Unitarity Condition and Anomalous Vertex Functions*

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An intuitive prescription is given in perturbation theory for the calculation of the absorptive part of a vertex function in the anomalous region from the unitarity condition. The relation of the present method to other known methods is also discussed.

1. INTRODUCTION

IN setting up the dynamical S-matrix theory of elementary particles by means of unitarity and dispersion relations, one of the complications that one is frequently confronted with is the presence of the anomalous threshold. In the presence of the anomalous threshold the unitarity condition serves to determine the absorptive part of a certain amplitude above the physical threshold in terms of other amplitudes, but it fails to give any direct information about the absorptive part below the normal threshold.

The determination of the absorptive part in the anomalous region was studied by several authors[1-3] for scattering amplitudes as well as for vertex functions. In this paper an intuitive method of making a continuation of the absorptive part of an anomalous vertex function from the normal region to the anomalous region is proposed and its connection with other methods is discussed.

First we shall point out the reality condition as the most characteristic feature of the absorptive part, and then, based on this condition general methods—continuation and dispersion methods—are proposed to compute the vertex function in the anomalous region (Sec. 2). Next the unitarity condition for the vertex function is formulated in perturbation theory (Sec. 3). As a typical example of anomalous vertex functions the deuteron form factor is studied in detail. First the continuation method is applied to this problem and it is verified that the result of the Feynman perturbation theory is reproduced. Also its connection with the method of deformation of the path of integration proposed by Blankenbecler and Nambu[2] is discussed. Then the second dispersion method and its connection to Cutkosky's method[3] are investigated (Sec. 4). Finally another simple example is studied by the dispersion method (Sec. 5).

2. REALITY CONDITION FOR THE ABSORPTIVE PART

If a function $f(z)$ is analytic in the upper half-plane of a complex variable z, $f(z)$ satisfies a dispersion relation

$$f(z)=\frac{1}{\pi}\int\frac{\mathrm{Im}f(x')}{x'-z}dx', \quad (\mathrm{Im}z>0) \tag{2.1}$$

provided that $f(z)$ falls off sufficiently rapidly at infinity. In this relation the absorptive part $\mathrm{Im}f(x)$ is a real function of x when x is a real variable, and when $f(z)$ stands for a physical amplitude $\mathrm{Im}f(x)$ is supposed to be expressed in terms of other amplitudes with the help of the unitarity condition.

Let us put

$$A(x)=\mathrm{Im}f(x), \tag{2.2}$$

and regard $A(x)$ as a function of a complex variable x. Then for real values of x, $A(x)$ is real and calculable from unitarity above the normal threshold, but below the normal threshold, which is the branch point of $f(x)$, $A(x)$ generally turns out to be complex when continued from above to below the normal threshold. The reality property of $A(x)$ above the normal threshold then implies that $A(x)$ be many-valued below the threshold.

The function $A(x)$ continued from above to below the branch point along a path in the upper half-plane will be called $A_+(x)$, and similarly it will be called $A_-(x)$ if it's continued in the lower half-plane (Fig. 1). If x is real $A_+(x)$ and $A_-(x)$ are the complex conjugates of each other as required by the reality condition

$$A_+^*(x)=A_-(x) \quad (x\text{: real}). \tag{2.3}$$

Since $A(x)$ calculated from unitarity is not necessarily real below the normal threshold we cannot identify it with the real absorptive part $\mathrm{Im}f(x)$, but perhaps the most reasonable ansatz would be the identification

$$\mathrm{Im}f(x)=\mathrm{Re}A(x). \tag{2.4}$$

The justification of this ansatz is the purpose of this paper and it will be verified in later sections by direct calculations.

In order to calculate $\mathrm{Re}A(x)$ below the normal threshold one can use two different methods. (1) Simply continue $\mathrm{Re}A(x)$ from above to below the normal threshold. This method, however, is not always appli-

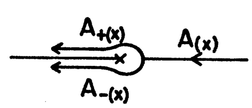

FIG. 1. The paths to define two branches of the function $A(x)$ below the normal threshold in the complex x plane.

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[1] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).
[2] R. Blankenbecler and Y. Nambu, Nuovo cimento 18, 595 (1960).
[3] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

cable as we shall see later and in some configurations the unitarity condition gives only the imaginary part of $A(x)$. In such a case one can use the second method. (2) Assume that $A_+(x)$ is analytic in the upper half-plane and likewise $A_-(x)$ in the lower half-plane, then one can write down a dispersion relation

$$\mathrm{Re}A(x)=\frac{P}{\pi}\int\frac{\mathrm{Im}A(x')}{x'-x}dx', \tag{2.5}$$

where

$$\mathrm{Re}A(x)=\mathrm{Re}A_+(x)=\mathrm{Re}A_-(x),$$

and

$$\mathrm{Im}A(x)=\mathrm{Im}A_+(x)=-\mathrm{Im}A_-(x).$$

At the anomalous threshold which is below the normal threshold $\mathrm{Re}A(x)$ jumps discontinuously and vanishes on the unphysical Riemann sheet. The dispersion relation (2.5) implies one subtraction and the subtraction constant is determined subject to the continuity condition that $\mathrm{Re}A(x)$ be continuous at the normal threshold when continued below it.

In this paper the above-mentioned methods are checked for the third order vertex function, but the extention of these methods to higher orders is feasible.

Furthermore the vertex function is a function of several physical parameters, and the connection of the present approach to other methods will be found by appropriate choices of the parameters as the variable x in this section.

