Corrections to the Impulse Approximation in Scattering from Deuterons*

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The effect of binding and multiple-scattering corrections to the impulse approximation, as well as of corrections due to the presence of off-the-energy-shell matrix elements, in the case of scattering from a deuteron or other two-particle system, is discussed. Those binding and multiple-scattering corrections which have the form of double scattering are evaluated explicitly, with the help of the phase shifts of Gammel and Thaler, for the case of quasi-elastic p-d scattering at 145 Mev, and their inclusion is found to give good agreement with the experimental results on the cross section and polarization despite the fact that estimates of the triple-scattering corrections show they are of comparable magnitude to the effects which were included. The double-scattering corrections to the cross section, polarization, and R and A parameters for quasi-elastic n-d scattering at the same energy are also given. The off-the-energy shell corrections are evaluated numerically, and are found to be too small to be observed because of experimental errors and uncertainty in the calculation of the multiple-scattering corrections.

I. INTRODUCTION

A DETAILED study has been made of the corrections which arise to the impulse approximation,[1,2] as applied to the case of inelastic scattering from deuterons, due to the effects of binding and multiple scattering, and due to the fact that the matrix elements of two particle scattering operators which enter into the calculation are "off the energy shell." The investigation was undertaken for several reasons. In the first place, it was hoped to gain additional insight into the validity of the impulse approximation, especially as applied to experiments involving inelastic scattering from deuterium; such experiments, analyzed by means of the impulse approximation, provide one of the major means of studying interactions involving neutrons. Secondly, the availability of the excellent data of Kuckes, Wilson, and Cooper[3] on p-p scattering studied by means of inelastic p-d scattering made possible a convenient comparison of our theory with experiment. Lastly, it was hoped that this experimental data would yield information on the change in the matrix elements of the p-p scattering matrix off the energy shell, and hence furnish a new type of information on nuclear forces. As we will see, this last hope proved largely illusory, due to the experimental errors and to the difficulty in making an accurate estimate of the other theoretical corrections.

Section II contains a summary of the impulse approximation expansion as applied to the deuteron problem. In Sec. III we discuss the estimation of the effect of those corrections to the simple impulse approximation which have the form of double scattering. Section IV contains a comparison of the results of our calculation with the results of KWC on the p-p cross section and polarization as measured in scattering from deuterons, as well as estimates of the errors which will be introduced by the use of the impulse approximation to find the n-p cross section, polarization, and R and A parameters from the results of p-d scattering experiments at a laboratory energy of 145 Mev. Section V discusses the calculation of the difference between the values of the two-nucleon scattering matrix elements which enter into the impulse approximation and the on-the-energy-shell values measured in free p-p scattering experiments. The phenomenological potential of Gammel and Thaler[4] is employed in these calculations. Section VI contains conclusions and some general remarks on the impulse approximation as applied to scattering from two-particle systems.

II. THEORY

This section summarizes the impulse approximation expansion and its application to the problem of inelastic scattering from deuterium. We consider a particle incident on a deuteron target. We will call the incident particle number 1, and the target proton and neutron, respectively, particles number 2 and 3. We assume two-body forces, and let V_2 and V_3 represent the interaction of the incident particle with the proton and neutron, and V_d the interaction between the two particles making up the deuteron; K_i represents the kinetic energy of particle i, and K the total kinetic energy operator of the system. The Hamiltonian for the system can then be written

$$H=K+V_2+V_3+V_d \tag{2.1a}$$

$$=H_0+V_2+V_3 \tag{2.1b}$$

$$=H_0+V. \tag{2.1c}$$

The cross section for scattering into a given final state is proportional to the square of the absolute value of

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[1] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[2] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[3] A. F. Kuckes, Ph.D. thesis, Harvard University (unpublished) and A. F. Kuckes, Richard Wilson, and Paul F. Cooper, Jr. (to be published); referred to in the text as KWC.

[4] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

the matrix element between initial and final states of an operator T' given by the equation[2]

$$T'=V+VGV, \tag{2.2}$$

where

$$G=\lim_{\epsilon\to 0}\frac{1}{E-H+i\epsilon}. \tag{2.3}$$

E is the total energy of the system, so that if we let b represent the binding energy of the deuteron

$$E=K_1+b. \tag{2.4}$$

Henceforth, the indications of the limiting process will be omitted in equations similar to Eq. (2.3).

