has constant velocity $\mathbf{v}$,

$$\boldsymbol{\alpha}=\boldsymbol{\alpha}'\pm\alpha'\mathbf{v}.$$

In terms of $\boldsymbol{\alpha}'$, the potential $A_\sigma(\mathbf{r},t)$ becomes identical with the Liénard-Wiechert potential for a source moving with constant velocity:

$$A_\sigma(\mathbf{r},t)=ev_\sigma/(\alpha'\pm\boldsymbol{\alpha}'\cdot\mathbf{v}).$$

Let

$$\alpha_0=\alpha'/(1-v^2)^{\frac{1}{2}}\pm\boldsymbol{\alpha}'\cdot\mathbf{v}/(1-v^2)^{\frac{1}{2}},$$

the Lorentz scalar invariant distance between the source point and field point in their rest frame. Then

$$A_\sigma(\mathbf{r},t)=ev_\sigma/\alpha_0(1-v^2)^{\frac{1}{2}}.$$

[6] B. Leaf, Physica **28**, 206 (1962).

Accordingly, using (3.4) and (3.5), we find from (2.6) and (2.7) that

$$\mathbf{p}=[\mu+\lim_{\alpha_0\to 0}e^2/\alpha_0]\mathbf{v}/(1-v^2)^{\frac{1}{2}}, \tag{3.6}$$

$$H=[\mu+\lim_{\alpha_0\to 0}e^2/\alpha_0]/(1-v^2)^{\frac{1}{2}}+V, \tag{3.7}$$

while, using (3.2) and (3.3) we find that the self-force in (2.9) becomes

$$\mathbf{F}/(1-v^2)^{\frac{1}{2}}=-\{\mathbf{1}+[(1-v^2)^{-\frac{1}{2}}-1]\mathbf{vv}/v^2\}\times\lim_{\alpha_0\to 0}(\partial/\partial\boldsymbol{\alpha}_0)(e^2/\alpha_0),$$

in agreement with the Lorentz transformation of the static Coulomb force, $-(\partial/\partial\boldsymbol{\alpha}_0)(e^2/\alpha_0)$ from the rest frame of the particle. But since $\lim_{\mathbf{x}\to 0}(\partial/\partial\mathbf{x})(1/x)=0$, therefore, finally,

$$\mathbf{F}[\mathbf{x}(t),t]=0. \tag{3.8}$$

According to (3.6) and (3.7), the effective mass of a free charge is

$$m=\mu+\lim_{\alpha_0\to 0}e^2/\alpha_0. \tag{3.9}$$

It is consistent to set the external potential $V=0$ and external force $\mathbf{F}_{\text{ext}}=0$ in this example. The absence of dissipative effects is attributable to the fact that the delta functions in (3.2) cannot be satisfied. Both retarded and advanced fields give the same result, since only principal-part terms remain.

4. NONRELATIVISTIC MOTION OF A CHARGED SIMPLE HARMONIC OSCILLATOR

Let

$$\begin{aligned}\mathbf{x}(t)&=a\mathbf{z}\cos\nu t,\\ \mathbf{v}(t)&=-a\nu\mathbf{z}\sin\nu t,\end{aligned} \tag{4.1}$$

where $\mathbf{z}$ is a unit vector along the line of motion.

According to (2.12),

$$\mathbf{F}^K=(e^2i/4\pi^3)\lim_{\alpha\to0}\int\cdots\int d\mathbf{k}d\omega\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})(\omega^2-k^2)^{-1}\int dt'\{\mathbf{k}[1-(a\nu)^2\sin\nu t\sin\nu t']$$
$$+\mathbf{z}a\nu\sin\nu t'[\mathbf{k}\cdot\mathbf{z}a\nu\sin\nu t+\omega]\}\exp i\{\mathbf{k}\cdot\mathbf{z}a(\cos\nu t-\cos\nu t')-\omega(t-t')\}. \quad (4.2)$$