3. UNITARITY CONDITION FOR THE VERTEX FUNCTION

The absorptive part of a vertex function can be expressed in terms of other amplitudes by means of the unitarity condition. Since, however, at least one of the three external momenta must be off the mass shell, the generalized rather than the ordinary unitarity condition has to be used. This was already discussed by Muraskin and the present author[4] in connection with the reproduction of the Feynman perturbation theory based on the generalized unitarity condition and parametric dispersion relations. In that paper this problem was discussed in configuration space and everything was straightforward, but from a practical point of view it is more desirable to study this problem in momentum space although one encounters some difficulties associated with the presence of the anomalous threshold, which motivated the present work.

Consider the meson-nucleon vertex function $\mathcal{G}$ corresponding to the Feynman diagram in Fig. 2. As for the definition of $\mathcal{G}$ we refer to MN.

The absorptive part of this third-order vertex function can be expressed in terms of lower order Green's functions as discussed in MN. Assuming the Yukawa-type interaction between the scalar nucleon field and neutral meson field with the coupling constant g, the absorptive part of $\mathcal{G}$ is given by

$$\mathrm{Im}\mathcal{G}=\frac{1}{2}\frac{g^3}{(2\pi)^2}\sum_{\mathrm{cycl}}\int d^4q\,\delta[(q+p_2)^2+M^2]$$
$$\times\delta[(q-p_1)^2+M^2]\left[\frac{\theta(p_{20}+q_0)\theta(p_{10}-q_0)}{q^2+m^2-i\epsilon}+\frac{\theta(-q_0-p_{20})\theta(q_0-p_{10})}{q^2+m^2+i\epsilon}\right], \tag{3.1}$$

where M and m denote the nucleon rest mass and meson mass, respectively, and $\sum_{\mathrm{cycl}}$ means a sum over the different ways of inserting intermediate states into the vertex function. In the present case there are three terms corresponding to three different channels

$$\begin{aligned}1&\rightleftarrows 2+3,\\ 2&\rightleftarrows 3+1,\\ 3&\rightleftarrows 1+2.\end{aligned} \tag{3.2}$$

It is easy to verify in this case that Eq. (3.1) reduces to

$$\mathrm{Im}\mathcal{G}=\tfrac{1}{2}\sum_{\mathrm{cycl}}\frac{g^3}{(2\pi)^2}\int d^4q\,\delta[(q+p_2)^2+M^2]$$
$$\times\delta[(q-p_1)^2+M^2]\frac{1}{q^2+m^2}\theta(-p_3{}^2-4M^2). \tag{3.3}$$

This equation gives the absorptive part of $\mathcal{G}$ correctly only in the physical configurations and the right-hand side fails to give the correct expression for $\mathrm{Im}\mathcal{G}$ in the unphysical configurations. Therefore let us define a new function A by

$$A=\tfrac{1}{2}\sum_{\mathrm{cycl}}\frac{g^3}{(2\pi)^2}\int d^4q\,\delta[(q+p_2)^2+M^2]$$
$$\times\delta[(q-p_1)^2+M^2]\frac{1}{q^2+m^2}\theta(-p_3{}^2-4M^2). \tag{3.4}$$

First of all one can recognize that in the physical configurations only one of the three terms can survice, since corresponding to the three channels in (3.2) one of the following three mutually exclusive conditions must be met:

$$\begin{aligned}(-p_1{}^2)^{\frac{1}{2}}&>(-p_2{}^2)^{\frac{1}{2}}+(-p_3{}^2)^{\frac{1}{2}}, &&\text{(channel 1)}\\ (-p_2{}^2)^{\frac{1}{2}}&>(-p_3{}^2)^{\frac{1}{2}}+(-p_1{}^2)^{\frac{1}{2}}, &&\text{(channel 2)}\\ (-p_3{}^2)^{\frac{1}{2}}&>(-p_1{}^2)^{\frac{1}{2}}+(-p_2{}^2)^{\frac{1}{2}}. &&\text{(channel 3)}\end{aligned}$$

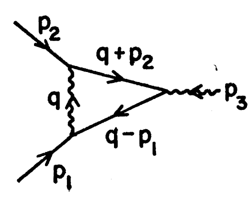

FIG. 2. Feynman diagram for the third-order vertex function. The straight lines represent nucleon lines and the wavy ones, meson lines.

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[4] M. Muraskin and K. Nishijima, Phys. Rev. 122, 331 (1961).

In the physical configurations one of these inequalities and the threshold condition must be satisfied, but in the unphysical configurations often none of them is satisfied. In order to continue A from a physical configuration to an unphysical configuration it is perhaps reasonable to retain only one term in the summation on the right-hand side of (3.4) provided that one selects the correct channel.[5] The selection of the correct channel is, however, not unique, and in some unphysical configurations A can be continued from any one of the three channels. Under such circumstances, we can give one convenient way of selecting the right channel. That prescription is to pick up the largest member of the three invariants

$$-p_1^2, \quad -p_2^2, \quad -p_3^2,$$

and if, for instance, $-p_3^2$ is the largest continue A to the unphysical configuration starting from

$$A_3=\frac{1}{2}\frac{g^3}{(2\pi)^2}\int d^4q\,\delta[(q+p_2)^2+M^2] \times\delta[(q-p_1)^2+M^2](q^2+m^2)^{-1}.$$

In this integrand we dropped $\theta(-p_3^2-4M^2)$ since $-p_3^2-4M^2>0$ is the threshold condition in the channel 3 and we are continuing A_3 down below the threshold into the unphysical region. In this way the problem is decomposed into two steps: (1) the selection of an appropriate channel, and (2) the continuation of A into the unphysical region.