Making use of the techniques employed by Chew and Goldberger in reference 2, we can expand T' in the following form:

$$T'=t_2+t_3+t_2gt_3+t_3gt_2+\sum{}''tgtgt+\sum{}''tgtgtgt+\cdots \tag{2.5}$$

where

$$t_2=V_2+V_2\frac{1}{E-K-V_2+i\epsilon}V_2, \tag{2.6}$$

$$g=\frac{1}{E-K+i\epsilon}, \tag{2.7}$$

with t_3 and t_d being defined similarly to t_2 but with V_2 replaced by V_3 or V_d. We use the notation $\sum''$ in Eq. (2.5) to indicate a summation over all possible combinations of t_2, t_3, and t_d in which no two adjacent t's have the same subscript, and in which t_d does not appear at either the extreme left or the extreme right. The scattering operator $t_p(e)$ which describes scattering of the incident particle by a free proton when the energy of the two-particle system is e is

$$t_p(e)=V_2+V_2\frac{1}{e-K_1-K_2-V_2+i\epsilon}V_2. \tag{2.8}$$

Letting t_n and t_{np} represent, respectively, the scattering operator for scattering of the incident particle by a free neutron, and for free neutron-proton scattering, we see we can make the following substitutions in Eq. (2.5).

$$\begin{aligned} t_2&=t_p(E-K_3),\\ t_3&=t_n(E-K_2),\\ t_d&=t_{np}(E-K_1). \end{aligned} \tag{2.9}$$

There is an additional complication which must be allowed for in our expansion in the case of inelastic scattering. Suppose we wish to calculate the cross section for a process in which the wave vectors of the three particles in the final state are $\mathbf{k}_1'$, $\mathbf{k}_2'$, and $\mathbf{k}_3'$. Let $\phi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')$ be the corresponding plane wave state, which is not, however, an eigenstate of the unperturbed Hamiltonian H_0, since H_0 includes V_d. The cross section is determined by the matrix element of T' to a state $\psi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')$ which is an eigenstate of H_0 and which contains in addition to plane waves *incoming* spherical waves. ψ is given by[2]

$$\psi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')=\left(1+\frac{1}{E-K-i\epsilon}t_d^{\dagger}\right)\phi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'). \tag{2.11}$$

From Eq. (2.10) we see that the cross section is determined by the matrix element between the initial state and the state $\phi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')$ of the operator T given by

$$\begin{aligned} T&=\left(1+\frac{1}{E-K-i\epsilon}t_d^{\dagger}\right)^{\dagger}T'\\ &=t_2+t_3+t_2gt_3+t_3gt_2+t_dgt_2\\ &\quad+t_dgt_3+\sum{}'tgtgt+\cdots, \end{aligned} \tag{2.12}$$

where the notation $\sum'$ has the same meaning as $\sum''$ in Eq. (2.5) except that t_d may now occur at the extreme left of the group of t's.

The simple impulse approximation, hereafter abbreviated by S.I.A., consists in keeping only the first two terms in Eq. (2.12) and in replacing them by their values for free scattering at the corresponding energy and angle. This simply approximates the wave scattered from a deuteron by saying it is the sum of the scattered waves from a free proton and a free neutron whose momentum distributions are those of the deuteron ground state. The corrections to this are of two types. The most obvious is the higher order terms in the expansion (2.12), which arise from multiple scattering of the incident particle, and from the fact that the struck particle is bound to its partner in the deuteron. The physical interpretation of the terms in Eq. (2.12) is of course obvious. The third term, for example, corresponds to double scattering of the incident particle first off the neutron and then off the proton; the fifth term represents scattering off the proton followed by an interaction in the final state between the two deuteron particles, and so on. In Sec. III, we discuss in detail the calculation of those correction terms involving only two of the t operators.

The second type of correction to the S.I.A. arises from the fact that the initial and final kinetic energies of the interacting particles are different, even if the collision is adequately described by the first two terms in (2.12). For instance, if the process in question is one in which the incident particle is scattered by the proton, it is easy to see that the initial kinetic energy of the two colliding particles is greater than the final kinetic energy by an amount ΔE given by

$$\Delta E=b+2E_3', \tag{2.13}$$

where E_3' is the final neutron energy. Thus the matrix element of t_p which should be substituted into Eq. (2.12) is "off the energy shell" and so differs from that measured in scattering from a free proton. The effects of this type of correction are discussed in Sec. V.