Using the expansions in Bessel's functions J_n of order n,

$$\begin{aligned}\exp(\pm ia\mathbf{k}\cdot\mathbf{z}\cos\nu t)&=\textstyle\sum_n J_n(a\mathbf{k}\cdot\mathbf{z})(\pm i)^n\exp(\pm in\nu t),\\ \sin\nu t\exp(\pm ia\mathbf{k}\cdot\mathbf{z}\cos\nu t)&=-\textstyle\sum_n(n/a\mathbf{k}\cdot\mathbf{z})J_n(a\mathbf{k}\cdot\mathbf{z})(\pm i)^n\exp(+in\nu t),\end{aligned}\quad (4.3)$$

we find

$$\mathbf{F}^K=(e^2i/4\pi^3)\lim_{\alpha\to0}\int\cdots\int d\mathbf{k}d\omega\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})(\omega^2-k^2)^{-1}\textstyle\sum_m\sum_n\displaystyle\int dt'\{\mathbf{k}[1-\nu^2nm/(\mathbf{k}\cdot\mathbf{z})^2]$$
$$+\mathbf{z}(\nu m/\mathbf{k}\cdot\mathbf{z})(\nu n-\omega)\}i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp i[(n\nu-\omega)t-(m\nu-\omega)t']$$
$$=-(e^2i/2\pi^2)\lim_{\alpha\to0}\int\int\int d\mathbf{k}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})\textstyle\sum_m\sum_n[k^2-(m\nu)^2]^{-1}\{\mathbf{k}[1-\nu^2nm/(\mathbf{k}\cdot\mathbf{z})^2]$$
$$+\mathbf{z}\nu^2(n-m)m/\mathbf{k}\cdot\mathbf{z}\}i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t]. \quad (4.4)$$

Similarly, from (2.12), (2.15), and (2.16),

$$\mathbf{F}^D=\mp(e^2/4\pi)\int\int\int d\mathbf{k}\,k^{-1}\textstyle\sum_m\sum_n[\delta(k-m\nu)-\delta(k+m\nu)]\{\mathbf{k}[1-\nu^2nm/(\mathbf{k}\cdot\mathbf{z})^2]$$
$$+\mathbf{z}\nu^2(n-m)m/\mathbf{k}\cdot\mathbf{z}\}i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t], \quad (4.5)$$

$$\mathbf{A}^K=(e/2\pi^2)\lim_{\alpha\to0}\int\int\int d\mathbf{k}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})\textstyle\sum_m\sum_n[k^2-(m\nu)^2]^{-1}(m\nu\mathbf{z}/\mathbf{k}\cdot\mathbf{z})i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t], \quad (4.6)$$

$$\phi^K=(e/2\pi^2)\lim_{\alpha\to0}\int\int\int d\mathbf{k}\exp(i\mathbf{k}\cdot\boldsymbol{\alpha})\textstyle\sum_m\sum_n[k^2-(m\nu)^2]^{-1}i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t], \quad (4.7)$$

$$\mathbf{A}^D=\mp(ei/4\pi)\int\int\int d\mathbf{k}\,k^{-1}\textstyle\sum_m\sum_n[\delta(k-m\nu)-\delta(k+m\nu)](m\nu\mathbf{z}/\mathbf{k}\cdot\mathbf{z})i^{n-m}$$
$$\times J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t], \quad (4.8)$$

$$\phi^D=\mp(ei/4\pi)\int\int\int d\mathbf{k}\,k^{-1}\textstyle\sum_m\sum_n[\delta(k-m\nu)-\delta(k+m\nu)]i^{n-m}J_n(a\mathbf{k}\cdot\mathbf{z})J_m(a\mathbf{k}\cdot\mathbf{z})\exp[i(n-m)\nu t]. \quad (4.9)$$

Equations (4.4) to (4.9) are Fourier series representations in harmonics of the oscillator frequency ν. Designating the combined contribution to $\mathbf{F}^K$ in (4.4) from the terms with $n=m+s$ and $n=m-s$ by $\mathbf{F}_s{}^K$, we have evaluated $\mathbf{F}_s{}^K$ in the nonrelativistic limit (see Appendix II) for $s=0, 1$,

$$\mathbf{F}_0{}^K=0,\quad \mathbf{F}_1{}^K=\lim_{\alpha\to0}(e^2/\alpha)\mathbf{z}a\nu^2\cos\nu t. \quad (4.10)$$

