Nonsingular Formulation of the Brueckner Approximation for an Infinite Fermi System*

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In the approximation neglecting any but single pair correlations, singularities of the reaction matrix render Brueckner's integral for the average energy per particle a singular integral. Several attempts to overcome this difficulty have been unsuccessful. By considering the infinite Fermi system to be a limit of finite systems, it is shown that the correct result merely involves replacing Brueckner's ordinary integral over diagonal reaction matrix elements by a principal value integral. In a finite system the level shift of a Bethe-Goldstone state differs from the diagonal reaction matrix element by a normalization factor which does not approach unity uniformly in the integration variable as the volume becomes infinite. In the neighborhood of a singularity the expression for the two-particle energy shift takes the form $cy/(y^2+c\mathcal{U}^{-1})$, where y is the unperturbed energy measured from the singularity, c is the square of a matrix element, and $\mathcal{U}$ is the quantization volume. Hence as $\mathcal{U}\to\infty$ the sum over the energy y indeed approaches a principal value integral. An alternative derivation, employing a modified reaction matrix for which there is no difference between level shift and matrix element, leads to the same result. The general derivations are preceded by a soluble example.

The connection of the Brueckner approximation with a phase-shift approximation for low-density systems is discussed. Some corrections to the higher order terms in existing derivations of the "separation method" expansion of the Brueckner reaction matrix are given.

I. INTRODUCTION

In the lowest order of Brueckner's theory[1] the average energy per particle in an infinite system of fermions (e.g., nuclear matter or liquid He^3) is given by

$$E=E_0+\frac{1}{2}\frac{1}{(2\pi)^3\rho}\int_F d^3K d^3p\langle\mathbf{Kp}|t|\mathbf{Kp}\rangle_a, \qquad (1)$$

where E_0 is the average energy per particle in the unperturbed system, $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2$ is the center-of-mass wave vector of a pair, $\mathbf{p}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$ is the relative wave vector, and

$$\langle\mathbf{Kp}|t|\mathbf{Kp}\rangle_a\equiv\langle\mathbf{k}_1\mathbf{k}_2|t|\mathbf{k}_1\mathbf{k}_2\rangle_a$$
$$=\langle\mathbf{k}_1\mathbf{k}_2|t|\mathbf{k}_1\mathbf{k}_2\rangle-\langle\mathbf{k}_1\mathbf{k}_2|t|\mathbf{k}_2\mathbf{k}_1\rangle, \qquad (1.1)$$

with t being the reaction operator. The integrals are over the Fermi sphere F. Dependence on internal variables, such as spin, has been suppressed (i.e., we shall discuss "spinless" Fermions). The reaction matrix in the infinite medium in Goldstone's formulation is determined by an integral equation which is symbolically represented as follows: letting Q_a be a projection operator onto the unperturbed pair state ϕ_a, the "ath column" of t, tQ_a, satisfies

$$tQ_a=vQ_a+v[Q^{\text{out}}/(\eta_a-h_0)]tQ_a, \qquad (1.2)$$

where Q^{out} is a projection operator onto the exterior of the Fermi sea defined by the unperturbed system, h_0 is the unperturbed Hamiltonian for the pair, and η_a is the unperturbed energy of the unperturbed pair state ϕ_a on which tQ_a operates (to the right).

The Brueckner approximation to the Rayleigh-Schrödinger perturbation series, which is embodied in Eq. (1), has been interpreted by Weisskopf and collaborators[2,3] in a simple intuitive way as an independent pair model. $\langle\mathbf{Kp}|t|\mathbf{Kp}\rangle_a$ is regarded as a two-particle level shift resulting from the interaction between the members of the pair. With this interpretation one expects Eq. (1) to be essentially a low-density approximation.

Soon after Cooper's discovery[4] that in a Fermi gas bound pair states with energy less than twice the Fermi energy could be produced by arbitrarily weak attractive forces effective near the Fermi level, it was recognized by Gottfried,[5] Goldstone,[6] and others that Brueckner's reaction matrix could have singularities. The existence of singularities has by now been proved for a wide class of two-particle interactions by Emery[7] and by Van Hove.[8] Brueckner and others have conjectured that Eq. (1) is essentially correct and that the difficulty arising from reaction matrix singularities could be eliminated merely by replacing the ordinary integral in Eq. (1) by a Cauchy principal value integral. Bloch[9] has stressed that up until now the principal value prescription has been an arbitrary one, completely lacking in derivation.

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[1] The "canonical" formulation of Brueckner's perturbation method is contained in J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957), from which also the earlier literature can be traced.

[2] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (New York) **3**, 241 (1958).

[3] A. de-Shalit and V. Weisskopf, Ann. Phys. (New York) **5**, 248 (1958).

[4] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).

[5] K. Gottfried, CERN report (unpublished).

[6] J. Goldstone, thesis, Cambridge University, 1958 (unpublished).

[7] V. J. Emery, Nuclear Phys. **12**, 69 (1959).

[8] L. van Hove, Physica **25**, 849 (1959).

[9] C. Bloch, Compt. rend. congr. intern. phys. nucléaire, Paris **1958**, 251 (1959).

The difficulty has been sharpened by the finding of Chisholm and Squires[10] that if the energy is computed for a finite system using the analog of Eq. (1), then there is no unique limit as the system becomes infinite because there are reaction matrix singularities for various values of the volume. The "limit" of the energy depends on the particular sequence of volumes chosen and may take on any value from minus to plus infinity. Chisholm and Squires also attempted to overcome the singularity problem by reformulating perturbation theory as an expansion in a more complicated reaction matrix than Brueckner's. Their reaction matrix, which contains "hole-hole" scatterings as well as the "particle-particle" scatterings already present in Brueckner's reaction matrix equation, is of definite interest. (A similar reaction matrix has also been proposed by Iwamoto, by Klein and Prange, and by Galitski and Migdal.) However, as Chisholm and Squires showed, the new reaction matrix also possesses singularities, for finite as well as for infinite systems.

In this article the technique of obtaining the energy for an infinite system by computing first for a finite system and then passing to the limit, is reinvestigated. At first it is assumed that the Brueckner–Goldstone reaction matrix of Eq. (1.2) is to be used for finite systems as well as for the infinite system. It is found that in a finite system the two-particle energy shift, defined as the energy of the interacting pair (in a medium) minus the unperturbed energy of the pair, differs from the diagonal reaction matrix element by a normalization factor. The considerations regarding the level shifts are somewhat different from those of Fukuda and Newton,[11] and of deWitt.[12] The difference arises from the presence of the projection operator Q^{out} in Eq. (1.2) and its absence in the reaction matrix equations discussed in references 11 and 12.

It is proposed that Eq. (1) for a finite system be replaced by the sum over the two-particle level shifts. This amounts to reformulating perturbation theory in terms of a renormalized reaction matrix, and assuming the lowest term in the modified expansion to give a good approximation. The fact that there is great freedom in the choice of a reaction matrix equation was pointed out several years ago by Tobocman,[13] and has been made use of by Chisholm and Squires and by others as described above.

As a singularity of the t matrix is approached, the part of the Bethe-Goldstone wave function which is orthogonal to the unperturbed component grows more rapidly than does the diagonal reaction matrix element. As described in the abstract this effect leads to a principal value integral over the diagonal elements in the limit of infinite volume, at constant average particle density.

It may be objected that the replacement of reaction matrix elements by "level shifts," for finite systems, is just as arbitrary as assuming the principal value prescription to begin with. In order to answer this objection with more than an appeal to the intuitive physical correctness of the previous method, a second independent formulation is also given.

In the second formulation the equation for the reaction matrix in a finite quantization volume is taken to be[2,3]

$$\tau Q_a=\left[v+v\frac{Q^{\text{out}}}{\eta_a+(a|\tau|a)/(a|a)-h_0}\tau\right]Q_a. \quad (1.3)$$

In the limit of infinite volume $(a|\tau|a)/(a|a)$ goes to zero, but not uniformly in $\mathbf{p}_a$. As is shown in Appendix A, the reaction matrix τ is a member of Tobocman's class of reaction matrices, the members of which satisfy[13]

$$tQ_a=[v+vg_at-\delta_avg_av^{-1}t]Q_a. \quad (1.4a)$$

For the reaction matrix of Eq. (1.3),

$$g_a=\frac{Q^{\text{out}}}{\eta_a-h_0} \quad \text{and} \quad \delta_a=\frac{(a|\tau|a)}{(a|a)}. \quad (1.4b)$$

These choices are the ones which Tobocman finds give rise to many cancellations in the perturbation series for the energy. This is understandable since Eq. (1.3) corresponds to a Brillouin-Wigner perturbation expansion of the two-body problem in a medium, whereas the iterative solution of Eq. (1.2) is the Rayleigh-Schrödinger expansion of the two-body problem.

The two-particle level shift is simply $(a|\tau|a)/(a|a)$ with no additional normalization factor. The energy of a state of the whole system is given by an expansion in terms of τ. Being careful to sum over diagonal reaction matrix elements before letting $\mathcal{V}\to\infty$, one again obtains Eq. (1) with a principal value integral for the Brueckner approximation to the energy per particle in an infinite Fermi system.

In Sec. II a soluble one-dimensional example is treated. The limiting processes leading to the principal value are explicitly exhibited. The formal proofs in the general case, given in Secs. III and IV, make use of an expansion of the Bethe-Goldstone wave function of a pair in a complete set of solutions of a "Hermitian Bethe-Goldstone equation" introduced by Luders.[14]

Section V contains a rather general discussion of the low-density limit of the Brueckner theory and its connection with the theory of a low-density system of distinguishable particles (the phase-shift approximation).