In order to study the integral (3.4) it is convenient to introduce an auxiliary expression I defined by

$$I=\int d^4q\,\delta[(q+p_2)^2+M^2] \times\delta[(q-p_1)^2+M^2]\delta(q^2+m'^2), \quad (3.4')$$

where m'^2 need not be positive. Then A_3 is given by

$$A_3=\frac{1}{2}\frac{g^3}{(2\pi)^2}\int\frac{dm'^2}{m^2-m'^2}I(m'^2), \quad (3.5)$$

and the problem reduces to the evaluation of the integral (3.4′). By combining the arguments of the three δ functions in the integrand, the integral (3.4′) can further be transformed into the standard form

$$Q=\int d^4q\,\delta(q^2+a)\delta(qP+b)\delta(q\Delta+c). \quad (3.6)$$

The two four-dimensional vectors P and Δ span a two-dimensional vector space L, and the metric in this space is given either by $(+,+)$ or by $(+,-)$. In the former case the integral Q diverges owing to the hyperbolic character of the argument of the factor $\delta(q^2+a)$, and therefore we shall study the latter case first. In the latter case we choose t and x axes in the space L and choose qP, $q\Delta$, q_y, and q_z as the variables of integration. Then one finds

$$Q=\int\delta(q^2+a)J^{-1}dq_ydq_z, \quad (3.7)$$

where

$$J=\left|\frac{\partial(qP,q\Delta)}{\partial(q_t,q_x)}\right|=\begin{vmatrix}P_t & P_x\\ \Delta_t & \Delta_x\end{vmatrix}, \quad (3.8)$$

and consequently

$$J^2=-\begin{vmatrix}P^2 & P\Delta\\ P\Delta & \Delta^2\end{vmatrix}. \quad (3.9)$$

We used the metric $AB=\mathbf{AB}-A_0B_0$. Q is then given by

$$Q=\pi\Big/\left(-\begin{vmatrix}P^2 & P\Delta\\ P\Delta & \Delta^2\end{vmatrix}\right)^{\frac{1}{2}}, \quad (3.10a)$$

if $a+q_x^2-q_t^2<0$, and

$$Q=0, \quad (3.10b)$$

if $a+q_x^2-q_t^2>0$. An algebraic calculation shows that the result can be expressed in a compact form

$$Q=\left[\pi\Big/\left(-\begin{vmatrix}P^2 & P\Delta\\ P\Delta & \Delta^2\end{vmatrix}\right)^{\frac{1}{2}}\right]\theta(-D), \quad (3.11)$$

where

$$D=\begin{vmatrix}q^2 & qP & q\Delta\\ qP & P^2 & P\Delta\\ q\Delta & P\Delta & \Delta^2\end{vmatrix}, \quad (3.12)$$

with the understanding that the expressions involving the vector q be expressed in terms of P, Δ, a, b, and c.

When the metric of the space L is given by $(+,+)$ the integral diverges, but we can *define* Q by (3.11) even in this case. Then Q is real for $(+,-)$ metric and purely imaginary and double-valued for $(+,+)$ metric.

With the help of the formula (3.11) the integral I is given by

$$I=\frac{\pi}{4}\left(-\begin{vmatrix}p_1^2 & p_1p_2\\ p_1p_2 & p_2^2\end{vmatrix}\right)^{-\frac{1}{2}}\theta(-D), \quad (3.13)$$

where D is defined by

$$D=\begin{vmatrix}q^2 & qp_1 & qp_2\\ qp_1 & p_1^2 & p_1p_2\\ qp_2 & p_1p_2 & p_2^2\end{vmatrix}. \quad (3.14)$$

It is worth noticing that the first determinantal factor is symmetric in p_1, p_2, and p_3, i.e.,

$$-\begin{vmatrix}p_1^2 & p_1p_2\\ p_1p_2 & p_2^2\end{vmatrix}=\tfrac{1}{4}(p_1^4+p_2^4+p_3^4 -2p_1^2p_2^2-2p_2^2p_3^2-2p_3^2p_1^2), \quad (3.15)$$

where we used $p_1+p_2+p_3=0$.

[5] The author owes this point to Dr. Muraskin.

Inserting (3.13) into (3.5) one finds the complete expression for the function A_3. In the next section we shall study this integral for the deuteron form factor.

4. DEUTERON FORM FACTOR

The method described in Sec. 2 will be illustrated by the calculation of the deuteron form factor in the perturbation theory. The corresponding Feynman diagram is given in Fig. 3, and we assume that all the particles participating in this process are of the scalar type.

If we select the channel corresponding to the virtual process

$$k \to N+\bar{N} \to d+\bar{d}, \tag{4.1}$$

then the function A defined in Sec. 3 is immediately given by

$$A=\frac{eg^2}{16\pi}\int\frac{dm'^2}{m^2-m'^2}\frac{\theta(-D(m'^2))}{[k^2(k^2+4M^2)]^{\frac12}}, \tag{4.2}$$

where M and m are the rest masses of the deuteron and nucleon, respectively, and g and e are the scalar coupling constants for the vertices $d\to N+N$ and $k\to N+\bar{N}$, respectively.

$$-D(m'^2)=k^2\left[\left(\frac{m'^2+m^2-M^2}{2}\right)^2 - m'^2\left(m^2+\frac{k^2}{4}\right)\right]>0. \tag{4.3}$$

This is the support condition expressed by the presence of the θ function in the integrand of (4.2).