Similarly, for $\mathbf{F}^D$, $\mathbf{A}^K$, $\mathbf{A}^D$ in (4.5), (4.6), and (4.8)

$$\mathbf{F}_0{}^D=0,\quad \mathbf{F}_1{}^D=\mp(2/3)\mathbf{z}e^2a\nu^3\sin\nu t, \quad (4.11)$$

$$\mathbf{A}_0{}^K=0,\quad \mathbf{A}_1{}^K=-\lim_{\alpha\to0}(e/\alpha)\mathbf{z}a\nu\sin\nu t, \quad (4.12)$$

$$\mathbf{A}_0{}^D=0,\quad \mathbf{A}_1{}^D=\mp\mathbf{z}ea\nu^2\cos\nu t. \quad (4.13)$$

In the case of ϕ^K and ϕ^D in (4.7) and (4.9), the contributions have been evaluated for $s=0, 1, 2$, as

$$\phi_0{}^K=\lim_{\alpha\to0}(e/\alpha)[1+(a\nu/2)^2],\quad \phi_1{}^K=0,$$
$$\phi_2{}^K=\lim_{\alpha\to0}(e/\alpha)(a\nu/2)^2\cos2\nu t, \quad (4.14)$$

$$\phi_0{}^D=0,\quad \phi_1{}^D=0,\quad \phi_2{}^D=\pm\tfrac{1}{2}ea^2\nu^3\sin2\nu t. \quad (4.15)$$

Combining these contributions we obtain, according to (2.8), (2.6), and (2.7), in the nonrelativistic case,

$$md\mathbf{v}/dt=\mathbf{F}_{\text{ext}}\mp(2/3)e^2d^2\mathbf{v}/dt^2, \quad (4.16)$$

$$\mathbf{p}=m\mathbf{v}\pm e^2d\mathbf{v}/dt, \quad (4.17)$$

$$H=m(1+\tfrac{1}{2}v^2)\pm e^2\mathbf{v}\cdot d\mathbf{v}/dt+V, \quad (4.18)$$

where the effective mass is

$$m=\mu+\lim_{\alpha\to0}e^2/\alpha, \quad (4.19)$$

just as in (3.9) for the first example of free motion of a charge. Equations (4.16), (4.17), and (4.18) have been obtained from only a few terms of the Fourier series (4.4) to (4.9). In every case the divergences arise from the contributions of the principal-part terms.

According to (4.16), the energy equation (2.10) becomes, in the nonrelativistic limit,

$$\tfrac{1}{2}mdv^2/dt = \mathbf{v}\cdot\mathbf{F}_{\text{ext}} \mp (2/3)e^2\mathbf{v}\cdot d^2\mathbf{v}/dt^2, \tag{4.20}$$

or

$$ma^2\nu^3 \sin\nu t \cos\nu t = \mathbf{v}\cdot\mathbf{F}_{\text{ext}} \pm (2/3)e^2a^2\nu^4 \sin^2\nu t. \tag{4.21}$$

Taking the time average in this equation over a whole number of oscillations, we find that the external force must provide energy at the average rate

$$\langle\mathbf{v}\cdot\mathbf{F}_{\text{ext}}\rangle = \mp\tfrac{1}{3}e^2a^2\nu^4. \tag{4.22}$$

This is the well-known[7] radiation damping term, often expressed as $\mp(2/3)e^2\nu^2U/m$, where $U=\frac{1}{2}m(a\nu)^2$ is the energy of a neutral oscillator of mass m. It is seen that the correct sign for the damping is given by the retarded potential.