II. SOLUBLE EXAMPLE

1. Solution in Terms of a Green Function on $(-\infty<x<\infty)$

In order not to have to introduce any approximations regarding the projections with respect to the Fermi sea

[10] J. Chisholm and E. Squires, Nuclear Phys. **13**, 156 (1959).
[11] N. Fukuda and R. G. Newton, Phys. Rev. **103**, 1558 (1956).
[12] B. S. deWitt, Phys. Rev. **103**, 1565 (1956).
[13] W. Tobocman, Phys. Rev. **107**, 203 (1957).
[14] G. Luders, Z. Naturforsch. **14a**, 1 (1959).

or regarding the dependence of the equations for the relative motion of a pair on their center-of-mass momentum, a one-dimensional example is chosen. Perhaps the simplest two-particle interaction is a delta-function potential. Requiring that the potential be symmetric so as to conserve parity, we choose

$$v(x)=-g[\delta(x-a)+\delta(x+a)], \quad a\geq 0. \tag{2.1}$$

For fixed center-of-mass momentum K, with $|K|\leq 2k_F$, the single-particle momenta k_1 and k_2 will both be outside the Fermi sea only if the relative momentum $k=\frac{1}{2}(k_1-k_2)$ satisfies $|k|>k_{FK}{}^+$, where

$$k_{FK}{}^+=k_F+\tfrac{1}{2}|K|. \tag{2.2a}$$

For a given value of K, values of the relative momentum corresponding to occupied states of the Fermi gas will be distinguished by writing p instead of k. One has

$$|p|<k_{FK}{}^-\equiv k_F-\tfrac{1}{2}|K|. \tag{2.2b}$$

The Bethe-Goldstone nonlocalized wave function $\psi_{Kp}(x)$ is related to the reaction matrix by

$$\langle k|t_K|p\rangle=\langle k|v|\psi_{Kp}\rangle, \tag{2.3}$$

where

$$\psi_{Kp}(x)=\phi_p(x)+w_{Kp}(x), \quad Q_K{}^{\text{out}}w_{Kp}=w_{Kp}, \tag{2.3a}$$

with

$$\begin{aligned} Q_K{}^{\text{out}}\phi_k(x)&=\phi_k(x), \quad |k|>k_{FK}{}^+,\\ &=0, \quad\quad |k|\leq k_{FK}{}^+. \end{aligned} \tag{2.3b}$$

If t is the Goldstone reaction matrix, satisfying Eq. (1.2), then $\psi_{Kp}(x)$ satisfies the Bethe-Goldstone (BG) equation.[15] In the effective-mass approximation in which the single-particle self-consistent interaction,

$$U(p_1)=U_0+\frac{1}{2}\left(\frac{1}{m^*}-\frac{1}{m}\right)p_1{}^2+U_2p_1{}^4+\cdots, \tag{2.4}$$

is approximated by the first two terms, the BG equation for the relative wave function is

$$-\frac{1}{m^*}\frac{d^2}{dx^2}\psi_{Kp}(x)+\left\{\int_{-\infty}^{-k_{FK}{}^+}+\int_{k_{FK}{}^+}^{\infty}\right\}dk \times\frac{e^{ikx}}{(2\pi)^{\frac{1}{2}}}\langle k|v|\psi_{Kp}\rangle=\epsilon_p\psi_{Kp}(x). \tag{2.5}$$

Taking the scalar product of Eq. (2.5) with respect to

$$\phi_p(x)=(2\pi)^{-\frac{1}{2}}e^{ipx}, \tag{2.6}$$

one obtains $\epsilon_p=p^2/m^*$. The solution of Eq. (2.5) can be written as

$$\psi_{Kp}(x)=\frac{e^{ikx}}{(2\pi)^{\frac{1}{2}}}+m^*\int_{-\infty}^{\infty}dx'\,G_{K\epsilon_p}(x,x')v(x')\psi_{Kp}(x'), \tag{2.7}$$

where the Green function is

$$G_{K\epsilon_p}(x,x')=\left\{\int_{-\infty}^{-k_{FK}{}^+}+\int_{k_{FK}{}^+}^{\infty}\right\}\frac{dk}{2\pi}\frac{e^{ik(x-x')}}{p^2-k^2}. \tag{2.8}$$

It is convenient to work directly with the symmetric and antisymmetric BG wave functions rather than to symmetrize later. As the two cases are very similar we shall be content to discuss only the spatially even state. For the delta-function potential of Eq. (2.1) the even wave function is

$${}^+\psi_{Kp}(x)=\pi^{-\frac{1}{2}}\cos px-2m^*g\,{}^+G_{K\epsilon_p}(x,a)\,{}^+\psi_{Kp}(a). \tag{2.9}$$

The corresponding reaction matrix elements are

$$\begin{aligned} {}^+t_{Kp}(k)&=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}dx\cos(kx)v(x)\,{}^+\psi_{Kp}(x)\\ &=-2\pi^{-\frac{1}{2}}g\cos ka\,{}^+\psi_{Kp}(a). \end{aligned} \tag{2.10}$$

Provided that the quantity

$${}^+D_{Kp}\equiv 1+2m^*g\,{}^+G_{K\epsilon_p}(a,a) \tag{2.11}$$

does not vanish, where

$${}^+G_{K\epsilon_p}(x,x')=\frac{1}{\pi}\int_{k_{FK}{}^+}^{\infty}dk\,\frac{\cos kx\cos kx'}{p^2-k^2}, \tag{2.12}$$

Eq. (2.9) has as its solution at $x=a$

$${}^+\psi_{Kp}(a)=\pi^{-\frac{1}{2}}\cos(pa)/{}^+D_{Kp}. \tag{2.13}$$

Hence at arbitrary x

$${}^+\psi_{Kp}(x)=\pi^{-\frac{1}{2}}\left\{\cos px-\frac{2m^*g}{{}^+D_{Kp}}\cos(pa)\,{}^+G_{K\epsilon_p}(x,a)\right\} \tag{2.14}$$

and

$${}^+t_{Kp}(p)=-(2/\pi)g\cos^2(pa)/{}^+D_{Kp}. \tag{2.15}$$

If ${}^+D_{Kp}$ vanishes then there exists a nontrivial localized (i.e., bound pair) solution of Eq. (2.5),

$${}^+\psi_{K\epsilon_p}(x)=-2m^*g\,{}^+G_{K\epsilon_p}(x,a)\,{}^+\psi_{K\epsilon_p}(a), \tag{2.16}$$

with an arbitrary value of ${}^+\psi_{K\epsilon_p}(a)$. Thus the singularities of the reaction matrix are seen to be connected[5–8] with the existence of bound pair states.

For $x=x'=a$, some properties of the Green function may be inferred by inspection of Eq. (2.12) (cf. also Fig. 1)

(i) ${}^+G_{K\epsilon}(a,a)<0$ for all $\epsilon<(k_{FK}{}^+)^2/m^*$. (2.17a)

(ii) ${}^+G_{K\epsilon}(a,a)$ is a decreasing function of ϵ with asymptotic values 0 as $\epsilon\to-\infty$ and $-\infty$ as $\epsilon\to(k_{FK}{}^+)^2/m^*$. (2.17b)

(iii) ${}^+G_{K\epsilon}(a,a)$ is an increasing function of $k_{FK}{}^+$. In particular for given k_F, ${}^+G_{0,\epsilon}(a,a)\leq{}^+G_{K,\epsilon}(a,a)\leq{}^+G_{2k_F,\epsilon}(a,a)$. (2.17c)

[15] H. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1956).

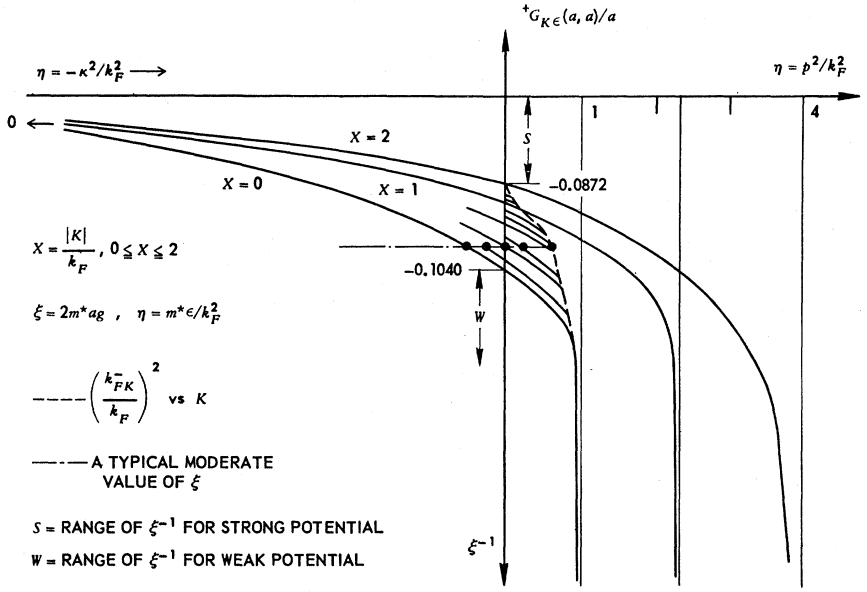

FIG. 1. Plot of $(a^{-1})\ ^+G_{K\epsilon}(a,a)$ vs $\eta=m^*\epsilon/k_F^2$ for $ak_F=1$ and for various values of $X=|K|/k_F$. The ranges of $-\xi^{-1}=-(2m^*ag)^{-1}$ corresponding to strong (S), moderate, and weak (W) attractions are displayed on the vertical axis. Bound states occur when $G/a=-\xi^{-1}$; for the case of a typical moderate attraction some of the bound states are indicated by dots. For a given value of X, the range of values of η for occupied states is $0\leq\eta\leq(k_{FK}^-/k_F)^2$. The upper limit as a function of K is shown by a dashed line.

The reaction matrix elements $^+t_{Kp}(p)$ which occur in the Brueckner energy formula, Eq. (1), have values of the energy parameter ϵ_p less than $(k_{FK}^-)^2/m^*$. Since $k_{FK}^-\longrightarrow k_{FK}^+$ as $K\longrightarrow 0$, there is a singularity of $^+G_{K\epsilon_p}(a,a)$ in the "occupied" region of momentum space at $K=0$, $p=k_F$. This kind of singularity has nothing to do with the form of interaction potential; it is a general feature of the BG equation. A singularity in G does not imply a singularity in the wave function or in reaction matrix elements. In the present case this is transparent, for we find from (2.13)–(2.15)

$$^+\psi_{0,k_F}(a)=0,\quad ^+t_{0,k_F}(k_F)=0,$$

and

$$^+\psi_{0,k_F}(x)=\pi^{-\frac{1}{2}}(\cos px-\cos pa).$$

A singularity in the reaction matrix results only at a node of D_{Kp}, i.e., when

$$(a^{-1})\ ^+G_{K\epsilon_p}(a,a)=-(2m^*ag)^{-1}\equiv-\xi^{-1}. \quad (2.18)$$

The strength of the attraction may be classified according to the kinds of bound states which occur. The potential will be said to be "weak" for given values of m^*, a, and k_F if

$$-\xi^{-1}<(1/a)\ ^+G_{0,0}{}^{\rm out}(a,a), \quad (2.18a)$$

moderate if

$$(1/a)\ ^+G_{0,0}{}^{\rm out}(a,a)\leq-\xi^{-1}\leq(1/a)\ ^+G_{2k_F,0}(a,a), \quad (2.18b)$$

and "strong" if

$$(1/a)\ ^+G_{2k,0}{}^{\rm out}(a,a)<-\xi^{-1}. \quad (2.18c)$$

In Fig. 1, curves of $(1/a)\ ^+G_{K,\epsilon}(a,a)$ vs $\eta\equiv m^*\epsilon/k_F^2$ are given for a specified value of $d=ak_F$. Regions of values of ξ corresponding to strong, moderate, and weak attractions divide the vertical axis into three intervals labeled S, M, and W, respectively.

It is seen that a "weak" attraction (small g) gives rise to no negative energy bound states, but to exactly one positive energy bound state for each $X\equiv|K|/k_F$ less than some maximum value $X_0(g)$, where $X_0(g)$ has the property that

$$X_0(g)\longrightarrow 0 \quad \text{as} \quad g\longrightarrow 0.$$

In particular, a bound state with $K=0$ exists for an arbitrarily weak attraction. For a "moderate" potential there are negative energy bound states for sufficiently small values of K, say for $0\leq X\leq X_1(g)<2$, and positive energy bound states for $X_1(g)<X\leq X_0(g)\leq 2$. For a "strong" attraction there are no positive-energy bound states for any K with $0\leq|K|\leq 2k_F$; instead there are bound states of negative energy for all K in this range.