The function A can be used to calculate the absorptive part $\mathrm{Im}\mathcal{G}$ only for time-like k since a space-like k implies the selection of a different channel. Therefore let us assume

$$k^2<0.$$

Then the inequality (4.3) reduces to

$$m'^4+2m'^2(m^2-M^2)+(m^2-M^2)^2-m'^2(4m^2+k^2)<0,$$

or

$$m_1^2<m'^2<m_2^2,$$

where

$$\begin{aligned} m_1^2&=M^2+m^2+\tfrac12k^2-\tfrac12[(4M^2+k^2)(4m^2+k^2)]^{\frac12},\\ m_2^2&=M^2+m^2+\tfrac12k^2\tfrac12+[(4M^2+k^2)(4m^2+k^2)]^{\frac12}. \end{aligned} \tag{4.4}$$

The m'^2 integration must be carried out from m_1^2 to m_2^2 and the result is given by

$$A=\frac{eg^2}{16\pi}\frac{1}{[k^2(k^2+4M^2)]^{\frac12}} \times\ln\left|\frac{k^2+2M^2-[(4M^2+k^2)(4m^2+k^2)]^{\frac12}}{k^2+2M^2+[(4M^2+k^2)(4m^2+k^2)]^{\frac12}}\right|. \tag{4.5}$$

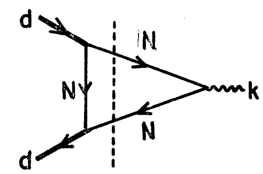

FIG. 3. Feynman diagram for the deuteron form factor. The thick straight lines represent deuteron lines, thin straight lines nucleon lines, and the wavy line the virtual photon. The dotted line indicates where the intermediate states should be inserted.

In what follows we shall put

$$s=-k^2$$

in order to make comparison with other papers easier.

A. Continuation Method

In the following arguments we introduce three domains of s as defined by

$$\begin{aligned} &\text{I.} && s>4M^2,\\ &\text{II.} && 4M^2>s>4m^2,\\ &\text{III.} && 4m^2>s, \end{aligned}$$

where $M>m$ is assumed. If we define $F(s)$ by

$$A=(eg^2/16\pi)F(s), \tag{4.6}$$

$F(s)$ is given in the domain I by

$$F_{\mathrm{I}}(s)=\frac{1}{[s(s-4M^2)]^{\frac12}} \times\ln\frac{s-2M^2+[(s-4M^2)(s-4m^2)]^{\frac12}}{s-2M^2-[(s-4M^2)(s-4m^2)]^{\frac12}}. \tag{4.7}$$

It is not hard to continue this function down into the domain II since the physical threshold $s=4M^2$ is not a branch point.

$$F_{\mathrm{II}}(s)=\frac{2}{[s(4M^2-s)]^{\frac12}}\tan^{-1}\left(\frac{[(4M^2-s)(s-4m^2)]^{\frac12}}{s-2M^2}\right). \tag{4.8}$$

$F_{\mathrm{II}}(s)$ vanishes at the normal threshold $s=4m^2$ if $2m^2>M^2$, but survives there if $2m^2<M^2$ as is in fact the case. In the latter case we have to continue $F(s)$ into the domain III passing through the branch point $s=4m^2$. It is sometimes more convenient to write

$$F_{\mathrm{II}}(s)=\frac{2}{[s(4M^2-s)]^{\frac12}} \times\left[\tan^{-1}\left(\frac{2M^2-s}{[(4M^2-s)(s-4m^2)]^{\frac12}}\right)+\frac{\pi}{2}\right], \tag{4.9}$$

in which case the $\tan^{-1}$ part can be understood in the sense of the principal value. In the domain III, the function F_{III} depends upon the path along which it is continued from the domain II. We shall call $F_{\mathrm{III}}(s)$ as $F_{\mathrm{III}}^{(+)}(s)$ or $F_{\mathrm{III}}^{(-)}(s)$ according to whether it is continued along a path in the upper half-plane or in the lower half-plane (see Fig. 4).

$$F_{\mathrm{III}}^{(\pm)}(s)=\frac{2}{[s(4M^2-s)]^{\frac{1}{2}}}\times\left(\pi\pm\frac{i}{2}\ln\frac{2M^2-s-[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}}{2M^2-s+[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}}\right). \tag{4.10}$$

This expression is valid as long as the numerator of the argument of the logarithm is positive, i.e.,

$$2M^2-s-[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}>0,$$

or

$$s>s_0=4M^2-M^4/m^2.$$

s_0 is the anomalous threshold and (4.10) is valid in the redefined domain III:

$$\mathrm{III.}\quad s_0<s<4m^2.$$

In order to call s_0 an anomalous threshold, however, it is necessary to show that $\mathrm{Re}F(s)$ vanishes at $s=s_0$. Examination of formula (4.10) shows that this is not the case, and a trick is needed in order to call s_0 an anomalous threshold. Continue $F_{\mathrm{III}}^{(+)}(s)$ around the point s_0 counter-clockwise and backward along the real s axis as shown in Fig. 5. Since s_0 is a logarithmic singular point, $\tilde{F}_{\mathrm{III}}^{(+)}$ defined by the above prescription is obtained as

$$\tilde{F}_{\mathrm{III}}^{(+)}(s)=\frac{i}{[s(4M^2-s)]^{\frac{1}{2}}}\times\ln\frac{2M^2-s-[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}}{2M^2-s+[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}}. \tag{4.11}$$

The function $\tilde{F}_{\mathrm{III}}^{(+)}(s)$ is purely imaginary and hence $\mathrm{Re}\tilde{F}_{\mathrm{III}}^{(+)}$ vanishes at $s=s_0$, although $\tilde{F}$ is on the unphysical Riemann sheet. This function vanishes at $s=4m^2$ as shown by Blankenbecler and Nambu.[2] Taking the real part of F_{III} we completed our continuation procedure.

$$\mathrm{Re}F_{\mathrm{III}}(s)=2\pi/[s(4M^2-s)]^{\frac{1}{2}}. \tag{4.12}$$

The absorptive part $\mathrm{Im}\mathcal{G}(s)$ is then given by

$$\mathrm{Im}\mathcal{G}(s)=\mathrm{Re}A(s)=(eg^2/16\pi)\,\mathrm{Re}F(s). \tag{4.13}$$

The function $F(s)$ is real in the domains I and II.