When the effective mass m is introduced according to (4.19), and interpreted as the experimental mass, this renormalization procedure removes the infinities arising from the principal part terms. The potential energy V and the force $\mathbf{F}_{\text{ext}}$, imposed on the system externally, can only contain experimental quantities, and, therefore, are taken to be renormalized, *ab initio*. In the present example, it must be remembered that we have postulated that the charge executes simple harmonic motion according to (4.1). We use the equation of motion (2.8) to obtain the external force $\mathbf{F}_{\text{ext}}$ required for this motion. Accordingly, we identify V as

$$V = \tfrac{1}{2}m\nu^2x^2 \pm e^2\nu^2\mathbf{v}\cdot\mathbf{x}. \tag{4.23}$$

With this choice of V, the energy of (4.18) reduces to

$$H = m(1+\tfrac{1}{2}v^2) + \tfrac{1}{2}m\nu^2x^2 = m(1+\tfrac{1}{2}a^2\nu^2), \tag{4.24}$$

the correct expression for the simple harmonic motion of an equivalent uncharged particle of mass m. The radiation damping term (4.22) accordingly takes its usual form as $\mp(2/3)e^2\nu^2U/m$ where U is now the energy H of (4.24) apart from the rest mass. A Lagrangian can be written as

$$\begin{aligned} L &= \mathbf{p}\cdot\mathbf{v} - H \\ &= m(-1+\tfrac{1}{2}v^2) - \tfrac{1}{2}m\nu^2x^2 \mp e^2\nu^2\mathbf{v}\cdot\mathbf{x}. \end{aligned} \tag{4.25}$$

We verify that, according to this Lagrangian,

$$\mathbf{p} = \partial L/\partial\mathbf{v} = m\mathbf{v} \mp e^2\nu^2\mathbf{x}, \tag{4.26}$$

in agreement with (4.17). The equation of motion obtained from the Lagrangian (4.25) is

$$(d/dt)(m\mathbf{v} \mp e^2\nu^2\mathbf{x}) = -m\nu^2\mathbf{x} \mp e^2\nu^2\mathbf{v}$$

or

$$md\mathbf{v}/dt = -m\nu^2\mathbf{x}, \tag{4.27}$$

the correct expression for a simple harmonic oscillator. Comparison with (4.16) shows that

$$\mathbf{F}_{\text{ext}} = -m\nu^2\mathbf{x} \pm (2/3)e^2d^2\mathbf{v}/dt^2, \tag{4.28}$$

which agrees with (4.22).

Two conditions must be provided externally for simple harmonic motion of a charge. The potential energy V of (4.23) must be supplied, and in addition the external force $\mathbf{F}_{\text{ext}}$ of (4.28) must be maintained. The potential energy does not provide for the dissipative term in the force.

5. DISCUSSION OF RESULTS

In both examples considered, the principal-part terms in the self-field give rise to the electromagnetic kinetic reaction; all divergences in the equations of motion stem from principal-part integrals. These can be removed by renormalization of mass, as in (3.9) and (4.19). No infinities requiring charge renormalization have appeared. The dissipative effects associated with motion of a charge derive from the δ-function terms, which are finite. As we have seen in (4.22) and (4.28), the retarded potential gives the correct sign for the radiation damping; the advanced potential, the opposite sign. Only the retarded field needs to be considered, in agreement with causality. As we have shown in (1.9), causality requires that the self-field be described by the retarded potential.

According to (1.11) and (1.12) the field obtained as half the sum of retarded and advanced potentials gives the same electromagnetic kinetic reaction as the retarded potential, but without the damping. On the other hand, half the difference of retarded field minus advanced field gives the correct damping without the divergent principal-part terms. A finite theory can be obtained by using this latter combination alone,[2] justification for this choice of field being adduced from the absorber theory of radiation.[8] However, this choice, in the first place, violates the causality principle of (1.9). In the second place, it neglects the important attribute of the field theory, that it can account for the electromagnetic mass (even though it is divergent in the present theory).

The important point with respect to the electromagnetic mass, $\lim_{\alpha\to 0}e^2/\alpha$, is that it is additive to the bare mass μ, permitting the sum m to be interpreted as the experimental (finite) mass. Any argument advanced for obtaining a finite quantity as the sum of bare mass and electromagnetic mass goes beyond the present classical theory. In conformity with renormalization procedure, our equations become finite if in (2.6) to (2.10) we replace the bare mass μ by $\mu-\lim_{\alpha\to 0}e^2/\alpha$. However, alternatively, since only the sum of the bare mass and the electromagnetic mass is an experimental quantity, we may agree to set the bare mass, $\mu=0$,

[7] R. Becker, *Theorie der Elektrizität* (B. G. Teubner, Leipzig, 1933), 6th ed., Vol. II.