The curves of Fig. 1 have been computed as follows. For $\epsilon=\epsilon_p\geq 0$ the integral in Eq. (3) may be evaluated analytically, with the result

$$\binom{1}{a}\ ^+G_{K\epsilon_p}(a,a)=-\frac{1}{4\pi ap}\left\{\ln\frac{1+p/k_{FK}^+}{1-p/k_{FK}^+}+\cos 2ap[\mathrm{Ci}(2a(k_{FK}^++p))-\mathrm{Ci}(2a(k_{FK}^+-p))]\right.$$
$$\left.+\sin 2ap[-\pi+\mathrm{Si}(2a(k_{FK}^++p))+\mathrm{Si}(2a(k_{FK}^+-p))]\right\}, \quad (2.19a)$$

where $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ are the cosine and sine integrals, respectively. In particular, for zero energy one has

$$\binom{1}{a}\ ^+G_{K,0}(a,a)=-\frac{1}{\pi}\left[\frac{\pi}{2}-\mathrm{Si}(2ak_{FK}^+)-\frac{\cos^2(ak_{FK}^+)}{ak_{FK}^+}\right]. \quad (2.19b)$$

For negative energies, $\epsilon=\epsilon_\kappa\equiv-\kappa^2/m^*$, one has to evaluate

$$\frac{1}{a}{}^+G_{K,\epsilon_\kappa}=-\frac{1}{\pi a}\int_{k_{FK}{}^+}^{\infty}dk\,\frac{\cos^2 ka}{k^2+\kappa^2}$$
$$=-\frac{1}{2\pi(ak_F)}\frac{1}{y}\left\{\frac{\pi}{2}-\arctan[(1+|K|/2k_F)/y]+\frac{\pi}{2}e^{-2(ak_F)y}-\int_0^{(1+|K|/2k_F)/y}dx\,\frac{\cos(2ak_Fyx)}{1+x^2}\right\},\quad(2.19c)$$

where $y=\kappa/k_F$. In spite of its simplicity the integral does not seem to have a simple representation in terms of tabulated functions. It was evaluated numerically for Fig. 1. The dependence of the energy of negative energy bound states on g and K/k_F may be seen from Fig. 1. For consideration of the dependence on the range a it is more convenient to consider k_FG than G/a. In the limit $a\to 0$ one has

$$\begin{aligned}k_F{}^+G_{K\epsilon}(0,0)&=-(k_F/2\pi p)\ln[(1+p/k_{FK}{}^+)/(1-p/k_{FK}{}^+)],&\epsilon&=\epsilon_p\\&=-(1/\pi)(1+|K|/2k_F),&\epsilon&=0\\&=-(k_F/\pi\kappa)[\pi/2-\arctan(k_{FK}{}^+/\kappa)],&\epsilon&=\epsilon_\kappa.\end{aligned}\quad(2.19d)$$

Bound states occur when

$$k_F{}^+G_{K\epsilon}(a,a)=-k_F/2m^*g.\quad(2.19e)$$

Since Fig. 1 is for the case $ak_F=1$ it may be reinterpreted as a graph of $k_F{}^+G_{K\epsilon}(k_F{}^{-1},k_F{}^{-1})$ with $\xi=k_F/2m^*g$. The corresponding curves for $a=0$ would lie somewhat lower than those for $a=k_F{}^{-1}$. From (2.19e) one sees that the curves for $K=0$ and $2k_F$ would cross the vertical axis at about -0.318 and -0.159, respectively. Thus, a value of g which gives rise to a "moderate" attraction at $a=k_F{}^{-1}$, provides a "strong" attraction when placed near $a=0$. Finally, for $k_F=0$ and $a=0$ we find from (2.19d) $\kappa=mg$, a well-known result for the one-dimensional Schrödinger equation.

A similar treatment can be given[16] if a delta-function repulsion is added to the delta-function attraction.

2. Solution on $-L<x<L$ and the Level Shift

The equations of Sec. II.1 apply equally well in a quantization volume $-L<x<L$ with periodic boundary conditions if one replaces $\int dx$ by $(\pi/L)\sum_k$ and $\int_{-\infty}^{\infty}dx$ by $\int_{-L}^{L}dx$. A singularity in ${}^+t_{Kp}(p)$ is accompanied by a singularity in the part of the wave function $\psi_{Kp}(x)$ containing "excited" components, previously denoted by $w_{Kp}(x)$. Denoting the scalar product on $-L\leq x\leq L$ by $(\ ,\ )$ to distinguish it from $\langle\ ,\ \rangle$ used on $-\infty\leq x\leq\infty$, one has

$$({}^+w_{Kp}{}^{(L)},{}^+w_{Kp}{}^{(L)})=\left(\frac{2m^*g}{\pi}\right)^2\left[\frac{\cos(pa)}{{}^+D_{Kp}{}^{(L)}}\right]^2{}^+A_{Kp}{}^{(L)},\quad(2.20)$$

where

$${}^+A_{Kp}{}^{(L)}=\pi\int_{-L}^{L}dx[{}^+G_{Kp}{}^{(L)}(x,a)]^2$$
$$=\frac{\pi}{L}\sum_{k>k_{FK}{}^+}\frac{\cos^2(ka)}{(p^2-k^2)^2}.\quad(2.21)$$

Taking the scalar product of Eq. (2.5) with respect to ψ_{Kp} and noticing that $\psi_{Kp}{}^*Q^{\text{out}}=\psi_{Kp}{}^*-\phi_p{}^*$, one finds

$$\frac{(\psi_{Kp}|-(1/m^*)(d^2/dx^2)+v|\psi_{Kp})}{(\psi_{Kp},\psi_{Kp})}=\frac{p^2}{m^*}+\frac{(p|v|\psi_{Kp})}{(\psi_{Kp},\psi_{Kp})}.\quad(2.22)$$

The left-hand side is the energy of the interacting pair. The level shift for symmetric states is then

$$\frac{[{}^+\psi_{Kp}|-(1/m^*)(d^2/dx^2)+v|{}^+\psi_{Kp}]}{({}^+\psi_{Kp},{}^+\psi_{Kp})}-\frac{p^2}{m^*}=\frac{\pi}{L}\frac{{}^+t_{Kp}(p)}{[1+(\pi/L)({}^+w_{Kp},{}^+w_{Kp})]}$$
$$=\frac{\pi}{L}\frac{-(2g/\pi)\cos^2(pa)\,{}^+D_{Kp}{}^{(L)}}{[{}^+D_{Kp}{}^{(L)}]^2+(\pi/L)[2m^*g\cos(pa)/\pi]^2\,{}^+A_{Kp}{}^{(L)}}\equiv\frac{\pi}{L}\delta E_{Kp}.\quad(2.23)$$

The expression for the average energy per particle analogous to Eq. (1) is, with the total number of particles equal to Lk_F/π,

$$E=E_0+\frac{1}{2}\frac{1}{2k_F}\frac{2\pi}{L}\sum_{K=-k_F}^{k_F}\frac{\pi}{L}\sum_p{}'\,\delta E_{K,p},\quad(2.24)$$

where the prime signifies $|p|\leq k_{FK}{}^-$. For large L it is true that ${}^+D_{Kp}{}^{(L)}$, ${}^+A_{Kp}{}^{(L)}$, and the value $p_K{}^{(L)}$ of p for which the reaction matrix element is singular are nearly independent of L. Using the Taylor expansions of the momentum-dependent quantities about $p_K\equiv p_K{}^{(\infty)}$, one sees that

$$\lim_{L\to\infty}(E-E_0)=\frac{1}{4k_F}P\int_F dK\,dp\,{}^+t_{Kp}(p),\quad(2.25)$$

where P signifies taking the principal value.

[16] R. L. Becker, Bull. Am. Phys. Soc. **5**, 434 (1960).

3. Alternative Treatment Using a Different Reaction Matrix

Using the reaction matrix τ of Eq. (1.3), the equation for the relative wave function of the interacting pair ψ_{Kp} defined by ${}^+\tau_K\phi_p=v\,{}^+\psi_{Kp}$ is

$$-\frac{1}{m^*}\frac{d^2}{dx^2}\psi_{Kp}(x)+\frac{\pi^{-\frac{1}{2}}\cos(px)({}^+\phi_p|v|{}^+\psi_{Kp})}{(L/\pi)}+\frac{\pi}{L}\sum_{k=k_{FK}+}^{\infty}\frac{\cos(kx)}{\pi^{\frac{1}{2}}}({}^+\phi_k|v|{}^+\psi_{Kp})=\epsilon_{Kp}\,{}^+\psi_{Kp}(x). \quad (2.26)$$

Taking the scalar product with respect to ${}^+\phi_p(x)=\pi^{-\frac{1}{2}}\cos px$, one finds that the energy eigenvalue is not equal to the unperturbed energy p^2/m^*, but instead to

$$\epsilon_{Kp}=\frac{p^2}{m^*}+\frac{\pi}{L}\,{}^+\tau_{Kp}(p), \quad (2.27)$$

where

$${}^+\tau_{Kp}(k)=({}^+\phi_k|v|{}^+\psi_{Kp}). \quad (2.28)$$

On the other hand, as may be verified by taking the scalar product with respect to ${}^+\psi_{Kp}$, the eigenvalue is equal to the energy of the interacting pair; hence the level shift is

$$\frac{({}^+\psi_{Kp}|-(1/m^*)(d^2/dx^2)+v|{}^+\psi_{Kp})}{({}^+\psi_{Kp},{}^+\psi_{Kp})}-\frac{p^2}{m^*}=\frac{\pi}{L}\,{}^+\tau_{Kp}(p). \quad (2.29)$$

For the delta-function attraction one finds

$${}^+\tau_{Kp}{}^{(L)}(p)=-2\pi^{-1}g\cos^2(pa)\Big/\left[1+\frac{2m^*g}{\pi}\frac{\pi}{L}\sum_{k>k_{Fk}+}\frac{\cos^2(pa)}{[p^2+m^*(\pi/L)\,{}^+\tau_{Kp}{}^{(L)}(p)]-k^2}\right]. \quad (2.30)$$

For finite L this equation has infinitely many solutions. It is shown in Appendix B that there is a unique physical solution in the limit $L\to\infty$.