One additional complication occurs if the incident particle is a proton or neutron, in that one must then consider the effects of the Pauli principle. In the case of an incident proton, for instance, particles 1 and 2 should be treated on an equal footing, while the formula for T', Eq. (2.2), obviously singles out those interactions involving particle 1. This problem may be treated in the following way. The initial state, Ψ_i, may be written as

$$\Psi_i=(\Phi_i-\Phi_i')/\sqrt{2}, \tag{2.14}$$

where Φ_i is an eigenstate of H_0 with particle 1 in an incident plane wave state and particles 2 and 3 in the deuteron ground state, and Φ_i' is the same state but with the roles of particles 1 and 2 reversed. Equation (2.2) is the appropriate formula for T' if we wish to calculate that part of the scattering arising from the state Φ_i. To calculate the scattering from Φ_i', we would need a similar formula for T', but with the roles of V_3 and V_d interchanged. If, however, we antisymmetrize the final state in the coordinates of the two protons, then the contributions due to scattering from states Φ_i and Φ_i' will be equal, so that all we need do is to calculate the matrix element of T as given in Eq. (2.12) between state Φ_i and the appropriately antisymmetric final state, and then multiply the answer by $\sqrt{2}$ to allow for the antisymmetrization of the initial state in accordance with Eq. (2.14). This factor is correctly included, without further comment, in the equations in Sec. III.

III. CALCULATION OF MULTIPLE SCATTERING CORRECTIONS

A. Spin-Independent Case

We turn now to the inclusion of the effects of binding and multiple scattering in the impulse approximation. We shall consider the case of an incident proton, and for the moment neglect the additional complications resulting from the spin of the particles; these are dealt with in Sec. B below. We will also use nonrelativistic kinematics, which is a good approximation at the energies at which we will be working. Our procedure will be to calculate explicitly the contributions of the next four terms beyond the S.I.A. terms in the expansion (2.12); we will refer to these as double scattering terms because of their form, although it should be remembered that two of them arise from the binding of the deuteron. The hope, of course, is that if these terms are small, it should be a good approximation to neglect the remaining terms, although we will see that the validity of this argument is somewhat questionable.

We write the wave function for the initial state of the system, ψ_i, in the form

$$\psi_i=\int a(\mathbf{k}_2)\phi(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_2)d\mathbf{k}_2, \tag{3.1}$$

where $a(\mathbf{k}_2)$ is the momentum space deuteron wave function, and the plane wave states ϕ are normalized to unit probability density per unit volume. In our calculations we took $a(\mathbf{k})$, following Moravscik,[5] to have the form

$$a(\mathbf{k})=C\left(\frac{1}{\mu_1^2+k^2}+\frac{1}{(2\mu_2)^2+k^2}-\frac{1}{\mu_2^2+k^2}-\frac{1}{(\mu_1+\mu_2)^2+k^2}\right), \tag{3.2}$$

where C is a constant, $\mu_1=2\mu b/\hbar^2$, μ $(=\frac{1}{2}M)$ is the reduced mass of the nucleon-nucleon system, and $\mu_2=6.9\mu_1$. The results of KWC with which we compare were computed using the more usual Hulthén wave function

$$a(\mathbf{k})=C\left(\frac{1}{\mu_1^2+k^2}-\frac{1}{(5.2\mu_1)^2+k^2}\right). \tag{3.2a}$$

The functions defined by Eqs. (3.2) and (3.2a) differ by amounts of the order of one percent in the range of momenta of interest to KWC; the difference between the two is somewhat significant for us in that in the calculation of the correction terms we have to integrate over momenta; the function (3.2) is presumably a better approximation for large values of k.