[8] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).

so that the entire mass is electromagnetic. The divergence in this case must be considered a defect of present theory. This procedure is consistent with the hypothesis of Abraham[7] that the mass of the electron (and positron) is entirely electromagnetic. When these particles are considered to be represented as point singularities in the field, rather than spherical shells of finite radius, there is no need to invoke extraneous forces to hold the charge together. For such a purely electromagnetic particle the equation of motion (2.8) takes the form (with $\mu=0$),

$$\mathbf{F}^K[\mathbf{x}(t),t]+\mathbf{F}^D[\mathbf{x}(t),t]+\mathbf{F}_{\text{ext}}=0. \tag{5.1}$$

The procedure of suppressing the bare mass μ in favor of the electromagnetic mass, even though the latter is infinite, is attractive because the electromagnetic mass is calculated from the theory.

Equation (4.16) for the retarded case (lower sign) is precisely the Dirac equation for one-dimensional motion of a charge in the nonrelativistic limit. It is sometimes[9] argued that this equation should be adopted as an exact mathematical representation for the force of radiative reaction within the framework of classical theory. As such, it introduces more freedom in specification of initial conditions than is allowed in Newtonian dynamics, where only the initial position and velocity of the particle are specified. This permits "runaway" solutions for (4.16) such as

$$\mathbf{x}(t)=\mathbf{z}(4e^2/9m^2)\exp(3mt/2e^2) \tag{5.2}$$

for the free-particle case in which one sets $\mathbf{F}_{\text{ext}}=0$. In our treatment of the oscillator it was shown that the dissipative term, $(2/3)e^2d^2\mathbf{v}/dt^2$, of (4.16) is cancelled by a term in the external force $\mathbf{F}_{\text{ext}}$ (4.28) required for the simple harmonic motion imposed by (4.1). The resulting equation of motion for the oscillator is then correctly the Newtonian equation (4.27). Run-away solutions such as (5.2) for free-particle motion are excluded according to the equation of motion (2.8) which is an integro-differential equation of second order, so that the motion of the charge is determined, as in Newtonian dynamics, once the initial position and velocity are specified. In this respect it is similar to Rohrlich's integro-differential equation.[1] But unlike Rohrlich's equation, which is obtained by adding a subsidiary condition to the Dirac equation, (2.8) is obtained directly from the Maxwell-Lorentz equations without the use of Dirac's equation or subsidiary conditions.

APPENDIX I. DERIVATION OF DISSIPATIVE POTENTIALS IN (1.1)

We start with the Lorentz gauge potentials in the form,[10]

$$A_\sigma(\mathbf{x},t)=\int d\mathbf{x}'\, j_\sigma(\mathbf{x}',t+R)/R,\quad R=|\mathbf{x}-\mathbf{x}'|, \tag{A1.1}$$

where the current four-vector for a point source is

$$j_\sigma(\mathbf{x},t)=ev_\sigma(t)\delta[\mathbf{x}-\mathbf{x}(t)], \tag{A1.2}$$

with $v_\sigma(t)$ defined in (1.2). Writing the Fourier integrals (with limits from $-\infty$ to $+\infty$),

$$A_\sigma(\mathbf{x},t)=\int A_\sigma(\mathbf{x},\omega)\exp(-i\omega t)d\omega, \tag{A1.3}$$

$$j_\sigma(\mathbf{x},t)=\int j_\sigma(\mathbf{x},\omega)\exp(-i\omega t)d\omega, \tag{A1.4}$$

and substituting them into (A1.1) gives

$$A_\sigma(\mathbf{x},\omega)=\int d\mathbf{x}'[j_\sigma(\mathbf{x}',\omega)/R]\exp(\mp i\omega R). \tag{A1.5}$$