Expanding the denominator in powers of τ/L one obtains

$${}^+\tau_{Kp}{}^{(L)}(p)=\frac{-2\pi^{-1}g\cos^2(pa)}{{}^+D_{Kp}{}^{(L)}-2\pi m^{*2}gA_{Kp}{}^{(L)}\,{}^+\tau_{Kp}(p)/L+O([{}^+\tau_{Kp}{}^{(L)}(p)/L]^2)}. \quad (2.31)$$

Neglecting the terms in the denominator of order $(\tau/L)^2$ one has, for sufficiently large L and $p\approx p_K$,

$${}^+\tau_{Kp}{}^{(L)}(p)=\frac{p-p_K}{|p-p_K|}\left\{\frac{\cos(p_Ka)}{\pi m^*}\left[\frac{L}{A_{KpK}{}^{(L)}}\right]^{\frac{1}{2}}+\frac{[(d/dp)D_{Kp}{}^{(L)}]_{p=pK}}{2\pi m^{*2}gA_{KpK}{}^{(L)}}L|p-p_K|\right\}. \quad (2.32)$$

Thus ${}^+\tau_{Kp}{}^{(L)}(p)$ takes on equal and opposite values in the neighborhood of the singularity, and so

$$\lim_{L\to\infty}\frac{\pi}{L}\sum_p{}^+\tau_{Kp}{}^{(L)}(p)=P\int{}^+t_{Kp}(p)dp. \quad (2.33)$$

III. GENERAL PROOF, USING GOLDSTONE'S REACTION MATRIX

To construct a general proof it is convenient to expand the Bethe-Goldstone wave function in terms of a complete set of solutions of the "Hermitian Bethe-Goldstone equation" which has been discussed by Luders.[14] This equation is

$$(h_0+Q^{\text{out}}vQ^{\text{out}})\chi=\eta\chi. \quad (3.1)$$

$$\chi=\chi_{\text{I}}+\chi_{\text{II}},\quad \chi_{\text{I}}=Q^{\text{out}}\chi,\quad \chi_{\text{II}}=(1-Q^{\text{out}})\chi; \quad (3.2)$$

Eq. (3.1) generates separate uncoupled equations for χ_{I} and χ_{II}, namely

$$(h_0+Q^{\text{out}}vQ^{\text{out}}-\eta)\chi_{\text{I}}=0=(h_0+Q^{\text{out}}v-\eta)\chi_{\text{I}} \quad (3.3a)$$

and

$$(h_0-\eta)\chi_{\text{II}}=0. \quad (3.3b)$$

Thus the solutions of Eq. (3.1) fall into two classes. The Class I solutions are solutions of the ordinary Bethe-Goldstone equation, consisting entirely of components lying outside the Fermi sea. The Class II solutions satisfy the unperturbed pair equation.

Let $E_{\min}(K)$ be the minimal energy of two unperturbed states with total momentum K. In the effective-mass approximation,

$$E_{\min}(K)-2U_0=K^2/[2(2m^*)].$$

Also let $E_F(K)$ be the two-particle Fermi energy for two unperturbed states with total momentum K. In one dimension in the effective-mass approximation, $E_F(K)=E_{\min}(K)+(k_{FK}^-)^2/[2(m^*/2)]$. The center-of-mass motion separates in the general case. One has

$$\chi_{\text{II},\mathbf{Kp}}(\mathbf{x}_1,\mathbf{x}_2)=\Phi_{\mathbf{K}}(\mathbf{X})\phi_{\mathbf{p}}(\mathbf{x}),\quad h_0\chi_{\text{II},\mathbf{Kp}}=\eta_{\mathbf{Kp}}\chi_{\text{II},\mathbf{Kp}} \quad (3.4a)$$

and

$$\chi_{\text{I},\mathbf{K}\eta\alpha}(\mathbf{x}_1,\mathbf{x}_2)=\Phi_{\mathbf{K}}(\mathbf{X})\chi_{\mathbf{K}\eta\alpha}{}^{\text{I}}(\mathbf{x}), \quad (3.4b)$$

where α stands for any additional quantum numbers needed to completely specify the wave function and where $\Phi_{\mathbf{K}}(\mathbf{X})=e^{i\mathbf{K}\cdot\mathbf{X}}/(2\pi)^{\frac{3}{2}}$. For fixed $\mathbf{K}$ the spectrum of

η for the Class II solutions is a continuum with

$$E_{\min}(K)\leq\eta_{\mathrm{II}}\leq E_F(K).$$

For the Class I solutions there is a continuum with η_{I} greater than $E_F(K)$ and possibly also a discrete spectrum below $E_F(K)$. The wave functions corresponding to the discrete spectrum are localized (i.e., bound state) solutions of the ordinary Bethe-Goldstone equation. In the example of Sec. II a singularity of the Goldstone reaction matrix was accompanied by the existence of a bound pair state. Thus by expanding the nonlocalized solutions of the ordinary Bethe-Goldstone equation with energies less than $E_F(K)$ in terms of the solutions of the Hermitian Bethe-Goldstone equation, the singularity will be explicitly exhibited.

For the nonlocalized solutions of the ordinary Bethe-Goldstone equation, the factor referring to the relative motion will be written, as in Sec. II, as

$$\psi_{\mathbf{Kp}}(\mathbf{x})=\phi_{\mathbf{p}}(x)+w_{\mathbf{Kp}}(\mathbf{x}),\quad Q_{\mathbf{K}}{}^{\mathrm{out}}w_{\mathbf{Kp}}=w_{\mathbf{Kp}}. \tag{3.5}$$

In the χ representation

$$w_{\mathbf{Kp}}(\mathbf{x})=\left\{\int_{E_F(K)}^{\infty}d\eta+\sum_{\eta<E_F(K)}\right\}\times\sum_{\alpha}\tilde{w}_{\mathbf{Kp}}(\eta\alpha)\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}(\mathbf{x}). \tag{3.6}$$

The ordinary Bethe-Goldstone equation reduces to

$$(h_0+Q^{\mathrm{out}}vQ^{\mathrm{out}})\Phi_{\mathbf{K}}w_{\mathbf{Kp}}=\Phi_{\mathbf{K}}[\epsilon_{\mathbf{Kp}}w_{\mathbf{Kp}}-Q_{\mathbf{K}}{}^{\mathrm{out}}v\phi_{\mathbf{p}}]. \tag{3.7}$$

Taking the scalar product with respect to $\Phi_{\mathbf{K}}\phi_{\mathbf{p}}$ verifies that the eigenvalue $\epsilon_{\mathbf{Kp}}$ is equal to the unperturbed energy $\eta_{\mathbf{Kp}}$ and taking the scalar product with respect to $\Phi_{\mathbf{K}}\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}$ yields

$$\tilde{w}_{\mathbf{Kp}}(\eta\alpha)=\langle\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}}\rangle/(\eta_{\mathbf{Kp}}-\eta), \tag{3.8}$$

where it has been assumed that the $\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}$ have been orthonormalized. It is apparent that $\psi_{\mathbf{Kp}}$ will be singular if $\eta_{\mathbf{Kp}}$ is equal to an energy η of one of the bound pair states unless[7]

$$\langle\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}}\rangle=0 \tag{3.9}$$

for all the bound states $\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}$ with energy η. This exception will occur if $v(1-Q^{\mathrm{out}})=0$ as in Cooper's original example.[4] But usually the Fredholm alternative property will hold: Either no bound state with energy η exists and the Bethe-Goldstone wave function is nonsingular, or a bound state does exist and the Bethe–Goldstone wave function is singular.

Going over to a finite system, the level shift derived as in Sec. II is, with $\mathcal{U}'\equiv\mathcal{U}/(2\pi)^3$,

$$\delta E_{\mathbf{Kp}}=\frac{t_{\mathbf{Kp}}(\mathbf{p})}{1+(w_{\mathbf{Kp}},w_{\mathbf{Kp}})/\mathcal{U}'}, \tag{3.10}$$

with

$$t_{\mathbf{Kp}}(\mathbf{p})=(\phi_{\mathbf{p}}|v|\psi_{\mathbf{Kp}}). \tag{3.11}$$

As $\epsilon_{\mathbf{Kp}}$ approaches the energy η_b of a particular bound pair state, or set of states $\chi_{\mathbf{K}\eta_b\alpha}{}^{\mathrm{I}}$

$$\delta E_{\mathbf{Kp}}=\frac{(\eta_{\mathbf{Kp}}-\eta_b)\left(\phi_{\mathbf{p}}|v|\phi_{\mathbf{p}}+\sum_{\eta\alpha}{}'\dfrac{(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})}{\eta_{\mathbf{Kp}}-\eta}\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}\right)+\sum_{\alpha}|(\chi_{\mathbf{K}\eta_b\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})|^2}{(\eta_{\mathbf{Kp}}-\eta_b)\left[1+\dfrac{1}{\mathcal{U}'}\sum_{\eta,\alpha}{}'\dfrac{(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})^2}{(\eta_{\mathbf{Kp}}-\eta)^2}\right]+\dfrac{1}{\mathcal{U}'}\sum_{\alpha}\dfrac{|(\chi_{\mathbf{K}\eta_b\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})|^2}{\eta_{\mathbf{Kp}}-\eta_b}}, \tag{3.12}$$

where the prime signifies excluding $\eta=\eta_b$ from the summation, as the contributions from the states $\chi_{\mathbf{K}\eta_b\alpha}{}^{\mathrm{I}}$ are given separately.

The energies η are only weakly dependent on the volume for large values of $\mathcal{U}$. Expanding $\eta(\mathcal{U})$ about $\eta(\infty)$, only the constant term will contribute in passing to the limit $\mathcal{U}\rightarrow\infty$. A similar remark is true for the matrix elements $(\chi_{K\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})$. Thus, letting $y=\eta_{\mathbf{Kp}}-\eta_b$, the dependence of $\delta E_{\mathbf{Kp}}$ on y and on the volume $\mathcal{U}$ is of the form

$$\delta E_{Kp}=\frac{ay+c}{y+dy\mathcal{U}^{-1}+cy^{-1}\mathcal{U}^{-1}}\approx c\frac{y}{y^2+c\mathcal{U}^{-1}} \tag{3.13}$$

for y small and $\mathcal{U}$ large. Taking the limit as $\mathcal{U}\rightarrow\infty$ of the sum over y gives a principal value integral. The usual divergent result corresponds to taking the limit of each term as $\mathcal{U}\rightarrow\infty$ and then integrating.

It would be of interest to have an exact expansion for the average energy per particle in terms of a renormalized reaction matrix T with diagonal elements

$$(\mathbf{k}|T_{\mathbf{K}}|\mathbf{k})=(\mathbf{k}|t_{\mathbf{K}}|\mathbf{k})/(\psi_{\mathbf{Kp}},\psi_{\mathbf{Kp}}). \tag{3.14}$$

Unfortunately $(\mathbf{k}|T_{\mathbf{K}}|\mathbf{k})$ does not represent a sum of terms corresponding to linked diagrams in t, for

$$\frac{(\psi_{\mathbf{Kp}},\psi_{\mathbf{Kp}})}{(\phi_{\mathbf{p}},\phi_{\mathbf{p}})}=1+(\mathcal{U}')^{-2}\sum_{\mathbf{k}}{}^{(\mathrm{out})}\frac{|(\mathbf{k}|t_{\mathbf{K}}|\mathbf{p})|^2}{(\eta_{\mathbf{Kp}}-\eta_{\mathbf{Kk}})^2}.$$

This does not mean, however, that the diagonal elements of T do not provide a perfectly good lowest order term for the system energy. It seems likely that in a complete formulation, principal value integrals will occur in the higher orders of the expansion in t, as well as in the lowest order approximation of Brueckner.