It is perhaps worth noticing that $\mathrm{Re}F_{\mathrm{III}}$ is given by

$$\mathrm{Re}F_{\mathrm{III}}(s)=F_{\mathrm{III}}^{(+)}(s)-\tilde{F}_{\mathrm{III}}^{(+)}(s). \tag{4.14}$$

This relation proves the equivalence of the present approach to that of Blankenbecler and Nambu[2] who proposed to deform the path of integration in the dispersion relation. The results obtained here are in accord with those of the Feynman perturbation theory.

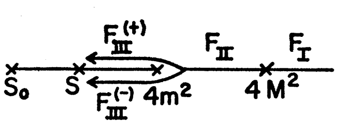

FIG. 4. The pahts in the complex s plane along which two branches $F_{\mathrm{III}}^{(+)}$ and $F_{\mathrm{III}}^{(-)}$ are defined.

B. Dispersion Method

In the previous subsection $\mathrm{Im}\mathcal{G}$ was determined as a function of s by means of the continuation method. This simple method fails, however, when $\mathrm{Im}\mathcal{G}$ is regarded as a function of some other variables, and in this subsection the second method will be introduced.[6]

The absorptive part A as determined by the unitarity condition is a function of s and M when M is regarded as a variable, and it will be denoted by $A(s,M^2)$. Then let us introduce $A(s\xi,M^2\xi)$ as a function of ξ with s and M^2 fixed. The analyticity properties of the $\mathcal{G}$ functions as functions of the scaling parameter ξ have been discussed in a series of papers and in this connection it is perhaps interesting to regard A as a function of ξ. In this calculation we confine ourselves to only positive values of ξ since otherwise we have to take care of channels other than the one considered so far.

First we shall define the function F from A by (6,4). so that

$$F(\xi)=\frac{1}{[s\xi(s\xi-4M^2\xi)]^{\frac{1}{2}}}\times\ln\left|\frac{2M^2\xi-s\xi-[(s\xi-4M^2\xi)(s\xi-4m^2)]^{\frac{1}{2}}}{2M^2\xi-s\xi+[(s\xi-4M^2\xi)(s\xi-4m^2)]^{\frac{1}{2}}}\right|$$
$$=\frac{1}{[s(s-4M^2)]^{\frac{1}{2}}}\frac{1}{\xi}\times\ln\left|\frac{2M^2-s-[(s-4M^2)(s-4m^2/\xi)]^{\frac{1}{2}}}{2M^2-s+[(s-4M^2)(s-4m^2/\xi)]^{\frac{1}{2}}}\right|. \tag{4.15}$$

In the domain I, i.e., for $s>4M^2$, the first factor is real and the absorptive part $\mathrm{Im}\mathcal{G}$ is simply given by

$$\mathrm{Im}\mathcal{G}_{\mathrm{I}}=(eg^2/16\pi)F_{\mathrm{I}}(1). \tag{4.16}$$

In the domains II and III the first factor is purely imaginary and double valued, whereas the second factor is real due to the absolute value symbol in the argument of the logarithm. In order to get $\mathrm{Re}F(1)$ in those domains we have to use the dispersion relation

$$\mathrm{Re}F_{\mathrm{II,III}}(1)=\frac{P}{\pi}\int\frac{d\xi}{\xi-1}\,\mathrm{Im}F_{\mathrm{II,III}}(\xi). \tag{4.17}$$

The scale transformation does not allow F to attain the physical domain I thus preventing the application of the continuation method. The imaginary part of

[6] This dispersion method has an interesting application to the deuteron problem. This will be discussed at another opportunity.

$F(\xi)$ in the domains II and III is given by

$$\operatorname{Im}F_{\mathrm{II,III}}(\xi)=\frac{-1}{[s(4M^2-s)]^{\frac{1}{2}}}\frac{1}{\xi}$$
$$\times\ln\left|\frac{2M^2-s-[(4M^2-s)(4m^2/\xi-s)]^{\frac{1}{2}}}{2M^2-s+[(4M^2-s)(4m^2/\xi-s)]^{\frac{1}{2}}}\right|. \quad (4.18)$$

The dispersion relation (4.17) is not the correct one, however. We generally have to make one subtraction in the dispersion relation in order to maintain the continuity of F at the boundary between the domains I and II, i.e.,

$$\operatorname{Re}F_{\mathrm{II}}(1)=\operatorname{Re}F_{\mathrm{I}}(1) \quad \text{at} \quad s=4M^2. \quad (4.19)$$

In the present case the dispersion integral converges without subtraction and we shall put an arbitrary additional constant to the dispersion relation. The first factor in (4.18) is a kinematical factor and this factor must be kept even in the presence of the subtraction, and, therefore, the correct dispersion relation is given by

$$\operatorname{Re}F_{\mathrm{II,III}}(1)$$
$$=\frac{1}{[s(4M^2-s)]^{\frac{1}{2}}}\left[\frac{P}{\pi}\int\frac{d\xi}{\xi(1-\xi)}\right.$$
$$\left.\times\ln\left|\frac{2M^2-s-[(4M^2-s)(4m^2/\xi-s)]^{\frac{1}{2}}}{2M^2-s+[(4M^2-s)(4m^2/\xi-s)]^{\frac{1}{2}}}\right|+C\right]$$
$$\equiv\frac{1}{[s(4M^2-s)]^{\frac{1}{2}}}\frac{I}{\pi}, \quad (4.20)$$

where C is the subtraction constant, and

$$I=P\int\frac{d\xi}{\xi(1-\xi)}\ln\left|\frac{a-(4m^2/\xi-s)^{\frac{1}{2}}}{a+(4m^2/\xi-s)^{\frac{1}{2}}}\right|+\pi C,$$

and

$$a=(2M^2-s)/(4M^2-s)^{\frac{1}{2}}. \quad (4.21)$$

If we put

$$m'^2=(m^2/\xi), \quad (4.22)$$

then the integral I looks like

$$I=P\int\frac{dm'^2}{m'^2-m^2}\ln\left|\frac{a-(4m'^2-s)^{\frac{1}{2}}}{a+(4m'^2-s)^{\frac{1}{2}}}\right|+\pi C. \quad (4.23)$$