We wish the matrix element of T, Eq. (2.12), between ψ_i and a final plane wave state ϕ_f where

$$\phi_f=C[\phi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')-p\phi(\mathbf{k}_2',\mathbf{k}_1',\mathbf{k}_3')]/\sqrt{2}, \tag{3.3}$$

with $p=\pm1$. We look first at the S.I.A. terms in Eq. (2.12). If we let $f_p(e,\theta)$ represent the amplitude for scattering of a proton by a proton through a center-of-mass angle θ when one of the protons has kinetic energy e in the rest system of the second, we may write for the matrix element of t_2 between the states $\phi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ and $\phi(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3')$ the following relation:

$$t_2(\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3';\,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=(2\pi)^3\delta(\mathbf{k}_3'-\mathbf{k}_3)t_p(\mathbf{k}_1',\mathbf{k}_2';\,\mathbf{k}_1,\mathbf{k}_2), \tag{3.4}$$

$$=-(2\pi)^7\hbar^2\delta(\mathbf{k}_1'+\mathbf{k}_2'-\mathbf{k}_1-\mathbf{k}_2)\times\delta(\mathbf{k}_3'-\mathbf{k}_3)f_p(e_0,\theta_0)/\mu, \tag{3.4a}$$

where

$$e_0=\hbar^2(\mathbf{k}_1'-\mathbf{k}_2')^2/2M, \tag{3.5}$$

and θ_0 is the angle between the vectors $\mathbf{k}_1-\mathbf{k}_2$ and $\mathbf{k}_1'-\mathbf{k}_2'$. We ignore for the moment the error in Eq. (3.4a) due to the fact that the matrix element of t_p is between states of unequal energy. We can thus write the matrix element of t_2 between initial and final states as

$$t_{2fi}=-(2\pi)^7\hbar^2a(k_3')f_p(e_0,\theta_0)/\mu. \tag{3.6}$$

There will be a similar contribution from the term t_3 in the expansion (2.12) but with $a(k_3')f_p(e_0,\theta_0)$ replaced by $a(k_2')f_n(e'',\theta'')$, where f_n is the n-p scattering

[5] M. J. Moravscik, Nuclear Phys. **7**, 113 (1958).

amplitude and e'' and θ'' are defined similarly to e_0 and θ_0 but with particle 3 replacing particle 2. We will, however, be interested in experimental cases where the momentum transfer to the struck proton, and hence k_2', are large enough that we will neglect the term involving t_3 because of the small size of $a(k_2')$. The error thus introduced will be much smaller than others involved in the calculation.

We now turn to the next four terms in Eq. (2.12). We label these as follows:

$$t_d g t_2 = C1 \quad t_d g t_3 = C2, \qquad t_3 g t_2 = C3 \quad t_2 g t_3 = C4. \tag{3.7}$$

We also define, for future convenience, the vectors $\mathbf{d}$, $\mathbf{a}$, and $\mathbf{h}$, where

$$\begin{aligned} \mathbf{d} &= \mathbf{k}_1 - \mathbf{k}_1' = \mathbf{k}_2' + \mathbf{k}_3', \\ \mathbf{a} &= \mathbf{k}_1 - \mathbf{k}_2' = \mathbf{k}_1' + \mathbf{k}_3', \\ \mathbf{h} &= \mathbf{k}_1' + \mathbf{k}_2' = \mathbf{k}_1 - \mathbf{k}_3'. \end{aligned} \tag{3.8}$$

We consider $C1$ as an example of the method of computation. The operator t_d satisfies relations analogous to Eqs. (3.4) and (3.4a) with p-p operators replaced by the corresponding n-p operators. $C1$ connects the final state with the state $\phi(\mathbf{k}_1, \mathbf{k}_2, -\mathbf{k}_2)$ through the two possible intermediate states $\phi(\mathbf{k}_1', \mathbf{k}_2+\mathbf{d}, -\mathbf{k}_2)$ and $\phi(\mathbf{k}_2', \mathbf{k}_2+\mathbf{a}, -\mathbf{k}_2)$. The matrix element of $C1$ can therefore be written