IV. GENERAL PROOF USING A MODIFIED REACTION MATRIX

Corresponding to the reaction matrix equation (1.3) there is an equation[2] for the wave function of the

interacting pair analogous to the Bethe-Goldstone equation in a finite quantization volume. A special case of this modified equation was given in Sec. II.3. Writing the relative wave function as in (3.5) the general equation, obtainable from (1.3), is

$$(h_0+Q^{\text{out}}v-\epsilon_{\mathbf{Kp}})\Phi_{\mathbf{K}}\psi_{\mathbf{Kp}} = -(\phi_{\mathbf{p}},\phi_{\mathbf{p}})^{-1}(\phi_{\mathbf{p}}|v|\psi_{\mathbf{Kp}})\Phi_{\mathbf{K}}\phi_{\mathbf{p}}. \quad (4.1)$$

Taking the scalar product with respect to $\Phi_{\mathbf{K}}\phi_{\mathbf{p}}$, one has

$$\epsilon_{\mathbf{Kp}}=\eta_{\mathbf{Kp}}+(\phi_{\mathbf{p}},\phi_{\mathbf{p}})^{-1}\tau_{\mathbf{Kp}}(\mathbf{p}) \quad (4.2)$$

with

$$\tau_{\mathbf{Kp}}(\mathbf{k})=(\phi_{\mathbf{k}}|v|\psi_{\mathbf{Kp}}). \quad (4.3)$$

The equation for $w_{\mathbf{Kp}}$ is then

$$[h_0+Q^{\text{out}}vQ^{\text{out}}-\epsilon_{\mathbf{Kp}}]\Phi_{\mathbf{K}}w_{\mathbf{Kp}}=-\Phi_{\mathbf{K}}Q_{\mathbf{K}}{}^{\text{out}}v\phi_{\mathbf{p}}, \quad (4.4)$$

together with the constraints implied in (4.2) and (4.3). Ignoring the constraints, a solution

$$\Phi_{\mathbf{K}}w_{\mathbf{Kp}}=-[h_0+Q^{\text{out}}vQ^{\text{out}}-\epsilon_{\mathbf{Kp}}]^{-1}\Phi_{\mathbf{K}}Q_{\mathbf{K}}{}^{\text{out}}v\phi_{\mathbf{p}} \quad (4.5)$$

of (4.4) exists for an arbitrary $\epsilon_{\mathbf{Kp}}<E_F(K)$ provided that there is no $\chi_{\mathrm{I},\mathbf{K}\eta\alpha}$ with $\eta=\epsilon_{\mathbf{Kp}}$ and with $(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})\neq 0$. Taking the scalar product of (4.4) with respect to $\chi_{\mathrm{I},\mathbf{K}\eta\alpha}$, and using (3.6), one finds

$$\tilde{w}_{\mathbf{Kp}}(\eta\alpha)=\frac{(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})}{\epsilon_{\mathbf{Kp}}-\eta}. \quad (4.6)$$

The solution of (3.5) and (4.1) is therefore the solution of the coupled equations

$$\psi_{\mathbf{Kp}}(\mathbf{x})=\phi_{\mathbf{p}}(\mathbf{x})+\sum_{\eta,\alpha}\frac{(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}(\mathbf{x})}{\epsilon_{\mathbf{Kp}}-\eta} \quad (4.7a)$$

and

$$(\epsilon_{\mathbf{Kp}}-\eta_{\mathbf{Kp}})(\phi_{\mathbf{p}},\phi_{\mathbf{p}})=(\phi_{\mathbf{p}}|v|\psi_{\mathbf{Kp}}). \quad (4.7b)$$

Since $(\phi_{\mathbf{p}},\phi_{\mathbf{p}})=\mathcal{V}'\equiv\mathcal{V}/(2\pi)^3$, one sees that $\epsilon_{\mathbf{Kp}}-\eta_{\mathbf{Kp}}=O(\mathcal{V}^{-1})$ except when $\eta_{\mathbf{Kp}}$ is near an energy η of a bound pair state, $\chi_{\mathrm{I},\mathbf{K}\eta\alpha}$. For $\eta_{\mathbf{Kp}}$ in the neighborhood of a particular bound state energy η_b, letting $x=\eta_{\mathbf{Kp}}-\eta_b$, one has

$$\tau_{\mathbf{Kp}}(\mathbf{p})=\left(\phi_{\mathbf{p}}|v|\phi_{\mathbf{p}}+\sum_{\eta\neq\eta_b}\sum_{\alpha}\frac{(\chi_{\mathbf{K}\eta\alpha}{}^{\mathrm{I}}|v|\phi_{\mathbf{p}})\chi_{K\eta\alpha}{}^{\mathrm{I}}}{(\mathcal{V}')^{-1}\tau_{\mathbf{Kp}}(\mathbf{p})+\eta_{\mathbf{Kp}}-\eta}\right)+\sum_{\alpha}\frac{|(\chi_{\mathbf{K}\eta_b\alpha}{}^{\mathrm{I}}|v|\phi_p)|^2}{(\mathcal{V}')^{-1}\tau_{\mathbf{Kp}}(\mathbf{p})+x}. \quad (4.8)$$

No difficulties arising from a hard repulsive core appear here, provided one does not further subdivide the first term, separating out $(\phi_{\mathbf{p}}|v|\phi_{\mathbf{p}})$. For large $\mathcal{V}$ and small x, (4.8) has the form

$$\tau=a+\frac{b}{(\mathcal{V}')^{-1}\tau+x}, \quad b>0.$$

For $|x|\gg b^{\frac{1}{3}}(\mathcal{V}')^{-\frac{1}{3}}$, the solution is $\tau\approx a+bx^{-1}$. Hence the solution everywhere in the neighborhood of η_b is

$$\tau_{\mathbf{Kp}}(\mathbf{p})=\tfrac{1}{2}\{a-x\mathcal{V}+\epsilon(x) \times[4b\mathcal{V}+x^2\mathcal{V}^2+2ax\mathcal{V}+a^2]^{\frac{1}{2}}\}, \quad (4.9)$$

where

$$\epsilon(x)=+1, \quad x>0$$
$$=-1, \quad x<0.$$

For $|x|<2b^{\frac{1}{3}}(\mathcal{V}')^{-\frac{1}{3}}$, this gives

$$\tau_{\mathbf{Kp}}(\mathbf{p})=\frac{a}{2}\left[1+\frac{1}{2}\frac{(\mathcal{V}')^{\frac{1}{3}}}{b^{\frac{1}{3}}}\frac{|x|}{(1+x^2\mathcal{V}'/4b)^{\frac{1}{2}}}\right] +\epsilon(x)b^{\frac{1}{3}}(\mathcal{V}')^{\frac{1}{3}}[(1+x^2\mathcal{V}'/4b)^{\frac{1}{2}} -|x|(\mathcal{V}')^{\frac{1}{3}}/b^{\frac{1}{3}}]+O[(\mathcal{V}')^{-\frac{1}{3}}]. \quad (4.10)$$

The unperturbed two-particle energy spectrum is of the form

$$\eta_{\mathbf{Kp}}=C_0(K)+C_1(K)p^2+\cdots;$$

hence the density of states $N_K(x)=p^2dp/dx$ is also a power series,

$$N_K(x)=d_0(K)+d_1(K)x+\cdots.$$

With $\alpha\ll 2b^{\frac{1}{3}}/\pi^{\frac{2}{3}}$ one has, therefore,

$$\int_{-\alpha\mathcal{V}^{-\frac{1}{3}}}^{\alpha\mathcal{V}^{-\frac{1}{3}}}N_K(x)\tau_{\mathbf{Kp}}(\mathbf{p})dx=O(\mathcal{V}^{-\frac{1}{3}}). \quad (4.11)$$

Consequently,

$$\lim_{\mathcal{V}\to\infty}\frac{1}{\mathcal{V}'}\sum_{\mathbf{p}}\tau_{\mathbf{Kp}}(\mathcal{V},\mathbf{p})=P\int d^3p\ \tau_{\mathbf{Kp}}(\infty,\mathbf{p}). \quad (4.12)$$

The equation for $\tau^{(\infty)}$ is just the equation for the Goldstone reaction matrix, so that the final result is the same as in Sec. III.

V. VERY LOW DENSITY REGION

Since in the limit of zero density the excluded volume in momentum space resulting from the Pauli principle vanishes, it is of interest to inquire whether the exclusion principle has a negligible effect on the energy of a gas of fermions at sufficiently low density. To answer this requires a comparison of a low-density Fermi gas with a similar system of distinguishable particles.

A. Distinguishable Particles

Fukuda and Newton[11] have shown that in a spherical quantization volume the energy shift of an unbound state of angular momentum l of a single particle in a potential well of finite range is proportional to the phase shift $\delta_l(k)$. They required that both the unperturbed and the perturbed radial wave functions must satisfy the same boundary condition at the radius R_0 of the

quantization volume. The wave number must change so as to exactly compensate for the phase shift produced by the potential, i.e.,

$$(k_l-k_{0l})R_0=-\delta_l(k_{0l}).$$

Hence

$$\Delta E_l=\frac{2E_{0l}}{k_{0l}}(k_l-k_{0l})=-\left(\frac{2E_{0l}}{k_{0l}}\right)\frac{\delta_l(k_{0l})}{R_0}.$$

These equalities hold, for fixed l, in the limit $R_0\to\infty$. Neglecting surface effects, the result also applies to the energy shift of an isolated interacting pair. It applies, furthermore, if the pair is immersed in a medium of particles which are distinguishable from the two particles of the interacting pair. The two interacting particles may, however, be indistinguishable from each other; then $\delta_l\to 2\delta_l$ for l even or odd, depending on the spin and isospin state, and $\delta_l\to 0$ for l with the opposite parity. Interactions between the medium and the particles of the interacting pair are being neglected, so that a low-density medium is assumed. In the Fukuda-Newton approximation, the medium is completely passive.

The case of distinguishable particles in the medium has also been discussed in terms of reaction matrices.[12] The distinguishability assumption is equivalent to setting $k_F=0$ in the projection operator Q^{out}. For $k_F=0$ the Goldstone reaction matrix t goes over into the Heitler reaction matrix K, familiar from scattering theory, which satisfies the integral equation

$$KQ_a=\left[v+v\frac{1-Q_a}{\eta_a-h_0}K\right]Q_a. \tag{5.1}$$

Similarly the matrix τ, which for a finite quantization volume contains the level shift $(a|\tau|a)/(a|a)$ in its propagator, goes over into a matrix R satisfying

$$RQ_a=\left[v+v\frac{1-Q_a}{\eta_a-h_0+(a|R|a)/(a|a)}R\right]Q_a. \tag{5.2}$$

For an infinite domain, with $\phi_{\mathbf{k}}=(2\pi)^{-\frac{3}{2}}e^{i\mathbf{k}\cdot\mathbf{x}}$, the well-known expression for a diagonal element of K is

$$\langle\phi_{\mathbf{k}}|K|\phi_{\mathbf{k}}\rangle=-\frac{\hbar^2}{4\pi^2\mu}\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\tan\delta_l(\eta_k), \tag{5.3}$$

where μ is the reduced mass and η_k is the unperturbed energy. It has been shown by deWitt[12] and by Riesenfeld and Watson[17] that the corresponding expression for R merely involves the substitution of δ_l for $\tan\delta_l$, i.e.,

$$\langle\phi_{\mathbf{k}}|R|\phi_{\mathbf{k}}\rangle=-\frac{\hbar^2}{4\pi^2\mu}\left(\frac{1}{k}\right)\sum_{l=0}^{\infty}(2l+1)\delta_l(\eta_k). \tag{5.4}$$

[17] W. B. Riesenfeld and K. M. Watson, Phys. Rev. **104**, 492 (1956); see, in particular, pp. 505–506. The Heitler reaction matrix is denoted by G.