This is nothing but the dispersion relation with respect to the internal nucleon mass. In order to simplify (4.23) we put

$$x=(4m'^2-s)^{\frac{1}{2}}, \quad (4.24)$$

so that I turns out to be

$$I=P\int_0^\infty\frac{2xdx}{x^2+s-4m^2}\ln\left|\frac{a-x}{a+x}\right|+\pi C$$
$$=P\int_0^\infty\frac{4x^2dx}{x^2+s-4m^2}\int_0^a\frac{da}{x^2-a^2}+\pi C. \quad (4.25)$$

Since the integrand is an even function of x we shall write I as

$$I=\int_0^a da\int_{-\infty}^\infty\frac{2x^2dx}{(x^2+s-4m^2)(x^2-a^2)}+\pi C. \quad (4.26)$$

In domain II, $4M^2>s>4m^2$.

Put $b=(s-4m^2)^{\frac{1}{2}}$, and I reduces to

$$I=\int_0^a da\int_{-\infty}^\infty\frac{2x^2dx}{(x^2+b^2)(x^2-a^2)}+\pi C$$
$$=2\pi b\int_0^a\frac{da}{a^2+b^2}+\pi C$$
$$=2\pi\tan^{-1}(a/b)+\pi C$$
$$=2\pi\tan^{-1}\frac{2M^2-s}{[(4M^2-s)(s-4m^2)]^{\frac{1}{2}}}+\pi C. \quad (4.27)$$

The boundary condition (4.19) enables us to determine C.

$$C=\pi.$$

In domain III, $4m^2>x$.

Put $b=(4m^2-s)^{\frac{1}{2}}$, and I reduces to

$$I=\int_0^a da\int_{-\infty}^\infty\frac{2x^2dx}{(x^2-b^2)(x^2-a^2)}+\pi C$$
$$=\pi^2\int_0^a[\delta(a-b)+\delta(a+b)]da+\pi^2$$
$$=\pi^2[\theta(a-b)+1]$$
$$=\pi^2[\theta(s-s_0)+1]. \quad (4.28)$$

Therefore in the domain III defined by $4m^2>s>s_0$, I is equal to $2\pi^2$. If we further continue I down below the anomalous threshold, then $F_{\mathrm{IV}}(s)$ in this fourth domain is given by

$$\operatorname{Re}F_{\mathrm{IV}}(s)=\pi/[s(4M^2-s)]^{\frac{1}{2}}, \quad (s_0>s>0). \quad (4.29)$$

This is exactly the same expression as that obtained from the continuation method if the path is taken along

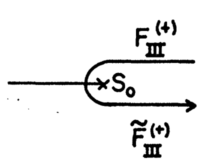

FIG. 5. Along the path indicated in the figure the function $F_{\mathrm{III}}^{(+)}$ is continued onto an unphysical Riemann sheet to define $\tilde{F}_{\mathrm{III}}^{(+)}$.

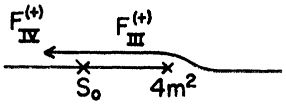

FIG. 6. The path along which $F_{III}^{(+)}$ is continued to $F_{IV}^{(+)}$.

the real axis in the upper half plane (Fig. 6). We must discard F_{IV}, however, since the discontinuity of ReF at $s=s_0$ is an indication that we continued the function F to this domain from a wrong channel.

C. Mixed Use of Both Methods

The absorptive part Im$\mathcal{G}$ was regarded as a function of s, ξ, and m^2, and two kinds of methods were proposed to complete the real part of F. In this subsection we shall propose the third method which lies between two previous treatments.

If one writes down the perturbation expression for $\mathcal{G}$ one finds that $\mathcal{G}$ is an analytic function of each internal mass variable, and so let us consider $\mathcal{G}$ as a function of the mass of the nucleon exchanged between the deuteron-antideuteron pair (Fig. 7). The variable mass of this nucleon will be denoted by m_a, then the function $F(m_a^2)$ is given by

$$F(m_a^2)=\int\frac{dm'^2}{m_a^2-m'^2}\frac{\theta(-D(m'^2))}{[s(s-4M^2)]^{\frac{1}{2}}}, \qquad (4.30)$$

where $\theta(-D(m'^2))$ has been defined by (4.3) or more explicitly by

$$\theta(-D(m'^2))=\theta(m'^2-m_1^2)-\theta(m'^2-m_2^2), \quad (4.31)$$

and m_1^2 and m_2^2 are given by (4.4).

In the domain I, (4.30) is immediately integrated and $F_I(m^2)$ is equal to the previous result as given by (4.7). Next we notice that the expressions for m_1^2 and m_2^2 imply

$$(s-4M^2)(s-4m^2)>0. \qquad (4.32)$$

Thus, if $s<4M^2$, s must be smaller than $4m^2$ no matter whether $[s(s-4M^2)]^{\frac{1}{2}}$ is real or imaginary and $F_{II}(m^2)=0$ results from (4.30). Therefore, in order to get $F_{II}(m^2)$ one has to continue F_I down to the domain II with respect to the variable s. Then in order to get F_{III} or more precisely ReF_{III} we can use either the continuation method as described in A or the dispersion method. We shall illustrate the latter for some later purpose.