$$\begin{aligned} C1_{fi} = \frac{1}{(2\pi)^3} \int d\mathbf{k}_2\, a(\mathbf{k}_2) \Big[& t_n(\mathbf{k}_2', \mathbf{k}_3'; \mathbf{k}_2+\mathbf{d}, -\mathbf{k}_2) \\ & \times \frac{1}{E - (\hbar^2/2M)[\mathbf{k}_1'^2 + (\mathbf{k}_2+\mathbf{d})^2 + \mathbf{k}_2^2] + i\epsilon} \\ & \times t_p(\mathbf{k}_1', \mathbf{k}_2+\mathbf{d}; \mathbf{k}_1, \mathbf{k}_2) \\ & - p t_n(\mathbf{k}_1', \mathbf{k}_3'; \mathbf{k}_2+\mathbf{a}, -\mathbf{k}_2) \\ & \times \frac{1}{E - (\hbar^2/2M)[\mathbf{k}_2'^2 + (\mathbf{k}_2+\mathbf{a})^2 + \mathbf{k}_2^2] + i\epsilon}, \\ & \times t_p(\mathbf{k}_2', \mathbf{k}_2+\mathbf{a}; \mathbf{k}_1, \mathbf{k}_2) \Big]. \end{aligned} \tag{3.9}$$

We might note that when one considers the sum $C1_{fi}+C3_{fi}$, one finds it is symmetric or antisymmetric combinations, e.g.,

$$t_p(\mathbf{k}_1', \mathbf{k}_2+\mathbf{d}; \mathbf{k}_1, \mathbf{k}_2) - p t_p(\mathbf{k}_2+\mathbf{d}, \mathbf{k}_1'; \mathbf{k}_1, \mathbf{k}_2)$$

of matrix elements of t_p which enter the double scattering correction.

In order to evaluate (3.9) one must replace the two-particle scattering operators t by constants and remove them from under the integral sign. This is necessary not only because the t's are available only in numerical, not analytic, form, but also because the variables on which the dependence of the t's is known, i.e., the energy and angle of scattering, are related in an algebraically very complicated way to the integration variable $\mathbf{k}_2$. Thus failure to make this approximation leaves one with a completely unmanageable numerical integration. The choice of appropriate average values for the operators t, as well as the error introduced by this approximation, will be discussed in detail below.

Once one has made the approximation of neglecting the dependence of the t operators on k_2, the integral $C1_{fi}$ can be evaluated. We will discuss the evaluation of the first of the two terms on the right of Eq. (3.9), which we denote by I_1, to illustrate the method used. The second term, as well as the matrix elements of the other double scattering correction terms, is obtained in the same way. In what follows it will be convenient for us to define the following quantities:

$$\begin{aligned} Q(x) &= k_1^2 - k_1'^2 - x^2 - 2Mb/\hbar^2, \\ z_1(x) &= \tfrac{1}{2}\{-x + [x^2 + 2Q(x)]^{\frac{1}{2}}\}, \\ z_2(x) &= \tfrac{1}{2}\{-x - [x^2 + 2Q(x)]^{\frac{1}{2}}\}, \\ p_1(x) &= [\mu_1^2 + \tfrac{1}{2}Q(x)]/x. \end{aligned} \tag{3.10}$$

We define also $p_2(x)$, $p_3(x)$, and $p_4(x)$ analogously to p_1 except with μ_1^2 replaced by μ_2^2, $(\mu_1^2+\mu_2^2)$, and $(2\mu_2)^2$, respectively. The first step in the evaluation of I_1 is to rewrite the energy denominator by observing

$$\begin{aligned} E - \hbar^2[k_1'^2 + (\mathbf{k}_2+\mathbf{d})^2 + k_2^2]/2M \\ = \hbar^2[-2k_2^2 - 2\mathbf{k}_2\cdot\mathbf{d} + Q(d)]/2M. \end{aligned} \tag{3.11}$$

We then make use of the well-known identity,

$$1/(E-H+i\epsilon) = -i\pi\delta(E-H) + P1/(E-H),$$

where P stands for principal value. The two resulting integrals are most conveniently done by working in a system of cylindrical coordinates with the z axis parallel to the vector $\mathbf{d}$. $I_{1\delta}$, the delta function part of I_1, can then be evaluated straightforwardly, and yields the result

$$I_{1\delta} = i\pi^2 C M \bar{t}_n \bar{t}_p L(d)/\hbar^2 d, \tag{3.12}$$

where

$$L(d) = \ln \frac{[z_1(d)-p_1(d)][z_1(d)-p_4(d)][z_2(d)-p_2(d)][z_2(d)-p_3(d)]}{[z_1(d)-p_2(d)][z_1(d)-p_3(d)][z_2(d)-p_1(d)][z_2(d)-p_4(d)]}, \tag{3.13}$$

and $\bar{t}_n$ and $\bar{t}_p$ indicate the average values of the two scattering operators.