This can be seen, following Riesenfeld and Watson, as follows: Eqs. (5.1) and (5.2) can be used to give

$$(a|R|a)=(a|K|a)-\frac{(a|R|a)}{(a|a)}\times\left[a\left|K^{\dagger}\frac{1-Q_a}{(\eta_a-h_0)\left[\eta_a-h_0+\dfrac{(a|R|a)}{(a|a)}\right]}R\right|a\right]. \tag{5.5}$$

In n dimensions $(\phi_{\mathbf{k}}|\phi_{\mathbf{k}})=\mathcal{V}/(2\pi)^n$, so that as $\mathcal{V}\to\infty$, the second term in (5.5) vanishes except for contributions from intermediate states with energies arbitrarily close to η_a. Thus the propagator of the correction term approaches a delta function. The only matrix element of R which then appears in the second term is $(\phi_{\mathbf{k}}|R|\phi_{\mathbf{k}})$, so that Eq. (5.5) becomes an algebraic equation. Using a spherical quantization volume of radius R_0 and the boundary condition $kR_0-\frac{1}{2}\pi l=m\pi$, one can then obtain (5.4) by using the identity

$$\left(\sum_{n=-\infty}^{-1}+\sum_{n=1}^{\infty}\right)\frac{1}{n(n+\sigma)}=\frac{\pi}{\sigma}\left[\frac{1}{\pi\sigma}-\frac{1}{\tan\pi\sigma}\right].$$

The result of Fukuda and Newton implies that it is correct to use R rather than K in computing the energy shift for the case under consideration. This also follows from the considerations of the present paper regarding level shifts: the substitution of $1-Q_a$ for Q^{out} does not alter the expressions for the level shift in terms of reaction matrices, i.e., the level shifts for a finite volume are

$$(\phi_{\mathbf{p}}|K|\phi_{\mathbf{p}})/(\psi_{\mathbf{p}},\psi_{\mathbf{p}})$$

or

$$(\phi_{\mathbf{p}}|R|\phi_{\mathbf{p}})/(\phi_{\mathbf{p}},\phi_{\mathbf{p}}).$$

B. Effects of the Exclusion Principle

An equation analogous to (5.5) holds for τ at arbitrary density. It is

$$(a|\tau|a)=(a|t|a)-\frac{(a|\tau|a)}{(a|a)}\times\left[a\left|t^{\dagger}\frac{Q^{\text{out}}}{(\eta_a-h_0)\left[\eta_a-h_0+\dfrac{(a|\tau|a)}{(a|a)}\right]}\tau\right|a\right]. \tag{5.6}$$

However, the relation

$$\frac{\langle\phi_{klm}|R|\phi_{klm}\rangle}{\langle\phi_{klm}|K|\phi_{klm}\rangle}=\frac{\delta_l(k)}{\tan\delta_l(k)}, \tag{5.7}$$

true in the limit of infinite volume, is special to the case of distinguishable particles. The result for $k_F>0$ and $k\equiv p<k_{FK}^-$ is in fact much simpler. As pointed out earlier, τ becomes identical with t in the limit of infinite

volume.[18] The reason is that for $p<k_{FK}^-$ there is a gap between η_p and the closest lying allowed intermediate states in (5.6). For a given finite volume the correction term in (5.6) has a much smaller effect than the correction term in (5.5), and vanishes as $\mathcal{V}\to\infty$. Even as $p\to k_F$ the allowed intermediate states lie entirely on one side of η_p, so there remains a difference between the correction terms of (5.5) and (5.6).

Actually the result $\tau=t$ for $\mathcal{V}=\infty$ can be regarded as consistent with (5.7) because for $k_F>0$ and $p<k_{FK}^-$ the phase shift $\delta_l(p)$ of the Bethe-Goldstone wave function vanishes.[15] This is perhaps the most striking effect of generalizing $1-Q_a$ to Q^{out} in the reaction matrix equation.

The second method of this paper, which is to use matrix elements of τ (for $\mathcal{V}<\infty$) in computing the average energy per particle, is seen to be the generalization to the case of indistinguishable fermions of the recommendation of Fukuda, Newton, deWitt, Riesenfeld, and Watson to use R instead of K for distinguishable particles.

The simple one-dimensional example of Sec. II illustrates the difference at low density between a system of distinguishable particles and a system of fermions. For low densities one has

$$\langle p|R|p\rangle\approx-\frac{p}{m}-\frac{2}{\pi m}\frac{1+mga}{mg}p^2[1-\tfrac{2}{3}mga(pa)^2]\times[1+\tfrac{1}{2}(pa)^2],$$

whereas for $k_{FK}^+\ll m^*g$

$$\langle p|t_K|p\rangle\approx-\frac{2}{\pi}g\frac{\cos^2(pa)}{1-(2mg/\pi k_{FK}^+)\Lambda(p)}\approx\frac{k_{FK}^+}{m}\frac{[1-\frac{1}{2}(pa)^2]}{\Lambda(p)},$$

with

$$\Lambda(p)\equiv\frac{k_{FK}^+}{2p}\ln\frac{1+(p/k_{FK}^+)}{1-(p/k_{FK}^+)}\approx 1 \text{ except for } p\approx k_{FK}^+.$$

For comparison, lowest order perturbation theory gives

$$\langle p|v|p\rangle=-(2g/\pi)\cos^2(pa)\approx-(2g/\pi)[1-\tfrac{1}{2}(pa)^2].$$

The integrals $I=\int_0^{k_{FK}^-}\delta E(p)dp$ for the three cases are

$$I[v]=-(2/\pi)gk_{FK}^-[1-(1/3)(k_{FK}^-a)^2+\cdots],$$

$$I[R]\approx-\frac{2}{\pi}gk_{FK}^-\left(\frac{\pi}{4}\frac{k_{FK}^-}{mg}\right)\left[1+\frac{4}{3}\frac{1+mga}{mga}k_{FK}^-a\right],$$

and

$$I[t]\approx-\frac{2}{\pi}gk_{FK}^-\frac{\pi k_{FK}^+}{2mg}.$$

[18] The difference between τ and t for finite volume is important for its effect on the sum over level shifts, however, as has been stressed in Secs. II.3 and IV.

The positive sign of $\langle p|t_K|p\rangle$ is discussed later. It is also of interest that the condition for a bound state of the Schrödinger equation with energy $-\kappa^2/m$ is

$$(mg/2\kappa)(1+e^{-2a\kappa})=1.$$

There are still several sources of possible confusion regarding the consistency of the results of the present paper with previous treatments of the very low density region. These confusing aspects can be clarified by a discussion of the various effects of the exclusion principle as incorporated in the Bethe-Goldstone equation.

1. Reduction of the Phase Shift

In the absence of the exclusion principle each two-particle phase shift attains the full value occurring for an isolated pair. For fermions, however, the exclusion principle greatly reduces the phase shift, and in the case of the Brueckner-Goldstone formulation (in which only the "particle-particle" interactions are included) the phase shift completely vanishes, i.e., the wave function "heals"[2] asymptotically. This is an extreme and special result which no longer holds when hole-hole interactions are included or when the Fermi surface is smoothed out. But, in general, the exclusion principle greatly reduces the phase shifts.

2. Effects Concerning Bound Pair States and Reaction Matrix Singularities

Because $\tau=t$ for $\mathcal{V}=\infty$ there is no need to consider τ unless there is a singularity in t. This is quite different from the case of distinguishable particles where there is a difference between $(a|R|a)$ and $(a|K|a)$ for $\mathcal{V}=\infty$ whether or not there is a singularity in $(a|K|a)$. Furthermore, when there is a singularity in t it is necessary to go to finite volumes in order to get a τ distinguishable from t.

The matrix K has a singularity if $\delta_l=\pm\pi/2$, which requires a sufficiently strong net attraction or repulsion. The critical strength of the interaction necessary to give a singularity is independent of the density of the system and is not related to the existence of a bound state of the Schrödinger equation. For calculating the energy per particle in a system of distinguishable particles, the singularity problem is completely eliminated by using R instead of K.

Turning to the case of fermions, one should recall that in the Brueckner theory the occupation number distribution in momentum space is assumed to be that of the ideal Fermi gas for the purposes of calculating reaction matrix elements. All bound state solutions of the Bethe-Goldstone equation are thereby required to be constructed solely of components lying outside the Fermi sea; they are Class I solutions of the Hermitian Bethe-Goldstone equation. A bound state solution of the Bethe-Goldstone equation may have either negative

or positive energy. It will be convenient to refer to the former as "molecular states"[16] and to the latter as "Cooper states."[4] The Cooper states exist mathematically in the Brueckner theory (i.e., are stable) only because the states into which they could decay are assumed to be completely filled.

For $k_F=0$ the "molecular states" are bound states of the Schrödinger equation. Their existence requires a sufficiently strong attraction. As k_F increases the molecular state has a rising energy and eventually becomes a Cooper state. As the density is further increased or the attraction is weakened the energy of the Cooper state approaches the Fermi energy, but remains below it; there is a singularity of t for an arbitrarily weak net attraction.

In the example of Sec. II the existence of a t matrix singularity is associated with a vanishing denominator. The diagonal elements for p less than the singularity are negative whereas those above the singularity are positive. As the energy of the Cooper state is lowered (by decreasing k_F or by increasing the coupling constant) more and more of the reaction matrix elements become positive. If the bound state is a "molecular" state, all the diagonal elements of t with $p<k_{FK}^-$ are positive. This is clearly an unphysical result, and shows the inapplicability of the Brueckner approximation for such strong attractions.

For a strong net attraction the Fermi sea is not a good first approximation and consequently the Bethe-Goldstone equation is not relevant. Usually the density will be high and clusters of three particles or more will become so important as to invalidate the restriction to two-body correlations. However, in case the forces saturate at two-body clusters, as with some gases of diatomic molecules, the independent-pair approximation may still be useful. In this case, instead of using Brueckner theory, it would seem more appropriate first to fill the bound states of the two-particle Schrödinger equation and then to build up the occupation of continuum states taking into account the partial occupation already implied by the filled bound states.

When the only bound-state solutions of the Bethe-Goldstone equation which exist at the saturation density are Cooper states lying fairly close to the Fermi level, the Brueckner approximation should be a good one. The present paper shows that at least the formal difficulty arising from the reaction matrix singularities can be avoided by taking into account the renormalization of the wave function. The very existence of the singularities indicates a weakness in the description of a Fermi gas provided by the Brueckner theory. The BCS theory has shown that this weakness is related to the failure to incorporate a smoothing of the discontinuity in occupation numbers at the Fermi surface. However, the energy given by the Brueckner approximation will presumably differ only slightly from the exact energy, provided the Cooper states lie close to the Fermi surface.