In applying the dispersion method the continuity of ReF at the border between domains II and III is required.

$$F_{II}(s=4m^2)=2\pi/[4m^2(4M^2-4m^2)]^{\frac{1}{2}} \qquad (4.33)$$

gives the boundary value of ReF_{III} at $s=4m^2$, and this must be incorporated into the dispersion relation. The imaginary part of $F_{III}(m_a^2)$ in the domain III is given by

$$\mathrm{Im}F_{III}(m_a^2)=\frac{-1}{[s(4M^2-s)]^{\frac{1}{2}}}\int\frac{dm'^2}{m_a^2-m'^2}\times[\theta(m'^2-m_1^2)-\theta(m'^2-m_2^2)]. \quad (4.34)$$

The real part of F_{III} is then given by the once-subtracted dispersion relation

$$\pi\,\mathrm{Re}F_{III}(m^2)=\frac{1}{[s(4M^2-s)]^{\frac{1}{2}}}\Bigg[\int\frac{dm_a^2}{m^2-m_a^2}\int\frac{dm'^2}{m'^2-m_a^2}\times[\theta(m'^2-m_2^2)-\theta(m'^2-m_1^2)]+\text{“subtraction constant”}\Bigg]. \quad (4.35)$$

Since the dispersion integral is convergent without subtraction, the once-subtracted dispersion relation can be written in the above form. If the integration over m_a^2 is carried out first, one finds

$$\mathrm{Re}F_{III}(m^2)=\frac{\pi}{[s(4M^2-s)]^{\frac{1}{2}}}\Bigg[\int dm'^2\delta(m^2-m'^2)\times[\theta(m'^2-m_2^2)-\theta(m'^2-m_1^2)]+C\Bigg]$$
$$=\frac{\pi}{[s(4M^2-s)]^{\frac{1}{2}}}[\theta(m^2-m_2^2)-\theta(m^2-m_1^2)+C]. \quad (4.36)$$

C must be independent of s, and if we put $s=4m^2$ then $m_1^2=m_2^2$ and ReF_{III} is given there by

$$\mathrm{Re}F_{III}(m^2)=\frac{\pi C}{[s(4M^2-s)]^{\frac{1}{2}}}, \quad \text{at} \quad s=4m^2. \quad (4.37)$$

Comparison of (4.37) with (4.33) yields

$$C=2, \qquad (4.38)$$

and hence we find for $s\leq 4m^2$

$$\mathrm{Re}F_{III}(m^2)=\frac{\pi}{[s(4M^2-s)]^{\frac{1}{2}}}\times[\theta(m^2-m_2^2)+\theta(m_1^2-m^2)+1], \quad (4.39)$$

where

$$2(m^2-m_2^2)=s-2M^2-[(4M^2-s)(4m^2-s)]^{\frac{1}{2}},$$
$$2(m_1^2-m^2)=2M^2-s-[(4M^2-s)(4m^2-s)]^{\frac{1}{2}}.$$

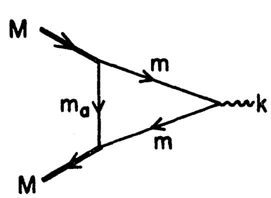

FIG. 7. The assignment of rest masses to lines in the Feynman diagram given in Fig. 3.

In the domain III the first expression is negative and the second one is positive if $s>s_0$. Thus we find

$$\mathrm{Re}F_{\mathrm{III}}(m^2)=2\pi/[s(4M^2-s)]^{\frac{1}{2}}, \tag{4.40}$$

which is identical with (4.12).

This result can be compared with the unsubtracted dispersion relation with $C=0$ which gives instead of (4.40)

$$-\pi/[s(4M^2-s)]^{\frac{1}{2}}.$$

This suggests the relation

$$\mathrm{Re}F_{\mathrm{III}}(m^2)=-\frac{2}{\pi}\int\frac{dm_a^2}{m_a^2-m^2}\,\mathrm{Im}F_{\mathrm{III}}(m_a^2), \tag{4.41}$$

and if we go back to the original expression (3.3) and use

$$\int\frac{dm_a^2}{m_a^2-m^2}\,\frac{1}{q^2+m_a^2}=\pi^2\delta(m^2+q^2),$$

we find that the absorptive part in the anomalous region is given by

$$A=-\frac{g^3}{4\pi}\,\mathrm{Im}\int d^4q\;\delta[(q+p_2)^2+M^2]$$
$$\times\delta[(q-p_1)^2+M^2]\delta(q^2+m^2). \tag{4.42}$$

This is exactly the same result as that of Cutkosky[3] who gave the simple rule that the replacement

$$1/(q^2+m^2)\rightarrow 2\pi i\delta(q^2+m^2) \tag{4.43}$$

reproduces the correct absorptive part in the anomalous region III.

5. ANOTHER EXAMPLE

In the previous section the deuteron form factor was studied in many different ways. In this section let us study the vertex function in a special configuration

$$p_1^2=p_2^2=p_3^2=-s. \tag{5.1}$$

In this case the function F is obtained from (4.7) by replacing M^2 by s, i.e.,

$$F(s)=\frac{1}{\sqrt{3}is}\ln\left|\frac{-s+[-3s(s-4m^2)]^{\frac{1}{2}}}{-s-[-3s(s-4m^2)]^{\frac{1}{2}}}\right|. \tag{5.2}$$

This expression is imaginary for real values of s, and the continuation method cannot be applied to this example. Therefore the dispersion method will be applied to it.

$$\mathrm{Im}F(s)=-\frac{1}{\sqrt{3}s}\ln\left|\frac{-s+[3s(4m^2-s)]^{\frac{1}{2}}}{-s-[3s(4m^2-s)]^{\frac{1}{2}}}\right|. \tag{5.3}$$

This expression survives only in the domain

$$0<s<4m^2.$$

In order to calculate $\mathrm{Re}F(s)$ we have to use the once-subtracted dispersion relation. In the presence of the physical region the subtraction constant is determined by the requirement that $\mathrm{Re}F$ be continuous at the border between physical and unphysical regions. In the absence of the physical region, however, this boundary condition is not available, and in order to overcome this difficulty the solution of the deuteron form factor problem will be utilized. $\mathrm{Re}F(M^2)$ is obtained by putting $s=M^2$ in the deuteron form factor, i.e.,

$$\mathrm{Re}F(M^2)=\mathrm{Re}F_{\mathrm{III}}(M^2)=2\pi/[M^2(4M^2-M^2)]^{\frac{1}{2}}. \tag{5.4}$$

This gives the proper boundary condition to determine the subtraction constant in the present problem.