The principal value integral, I_{1p}, may also be evaluated in the same coordinate system. The integrand is

proportional to the expression

$$d\phi dk_{2z}dk_{2r}\,k_{2r}\frac{1}{\mu_1^2+k_{2z}^2+k_{2r}^2}\times\frac{1}{k_{2z}^2+k_{2r}^2+dk_{2z}-\frac{1}{2}Q(d)}, \quad (3.14)$$

plus similar terms arising from the other terms in the momentum distribution. k_{2z} and k_{2r} in (3.14) refer to the components of $\mathbf{k}_2$ parallel and perpendicular to the z axis, that is, to $\mathbf{d}$. To carry out the integration, we make a partial fraction separation of the two fractions. The partial fraction separation fails for the case where the denominators of the two fractions are equal so that in integrating the terms separately we must take the principal value at the point $k_{2z}=[\mu_1^2+\frac{1}{2}Q(d)]/d$ as well as at the pole in the energy denominator. The integration over the azimuthal angle ϕ and over k_{2r} can be done immediately. One is then left, for the k_{2z} integration, with a number of terms of the form $\int[\ln(c+x)/x]dx$, which can be evaluated approximately by expanding the integrand in power series in x/c or c/x. The integrals corresponding to the separate terms in the momentum distribution are divergent at infinity, so that in order to carry out the integration term by term, an upper cutoff for k_2 must be introduced.

The evaluation of the principal value integrals is quite tedious. It is, however, simplified by several considerations. First, the major contribution to the integrals comes from k_2 lying in a sphere of radius $3\mu_1$. In view of this, one can make use of the simpler momentum distribution (3.2a) without making a serious error. This also means that approximating, as we do, the matrix elements of the t operators by their average value on the energy shell is reasonable, since most of the integral comes from a range of k_2 which does not involve large violations of energy conservation in the individual collisions. Moreover, the dependence on kinematic parameters describing the particular final state in question is not strong. Thus, in comparing with the KWC data, the principal value integrals were calculated for a particular final state (one in which the two protons scattered through 90° in their center-of-mass system and the spectator neutron came off with zero energy) which lay in the middle of the range of experimental data, and this value used throughout. Lastly, but most important, the contribution of the principal value integrals is only about 20% of the contribution of the delta function terms, so that a rather large error in the principal value terms can be tolerated. The final result for I_{p1} is, for the final state chosen,

$$I_{p1}=2.1M\pi C\bar{t}_n\bar{t}_p/\hbar^2d. \quad (3.15)$$

We now need to evaluate $\bar{t}_p$ and $\bar{t}_n$. We have

$$\bar{t}_p=-(2\pi)^4\hbar^2\bar{f}_p(\hbar^2(\mathbf{k}_1'-\mathbf{k}_2-\mathbf{d})^2/2M,\theta)/\mu,$$
$$\bar{t}_n=-(2\pi)^4\hbar^2\bar{f}_n(\hbar^2(\mathbf{k}_2'-\mathbf{k}_3')^2/2M,\theta')/\mu, \quad (3.16)$$

TABLE I. Approximate laboratory energies and center-of-mass angles at which the scattering amplitudes in Eq. (3.18) are evaluated. f_{pi} (θ,ϕ) means f_{pi} $(\theta,\phi)-pf_{pi}(\pi-\theta,\phi+180°)$.

Amplitude	Energy (Mev)	Polar angle	Aximuthal angle
f_{pi} $(i=1,2,3)$	130	90°	0°
f_{ni} $(i=1,4)$	70	20°	0°
f_{n2}	130	90°	0°
f_{ni} $(i=3,6)$	70	160°	0°
f_{n5}	130	90°	180°
f_{n7}	130	0°	0°

where θ is the angle between the vectors $\mathbf{k}_1-\mathbf{k}_2$ and $\mathbf{k}_1'-\mathbf{k}_2-\mathbf{d}$, and θ' the angle between the vectors $2\mathbf{k}_2+\mathbf{d}$ and $\mathbf{k}_2'-\mathbf{k}_3'$. The quantities appearing in (3.16) all have to be averaged over $\mathbf{k}_2$ wherever it appears. Because of the momentum-space wave function of the deuteron, the main contribution to the integrals comes from the region $k_2<3\mu_1$, while for most of the experimental data with which we are concerned, k_1', k_2', and d are all of order $8\mu_1$; consequently the variation in the energy and angle arguments of the scattering amplitudes in (3.16) as $\mathbf{k}_2$ varies is not too large. A more careful consideration of the kinematics leads to the conclusions