3. *Effect on the Magnitude of the Level Shifts at Low Density*

For a potential consisting solely of a hard-core repulsion the average energy shift per particle is[19]

$$E=(k_F^2/2M)[0.63k_FC+0.32(k_FC)^2+O(k_F^3C^3)],$$

where C is the core radius. The first term corresponds to using the R matrix while the second term comes from the exclusion principle. Thus for a purely repulsive potential, for which there are no t-matrix singularities, the effect of the exclusion principle becomes negligible at sufficiently small densities, as expected because of the small "excluded volume" in momentum space. With the core radius of nuclear forces and with the saturation density of nuclear matter, $k_FC\sim0.6$ so that the effect of the Pauli term on the hard-core contribution to the nuclear reaction matrix is quite significant.

Turning to potentials which give a net attraction, attention will be restricted to an intermediate range of densities sufficiently high to avoid "molecular" states, but sufficiently low that the excluded volume in momentum space is small enough that an expansion of t in terms of R would converge rapidly. If the net attraction is so weak that $\tan\delta\approx\delta$ then an expansion in terms of K is adequate. In practice this is much easier to do than to expand in terms of R. Moszkowski[20] has developed a practical method of this sort, termed the "separation method," applicable when the potential is repulsive at short distances and attractive for larger distances. It is separated into two parts, a short-range part v_s, consisting of the repulsion plus part of the attraction chosen so that the free-particle phase shifts are very small (e.g., vanish for a particular relative momentum), and a long-range part v_l. The method requires that v_l be weak enough to be treated by perturbation theory. The Goldstone reaction matrix t_s produced by the short-range potential v_s is expanded in terms of the corresponding distinguishable particle reaction matrix K_s. Some improvements have recently been made by Kohler.[21] Levinger *et al.*[22] have used a similar expansion, but choose for v_s the repulsion alone; consequently the series converges more slowly.

In deriving the expansion of t by the separation method one makes use both of reaction matrices operating to the right and of their adjoints, which operate to the left. The energy eigenvalue η appearing in the integral equation for an adjoint reaction operator refers to the unperturbed state standing to the left, on which

[19] This case is reviewed in J. S. Bell and E. J. Squires, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis, Ltd., London, 1961), Vol. 10, p. 211; see, in particular, pp. 251–252.

[20] S. A. Moszkowski, Office of Ordnance Research Technical Rept. No. 2, University of California at Los Angeles, 1959 (unpublished); S. A. Moszkowski and B. L. Scott, Ann. Phys. (New York) **11**, 65 (1960); **14**, 107 (1961); Nuclear Physics **29**, 665 (1962).

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the adjoint reaction operator acts. In the existing derivations the eigenvalue η has not been identified with sufficient care. Consequently certain equations are given as operator equations which actually hold only for diagonal matrix elements, and some terms of third and higher order are not correct even for diagonal elements. Diagonal elements are not affected to second order, however, so that existing numerical calculations stand. Because of the practical importance of these series solutions of the Goldstone reaction matrix, a derivation of the basic formulas with attention to the identification of η is given in Appendix C. Moszkowski and Scott have given a detailed and quantitative discussion of the effect of the exclusion principle on the magnitude of the level shifts for the case of nuclear matter.

The present discussion indicates how the Brueckner approximation goes over into the phase-shift approximation in the low-density limit when the potential is incapable of giving a bound state of the Schrödinger equation. It has also been shown why the phase-shift approximation and the Brueckner approximation are invalid when the density is so low and the net attraction so great as to support "molecular" (negative energy) bound states. The corrections to the phase-shift approximation resulting from the exclusion principle as given by the Brueckner theory at low density have been discussed and some modifications of higher order terms in the "separation method" pointed out. The second method of the present paper has been shown to be the generalization to the case of fermions of the use of the matrix R ($R_l \propto \delta_l$) instead of Heitler's matrix K ($K_l \propto \tan\delta_l$) for distinguishable particles.

Note added in proof. Additional insight into the relations between the formulations using reaction matrices with and without a modified propagator is given by a direct comparison of the level shifts. As has been pointed out to the author by Dr. T. Terasawa one can easily obtain from (5.5) or (5.6)

$$\begin{aligned}(a|T'|a)&=(a|T|a)-[(a|T'|a)/(a|a)](w_a|w_a')\\&=(a|T|a)/[1+(w_a|w_a')/(a|a)],\end{aligned}$$

where T is either K or t and T' is R or τ, respectively. The level shift computed with T' is then in a form very similar to that obtained with T. The two differ only by a term in the denominator, namely by

$$\begin{aligned}&(\mathcal{U}')^{-1}(w_a|w_a'-w_a)\\&\quad=(\mathcal{U}')^{-2}(a|T'|a)(w_a|(\eta_a-h_0-QvQ)^{-1}|w_a')\\&\quad=-(\mathcal{U}')^{-2}(a|T'|a)\textstyle\sum_\eta|(a|v|\chi_\eta^I)|^2/(\eta_a-\eta)^2\\&\qquad\qquad\times[\eta_a+(\mathcal{U}')^{-1}(a|T'|a)-\eta],\end{aligned}$$

where $Q=1-Q_a$ or Q^{out}.

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APPENDIX A. EQUIVALENCE OF EQ. (1.3) AND EQS. (1.4)[23]

$$\begin{aligned}&\left[\tau-v-v\frac{Q^{\text{out}}}{e_a-h_0}\tau\right]Q_a\\&=v\left[\frac{Q^{\text{out}}}{e_a+(a|\tau|a)/(a|a)-h_0}-\frac{Q^{\text{out}}}{e_a-h_0}\right]\tau Q_a \qquad (1.3)\\&=-\frac{(a|\tau|a)}{(a|a)}v\frac{Q^{\text{out}}}{e_a-h_0}\frac{Q^{\text{out}}}{e_a+(a|\tau|a)/(a|a)-h_0}\tau Q_a\\&=-\frac{(a|\tau|a)}{(a|a)}v\frac{Q^{\text{out}}}{e_a-h_0}[v^{-1}\tau-1]Q_a\\&=-\frac{(a|\tau|a)}{(a|a)}v\frac{Q^{\text{out}}}{e_a-h_0}v^{-1}\tau Q_a. \qquad (1.4)\end{aligned}$$

APPENDIX B. UNIQUENESS OF THE SOLUTION FOR $^+\tau$ OF SEC. II.

If we let

$$\begin{aligned}f(p,\tau(p))&=1-\frac{2m^*g}{\pi}\frac{\pi}{L}\\&\quad\times\sum_{k>k_{FK^+}}\frac{\cos^2(pa)}{k^2-[p^2+(\pi m^*/L)\tau(p)]}, \qquad (B.1)\end{aligned}$$

Eq. (2.30) may be rewritten as

$$f(p,\tau(p))=-\frac{2\pi^{-1}g\cos^2(pa)}{\tau(p)}.$$

Regarding τ as an independent variable, $f(p,\tau)$ has a singularity whenever $\tau=T(k,p)$ where

$$T(k,p)=(L/\pi m^*)(k^2-p^2), \quad k\geq k_{FK^+}. \qquad (B.2)$$

Since $f(p,\tau)\to-\infty$ as $\tau\to T(k,p)$ from below and $f(p,\tau)\to+\infty$ as $\tau\to T(k,p)$ from above, it is apparent that for fixed p the graph of $f(p,\tau)$ intersects the graph of $-2\pi^{-1}g\cos^2(pa)/\tau$ at an infinite number of points with $\tau>T(k_{FK^+},p)$. As $L\to\infty$, $T(k_{FK^+},p)\to\infty$ so that all these solutions become singular for all values of p, and are therefore unphysical. In addition, there are solutions with $\tau<T(k_{FK^+},p)$. Denoting by p_K the value

[23] To facilitate comparison with Tobocman's paper (reference 13) his notation $v^{-1}\tau$ has been used. The notations w_a and $\Omega-1$ as defined in the present paper are to be preferred since, while $v^{-1}\tau$ is well-defined, v^{-1} may not exist.

of the relative momentum for which

$$f(p_K,0)\equiv {}^{+}D_{K_PK}{}^{(L)}=0,$$

consider first the case $p>p_K$, for which $f(p,0)<0$. There is a solution of Eq. (2.30) with positive τ, and also one for a large negative τ since $f(p,\tau)\to 1$ as $\tau\to-\infty$. The latter solution moves off to $-\infty$ as $L\to\infty$ for all p. A unique physical solution $\tau_{Kp}{}^{(L)}(p)>0$ $(p>p_K)$ remains. The positivity is in agreement with the result for the Goldstone reaction matrix t. If $p<p_K$, then $f(p,0)>0$ and there is a solution with $\tau<0$ and also one for a large positive τ which is near, but less than, $T(k_{FK}{}^{+},p)$. The latter moves off to $+\infty$ as $L\to\infty$. Therefore for $p<p_K$ there is also a unique physical solution, with ${}^{+}\tau_{Kp}(p)<0$, the negativity again being in agreement with the result for t.

APPENDIX C. ON MOSZKOWSKI'S "SEPARATION METHOD"

To conform to the literature $\eta-h$ will be denoted by e and Q^{out} will be written Q. However, reaction matrices for distinguishable particles will be denoted by K as in the rest of this paper rather than by the t_{nx} (no exclusion) of Levinger *et al.* or various notations of Moszkowski, Scott, and Kohler. One has

$$t|a)=v|a)+v\frac{Q}{e_a}t|a), \tag{C.1a}$$

$$e_a|a)=(h-\eta_a)|a)=(h_0+V-\eta_a)|a)=0, \tag{C.1b}$$

$$t|a)=v\Omega_t|a)=v|\psi_a), \tag{C.1c}$$

$$\Omega_t|a)=|a)+\frac{Q}{e_a}t|a)=|a)+\frac{Q}{e_a}v\Omega_t|a), \tag{C.1d}$$

where V is the self-consistent single-particle interaction produced by v and $|\psi_a)$ is the Bethe-Goldstone wave function. The adjoint equations are

$$(b|t^{\dagger}=(b|v+(b|t^{\dagger}\frac{Q}{e_b}v, \tag{C.1e}$$

$$(b|t^{\dagger}=(b|\Omega_t{}^{\dagger}v. \tag{C.1f}$$

It is assumed that the unperturbed states $|a)$, $|b)$, $\cdots$ are real so that $(b|t^{\dagger}|a)\equiv(a|t|b)$; in the plane-wave basis this requires symmetric or antisymmetric combinations.

Following Kohler,[21] expansion of t is first made in terms of a matrix, here denoted by T_s (Kohler's $G_s{}^N$), satisfying

$$T_s|a)=v_s|a)+v_s\frac{Q}{e_a}T_s|a), \tag{C.2}$$

where

$$v=v_s+v_l. \tag{C.3}$$

The equation for T_s differs from the equation for the reaction matrix t_s determined solely by v_s, satisfying

$$t_s|a)=v_s|a)+v_s\frac{Q}{e_a{}^s}t_s|a), \tag{C.4}$$

by involving h rather than $h_s=h_0+V_s$, where V_s is produced solely by the potential v_s.