The unsubtracted dispersion relation with respect to the scaling parameter ξ gives

$$\text{“}\mathrm{Re}F(s)\text{”}=\frac{1}{\pi}\int\frac{d\xi'}{\xi'-1}\,\frac{-1}{\sqrt{3}s\xi'}\ln\left|\frac{-s+[3s(4m^2/\xi'-s)]^{\frac{1}{2}}}{-s-[3s(4m^2/\xi'-s)]^{\frac{1}{2}}}\right|$$
$$=\frac{1}{\sqrt{3}s}\,\frac{1}{\pi}\int_0^{4m^2/s}\frac{d\xi'}{\xi'(1-\xi')}$$
$$\times\ln\left|\frac{-s+[3s(4m^2/\xi'-s)]^{\frac{1}{2}}}{-s-[3s(4m^2/\xi'-s)]^{\frac{1}{2}}}\right|. \tag{5.5}$$

One could write down a dispersion relation in the variable s as well.[7] The correct once-subtracted dispersion relation gives

$$\mathrm{Re}F(s)=\frac{1}{\sqrt{3}s}\left[\frac{1}{\pi}\int_0^{4m^2/s}\frac{d\xi'}{\xi'(1-\xi')}\right.$$
$$\left.\times\ln\left|\frac{(s/3)^{\frac{1}{2}}-(4m^2/\xi'-s)^{\frac{1}{2}}}{(s/3)^{\frac{1}{2}}+(4m^2/\xi'-s)^{\frac{1}{2}}}\right|+C\right], \tag{5.6}$$

where the subtraction must be done so as to keep the kinematical factor $1/\sqrt{3}s$. Introducing a new variable of integration by

$$x=(4m^2/\xi'-s)^{\frac{1}{2}}, \tag{5.7}$$

the integral then looks like

$$\mathrm{Re}F(s)=(1/\sqrt{3}s)I,$$

where

$$I=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xdx}{x^2+s-4m^2}\ln\left|\frac{a-x}{a+x}\right|+C, \tag{5.8}$$

and

$$a=(s/3)^{\frac{1}{2}}.$$

[7] If we write down the dispersion relation in the variable s and put $s'=s\xi'$, we get the dispersion relation in ξ' as given by (5.5) or (5.6).

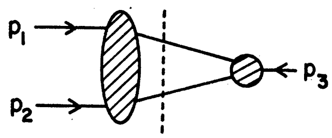

FIG. 8. Generalization of the Feynman diagram given in Fig. 2. The dotted line indicates where the intermediate states should be inserted.

This integral is exactly the same as (4.25) and is immediately evaluated. $s>4m^2$

$$\mathrm{Re}F(s)=\frac{1}{\sqrt{3}s}\left\{2\tan^{-1}\left[\left(\frac{s}{3(s-4m^2)}\right)^{\frac{1}{2}}\right]+C\right\}, \quad (5.9\text{a})$$

$4m^2>s$

$$\mathrm{Re}F(s)=(1/\sqrt{3}s)[\pi\theta(s-3m^2)+C]. \quad (5.9\text{b})$$

Comparison of (5.9b) with (5.4) gives

$$C=\pi. \quad (5.10)$$

We shall summarize the results as follows:

$$\mathrm{Re}F_{\mathrm{II}}(s)=\frac{1}{\sqrt{3}s}\left\{2\tan^{-1}\left[\left(\frac{s}{3(s-4m^2)}\right)^{\frac{1}{2}}\right]+\pi\right\},$$
$$\text{for}\quad s>4m^2, \quad (5.11)$$

$$\mathrm{Re}F_{\mathrm{III}}(s)=2\pi/\sqrt{3}s, \quad \text{for}\quad 4m^2>s>3m^2, \quad (5.12)$$

and

$$\mathrm{Re}F_{\mathrm{IV}}(s)=\pi/\sqrt{3}s, \quad \text{for}\quad 3m^2>s>0. \quad (5.13)$$

In this example the physical region is absent. $s=4m^2$ is the normal (but unphysical) threshold as before, and $s=3m^2$ is the anomalous threshold. Below $3m^2$, the function $\mathrm{Re}F_{\mathrm{IV}}$ should be omitted as before; then the above results are again in accord with those of the Feynman perturbation theory.

6. DISCUSSION

1. In the text we limited ourselves to discussion of the third order vertex functions, but it is easy to extend our methods so as to cover more general diagrams as those given in Fig. 8. Even in those cases the phase-space integration reduces to the form (3.6) provided that further integrations are to be done over a, b, and c with a weight function of a, b, and c. Then we first carry out the q integration, apply the present method and then integrate over a, b, and c. This enables us to include the rescattering corrections into the deuteron form factor just as done in the work of Blankenbecler and Nambu.

2. When no anomalous threshold is present, the function F becomes purely imaginary just below the normal threshold, whereas in the anomalous case F becomes complex, i.e., neither real nor purely imaginary.

3. From the expression (4.2) it is clear that a branch point is determined by

$$D(m^2)=0. \quad (6.1)$$

This is exactly the Landau-Cutkosky condition[3,8] to determine the singularities of a given Feynman amplitude (the third order vertex function in this case).

Most of the results obtained in this paper are already known, but the author hopes that in this paper understanding of the anomalous vertex functions is served in an intuitive pedestrian way.

ACKNOWLEDGMENT

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[8] L. D. Landau, Nuclear Phys. 13, 181 (1959).