Using (A1.2) and the inversion of (A1.4) we find

$$\begin{aligned} j_\sigma(\mathbf{x}',\omega)&=(e/2\pi)\int dt\, v_\sigma(t)\delta[\mathbf{x}'-\mathbf{x}(t)]\exp(i\omega t)\\ &=(e/16\pi^4)\int\cdots\int d\mathbf{k}dt\, v_\sigma(t)\\ &\qquad\times\exp i\{\mathbf{k}\cdot[\mathbf{x}'-\mathbf{x}(t)]+\omega t\} \end{aligned} \tag{A1.6}$$

so that

$$\begin{aligned} A_\sigma(\mathbf{x},\omega)&=(e/16\pi^4)\int\cdots\int d\mathbf{k}dt\, v_\sigma(t)\\ &\qquad\times\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t)]+\omega t\}I(\mathbf{k},\omega), \end{aligned} \tag{A1.7}$$

where, with $\mathbf{R}=\mathbf{x}'-\mathbf{x}$

$$I(\mathbf{k},\omega)=\int d\mathbf{R}\, R^{-1}\exp[i(\mathbf{k}\cdot\mathbf{R}\mp\omega R)].$$

Evaluation of $I(\mathbf{k},\omega)$ gives, with the notation of (1.3),

$$I(\mathbf{k},\omega)=\mp(2\pi i/k)[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)]. \tag{A1.8}$$

Substituting (A1.7) into (A1.3) yields (1.1).

APPENDIX II. EVALUATION OF SOME INTEGRALS IN (4.4) TO (4.9)

In spherical polar coordinates,

$$\iiint d\mathbf{k}=\int_0^\infty k^2dk\iint d\boldsymbol{\kappa}, \tag{A2.1}$$

where the unit vector $\boldsymbol{\kappa}$ ranges over the unit sphere. Since k is always positive, $\delta(k-mv)$ can only be satisfied for positive m; $\delta(k+mv)$ for negative m. The term with $n=m$ in $\mathbf{F}^D$, Eq. (4.5), clearly vanishes, so that $\mathbf{F}_0{}^D=0$, as stated in (4.11).

[9] G. N. Plass, Revs. Modern Phys. **33**, 37 (1961).

[10] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), pp. 174, 72.

The term with $n=m$ in $\mathbf{F}^K$, Eq. (4.4), is

$$\mathbf{F}_0{}^K=-(e^2i/2\pi^2)\lim_{\alpha\to 0}\int\int\int d\mathbf{k}\ \exp(i\mathbf{k}\cdot\boldsymbol{\alpha})\mathbf{k}\sum_m[k^2-(m\nu)^2]^{-1}[1-(m\nu/\mathbf{k}\cdot\mathbf{z})^2]J_m{}^2(a\mathbf{k}\cdot\mathbf{z}). \tag{A2.2}$$

Let $\mathbf{k}=k\boldsymbol{\kappa}$, and then let $k=K\nu$. Then,

$$\mathbf{F}_0{}^K=-(e^2\nu/2\pi^2)\lim_{\alpha\to 0}\frac{\partial}{\partial\boldsymbol{\alpha}}\int\int d\boldsymbol{\kappa}[I_1-I_2/(\boldsymbol{\kappa}\cdot\mathbf{z})^2], \tag{A2.3}$$

where

$$I_1=\frac{1}{2}\int_{-\infty}^{\infty}dK\ \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}K\nu)\sum_m[K^2/(K^2-m^2)]J_m{}^2(\boldsymbol{\kappa}\cdot\mathbf{z}Ka\nu), \tag{A2.4}$$

$$I_2=\frac{1}{2}\int_{-\infty}^{\infty}dK\ \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}K\nu)\sum_m[m^2/(K^2-m^2)]J_m{}^2(\boldsymbol{\kappa}\cdot\mathbf{z}Ka\nu). \tag{A2.5}$$