If the separation of v into v_s and v_l is allowed to be momentum dependent in order to keep the phase shifts $\delta_l{}^s$ small, then v_s and v_l are not Hermitian. If an average separation distance is used for ease of computation, then v_s and v_l are Hermitian and various terms given below will vanish; however, certain other terms will acquire larger values. $T_s{}^{\dagger}$ and $\Omega_{T_s}{}^{\dagger}$ satisfy

$$(b|T_s{}^{\dagger}=(b|v_s{}^{\dagger}+(b|T_s{}^{\dagger}\frac{Q}{e_b}v_s{}^{\dagger}, \tag{C.5a}$$

$$T_s{}^{\dagger}=\Omega_{T_s}{}^{\dagger}v_s{}^{\dagger}, \tag{C.5b}$$

$$(b|\Omega_{T_s}{}^{\dagger}=(b|+(b|T_s{}^{\dagger}\frac{Q}{e_b}=(b|+(b|\Omega_{T_s}{}^{\dagger}v_s{}^{\dagger}\frac{Q}{e_b}. \tag{C.5c}$$

An expansion of t in terms of Ω_{T_s} and v_l is given by

$$\begin{aligned}(b|t|a)&=(b|(v_s+v_l)\Omega_t|a)=(b|[\Omega_{T_s}{}^{\dagger}-\Omega_{T_s}{}^{\dagger}v_s{}^{\dagger}(Q/e_b)](v_s+v_l)\Omega_t|a)\\&=(b|\Omega_{T_s}{}^{\dagger}v_s[1+(Q/e_a)t]|a)+(b|\Omega_{T_s}{}^{\dagger}v_l\Omega_t|a)-(b|\Omega_{T_s}{}^{\dagger}v_s{}^{\dagger}(Q/e_b)t|a)\\&=(b|\Omega_{T_s}{}^{\dagger}v_s|a)+(b|\Omega_{T_s}{}^{\dagger}v_l\Omega_t|a)+\left(b\left|\Omega_{T_s}{}^{\dagger}\left[v_s\frac{Q}{e_a}-v_s{}^{\dagger}\frac{Q}{e_b}\right]t\right|a\right),\end{aligned} \tag{C.6}$$

which differs from the operator expression in Kohler's Eq. (6) by distinguishing e_a and e_b in the third term. The second term can be expanded to give

$$(b|\Omega_{T_s}{}^{\dagger}v_l\Omega_t|a)=(b|v_l|a)+\left(b\left|T_s{}^{\dagger}\frac{Q}{e_b}v_l+v_l\frac{Q}{e_a}t\right|a\right)+\left(b\left|T_s{}^{\dagger}\frac{Q}{e_b}v_l\frac{Q}{e_a}t\right|a\right). \tag{C.6a}$$

If v_s is Hermitian the diagonal elements of t are

$$(a|t|a)=(a|T_s+v_l|a)+\left(a\left|v_l\frac{Q}{e_a}(2T_s+v_l)\right|a\right)+\left(a\left|v_l\frac{Q}{e_a}(t-T_s-v_l)\right|a\right)+\left(a\left|T_s{}^{\dagger}\frac{Q}{e_a}v_l\frac{Q}{e_a}t\right|a\right). \tag{C.6b}$$

It is desirable to express the first term of Eq. (C.6) in terms of $T_s^\dagger$ or T_s and $v_s-v_s^\dagger$. One has

$$(b|\Omega_{T_s}^\dagger v_s|a)=(b|T_s^\dagger|a)+(b|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)|a). \tag{C.7a}$$

Also

$$(b|\Omega_{T_s}^\dagger v_s|a)=(b|T_s^\dagger|a)+\left(b\left|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)\left[\Omega_{T_s}-\frac{Q}{e_a}T_s\right]\right|a\right)$$

$$=(b|T_s+\Omega_{T_s}^\dagger T_s-T_s^\dagger\Omega_{T_s}|a)-\left(b\left|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)\frac{Q}{e_a}T_s\right|a\right)$$

$$=(b|T_s|a)+\left(b\left|T_s^\dagger\left[\frac{Q}{e_b}-\frac{Q}{e_a}\right]T_s\right|a\right)-\left(b\left|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)\frac{Q}{e_a}T_s\right|a\right) \tag{C.7b}$$

$$=(b|T_s|a)+\left(b\left|T_s^\dagger\left[\frac{Q}{e_b}-\frac{Q}{e_a}\right]T_s\right|a\right)-(b|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)(\Omega_{T_s}-1)|a). \tag{C.7c}$$

Equation (C.7b) reduces to Kohler's Eq. (7) if $\eta_b=\eta_a$. Subtracting (C.7a) from (C.7c) gives

$$(b|T_s-T_s^\dagger|a)=(b|\Omega_{T_s}^\dagger(v_s-v_s^\dagger)\Omega_{T_s}|a)-\left(b\left|T_s^\dagger\left[\frac{Q}{e_b}-\frac{Q}{e_a}\right]T_s\right|a\right), \tag{C.8}$$

which does not vanish if v_s is Hermitian unless $\eta_b=\eta_a$.

In order to isolate the effect of the exclusion principle, T_s may be expanded in terms of a reaction matrix K_s^D (MS's t_D, Kohler's G_s^D; D signifies that dispersion is included in the propagator) satisfying

$$K_s^D|a)=v_s|a)+v_s\frac{1-Q_a}{e_a}K_s^D|a). \tag{C.9}$$

One finds

$$(b|T_s|a)=(b|K_s^{D\dagger}|a)-\left(b\left|K_s^{D\dagger}\left[\frac{1-Q_b}{e_b}-\frac{Q}{e_a}\right]T_s\right|a\right). \tag{C.10}$$

The chief effect of the exclusion principle on the level shift is through the "Pauli term"

$$P_a=-\left(a\left|K_s^{D\dagger}\frac{1-Q_b-Q}{e_a}T_s\right|a\right)=-\left(a\left|K_s^{D\dagger}\frac{1-Q_a-Q}{e_a}K_s^{D\dagger}\right|a\right)$$

$$+\sum_b\left(a\left|K_s^{D\dagger}\frac{1-Q_b-Q}{e_a}\right|b\right)\left(b\left|K_s^{D\dagger}\left[\frac{1-Q_b}{e_b}-\frac{Q}{e_a}\right]T_s\right|a\right). \tag{C.11}$$

The analog of Eq. (C.8) for K_s^D is

$$(b|K_s^D-K_s^{D\dagger}|a)=(b|\Omega_{K_s^D}^\dagger(v_s-v_s^\dagger)\Omega_{K_s^D}|a)-\left(b\left|K_s^{D\dagger}\left[\frac{1-Q_b}{e_b}-\frac{1-Q_a}{e_a}\right]K_s^D\right|a\right). \tag{C.12}$$

The lowest approximation to P_a is therefore, if $v_s^\dagger=v_s$,

$$P_a^{(1)}=-\left(a\left|K_s^{D\dagger}\frac{1-Q_a-Q}{e_a}K_s^D\right|a\right) \tag{C.13a}$$

$$=-(a|(\Omega_{K_s^D}^\dagger-1)e_a(1-Q_a-Q)(\Omega_{K_s^D}-1)|a), \tag{C.13b}$$

in agreement with Kohler's Eq. (20).

For ease of computation K_s^D has been further expanded in terms of a reaction matrix K_s for freely propagating particles (no dispersion); its integral equation is

$$K_s|a)=v_s|a)+v_s\frac{1-Q_a}{e_a^0}K_s|a), \tag{C.14a}$$

where

$$e_a^0=\eta_a^0-h_0. \tag{C.14b}$$

The expansion of K_s^D in terms of K_s is

$$(b|K_s^D|a)=(b|K_s^\dagger|a)+\left(b\left|K_s^\dagger\left[\frac{1-Q_a}{e_a}-\frac{1-Q_b}{e_a^0}\right]K_s^D\right|a\right). \tag{C.15}$$

The "dispersion term" for the level shift is

$$(a|K_s^D-K_s|a)=\left(a\left|(\Omega_{K_s}^\dagger-1)e_a^0\left[\frac{1}{e_a}-\frac{1}{e_a^0}\right]e_a(\Omega_{K_s^D}-1)\right|a\right). \tag{C.16}$$

One could approximate $\Omega_{K_s^D}$ in Eq. (C.16) by $\Omega_{K_s}^\dagger$ or by Ω_{K_s}, leaving correspondingly different correction terms.

The equations as given here contain corrections to terms beyond the second order, which should be included if detailed calculations are extended to third order.

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Model for the Two-Pion and Three-Pion Resonances

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A relativistic model of π-π interaction is proposed from analogy with field theory. The main assumption in the model is that the interaction kernel is separable in momentum space. A Frazer-Fulco type resonance formula for the isovector 2π resonance is exactly derived on this model and its parameters are determined from the observed 2π resonance at 750 MeV.

The 3π problem is then solved with this model of π-π interaction and the exact isoscalar 3π wave function expressed in terms of a single-parameter function which satisfies a one-dimensional integral equation for the cases of "scalar" and "axial vector" forms of the wave function. It is found that it is possible to understand the observed energy (780 MeV) of the 3π resonance, on the above model, only for the case of the axial vector wave function and not for the scalar case. The model thus predicts a vector isoscalar meson at the observed energy.

1. INTRODUCTION

THE recent experimental discovery by Maglić *et al.*[1] of an isoscalar three-pion resonance at 787±15 MeV must be welcome to a large number of persons who, for various reasons (electromagnetic structure, resonances in π-N and K-N scattering, spin-orbit potentials, etc.), found it necessary to postulate the existence of an isoscalar vector meson. In the same analysis,[1] these authors have also confirmed the existence of the previously known[2] isovector two-pion resonance at about 750±50 MeV, from which, they feel confident, the 3π resonance can be distinguished, in spite of the anomalously small separation of the two resonances. Their analysis, based on Dalitz-type plots, though lacking detailed statistics, suggests further that the resonance can be interpreted more likely in terms of an axial vector "matrix element" (corresponding to a vector meson) than as vector or scalar elements (axial vector or pseudoscalar mesons, respectively). While such an interpretation would no doubt be most desirable from the point of view of the electromagnetic structure of the nucleon, there still remains the important question of whether the *mass* of the observed particle fits in with the value needed to explain the isoscalar form factor. For example, the recent findings of Littauer *et al.*[3] on the nucleon model proposed by Bergia *et al.*[4] suggest that the two-pion state should have a mass of about 4.0 μ, and the three-pion state an even lower mass, viz. 2.9 μ—a quasi-bound state. If this interpretation comes to be accepted, the resonances observed by Maglić *et al.*[1] should apparently have nothing to do with the electromagnetic form factors. However, the very fact that they have been *observed* must put their existence on a stronger footing than those of hypothetical particles *deduced* from specific models of the type discussed in reference 4, at least until such time as the latter are also "seen." In the meantime, one hopes that a more careful investigation would be in order before such a delicate question, as to the exact masses of the intermediate $T=1$ and $T=0$ vector mesons needed to explain the electromagnetic form factors, is finally answered.

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