Now[5]

$$\sum_m[m^2/(K^2-m^2)]J_m{}^2=-1+(\pi K/\sin\pi K)J_{-K}(\boldsymbol{\kappa}\cdot\mathbf{z}Ka\nu)J_K(\boldsymbol{\kappa}\cdot\mathbf{z}Ka\nu)$$
$$=\sum_{m=1}^{\infty}\frac{(2m)!(\boldsymbol{\kappa}\cdot\mathbf{z}a\nu/2)^{2m}}{(m!)^2}\left(1+\frac{1^2}{K^2-1^2}\right)\left(1+\frac{2^2}{K^2-2^2}\right)\cdots\left(1+\frac{m^2}{K^2-m^2}\right). \tag{A2.6}$$

But on integration over K, the only nonvanishing integral is the divergent integral,

$$\int_{-\infty}^{\infty}dK\frac{m^2}{K^2-m^2}J_m{}^2=\int_{-\infty}^{\infty}dK\frac{(2m)!(\boldsymbol{\kappa}\cdot\mathbf{z}a\nu/2)^{2m}}{(m!)^2}=\int_{-\infty}^{\infty}dK\{[1-(a\nu\boldsymbol{\kappa}\cdot\mathbf{z})^2]^{-\frac{1}{2}}-1\}. \tag{A2.7}$$

Accordingly,

$$I_1=(\pi/\nu\alpha)\delta(\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}/\alpha)[1-(\boldsymbol{\kappa}\cdot\mathbf{z}a\nu)^2]^{-\frac{1}{2}},$$
$$I_2=(\pi/\nu\alpha)\delta(\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}/\alpha)\{[1-(\boldsymbol{\kappa}\cdot\mathbf{z}a\nu)^2]^{-\frac{1}{2}}-1\}.$$

We find, in the nonrelativistic limit, with $a\nu=v_{\max}\ll 1$,

$$\mathbf{F}_0{}^K=-(e^2/2\pi)\lim_{\alpha\to 0}(1/\alpha)\int\int d\boldsymbol{\kappa}\ \delta(\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}/\alpha)=-\lim_{\alpha\to 0}(\partial/\partial\boldsymbol{\alpha})(e^2/\alpha)=0, \tag{A2.8}$$

as stated in (4.10).

To calculate the terms in the Fourier series (4.4) to (4.9) for $n=m+1$ and $n=m-1$ (combined), we replace $\exp[i(n-m)\nu t]i^{n-m}J_mJ_n$ by $-(2m/a\mathbf{k}\cdot\mathbf{z})J_m{}^2\sin\nu t-(i/\mathbf{k}\cdot\mathbf{z})(\partial/\partial a)J_m{}^2\cos\nu t$, and $mn\exp[i(n-m)\nu t]i^{n-m}J_mJ_n$ by $-(i/\mathbf{k}\cdot\mathbf{z})\cos\nu t\partial J_m{}^2/\partial a-(2m^3/a\mathbf{k}\cdot\mathbf{z})J_m{}^2\sin\nu t+(2im/a\mathbf{k}\cdot\mathbf{z})J_m{}^2\cos\nu t+(m/\mathbf{k}\cdot\mathbf{z})\sin\nu t\partial J_m{}^2/\partial a$.

To calculate the terms in (4.14) and (4.15) for $n=m+2$ and $n=m-2$ (combined), we replace $i^{n-m}J_nJ_m\times\exp[i(n-m)\nu t]$ by

$$2\cos 2\nu t[J_m{}^2-2m^2J_m{}^2/a^2(\mathbf{k}\cdot\mathbf{z})^2+(\partial J_m{}^2/\partial a)/a(\mathbf{k}\cdot\mathbf{z})^2]+2mi\sin 2\nu t[-2J_m{}^2/a^2(\mathbf{k}\cdot\mathbf{z})^2+(\partial J_m{}^2/\partial a)/a(\mathbf{k}\cdot\mathbf{z})^2].$$

A new integral appears, similar to I_1 and I_2 in (A2.4), (A2.5), namely,

$$I_3=\frac{1}{2}\int_{-\infty}^{\infty}dK\ \exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}K\nu)\sum_m(K^2-m^2)^{-1}J_m{}^2(\boldsymbol{\kappa}\cdot\mathbf{z}Ka\nu). \tag{A2.9}$$

This integral vanishes in the limit $\boldsymbol{\alpha}\to 0$.