$$e_0-40\text{ Mev}<\hbar^2(k_1'-k_2-d)^2/2M<e_0+20\text{ Mev},$$
$$\theta_0-20°<\theta<\theta_0+20°, \quad (3.17)$$
$$-45°<\theta'<45°,$$

so that the scattering amplitudes in Eqs. (3.16) need to be averaged over these ranges of their arguments.

Carrying out a similar calculation for the remainder of the double scattering terms, one finally arrives at the following relation for T_{2fi}, the sum of the double scattering contributions to the matrix element of T.

$$T_{2fi}=[(2\pi)^7\hbar^2/\mu][H(d)(f_{p1}f_{n1}+f_{n2}f_{n3})+H(a)(f_{p2}f_{n4}-pf_{n5}f_{n6})+H(h)f_{n7}f_{p3}], \quad (3.18)$$

where

$$H(x)=[iL(x)+2.1/\pi]/2x. \quad (3.19)$$

The scattering amplitudes appearing in (3.19) must be averaged over ranges of their arguments similar to those given in relations (3.17). The amplitudes in (3.19) may be approximated by being evaluated at the energies and angles shown in Table I, while once the final state is specified, the quantities d, a, and h, and thus $H(d)$, $H(a)$, and $H(h)$ may be found.

The calculation can be carried out in a similar way if we remove the restriction of nonrelativistic kinematics. Two changes are needed in the calculation. In the first place, in the relation between the matrix elements of the operators t and the scattering amplitudes f, the reduced mass μ must be replaced by the relativistic expression $E_1E_2/[c^2(E_1+E_2)]$, where E_1 and E_2 are the total energies of the two particles in their center of mass system. Secondly, the relativistic relation between

energy and momentum must be used in the energy denominators. As an example, consider the process in which a relativistic pion of mass m undergoes double scattering from the deuteron, with the first scattering being from the proton. We will assume the nucleons remain nonrelativistic for simplicity; no additional complication is introduced if this is not so. The energy denominator occurring in the calculation of the matrix element for this process is

$$E-H=2Mc^2-b+(m^2c^4+\hbar^2k_1^2c^2)^{\frac{1}{2}} -[m^2c^4+\hbar^2c^2(\mathbf{k}_2+\mathbf{a})^2]^{\frac{1}{2}} -2Mc^2-\hbar^2(k_2'^2+k_2^2)/2M. \quad (3.20)$$

The terms involving $\mathbf{k}_2$ in the second square root in Eq. (3.20) are relatively small compared with the remaining terms, so that the square root may be expanded in a power series. The variable of integration $\mathbf{k}_2$ then no longer appears under a square root sign, and the integration can be carried out by a procedure similar to that for the nonrelativistic case.

We can now get an estimate of the double scattering corrections by using Eqs. (3.6) and (3.18) and replacing scattering amplitudes which appear in them by square roots of the corresponding p-p and n-p differential cross sections, experimental values for which are tabulated in reference 6. Doing this for the case of scattering of the incident proton from the target proton at rest through an angle of 90° in their center-of-mass system, and choosing the values of d, a, and h in Eq. (3.18) to correspond to this case, one finds $T_{2fi}/t_{2fi}=0.17$. Estimating the amplitudes in this way allows us to say nothing about the sign of the correction, since it arises principally from interference between single and double scattering and their relative phase is unknown. If we were to assume that the first- and second-order terms are of opposite sign, then our estimate would say that the experimental cross section for this case should be about 69% of the value predicted by the simple impulse approximation; KWC find their measured cross section to be about 83% of the S.I.A. value.

B. Spin Dependence

To obtain the sign and a more accurate value of the magnitude of the double-scattering correction, as well as to find its effect on the polarization and triple scattering parameters, we need to take into account the spin of the three nucleons involved in the process. Since we are dealing with a system of three spin-$\frac{1}{2}$ particles, there are eight possible spin states for the system. The cross section for scattering from initial to final coordinate or momentum space states labeled by indices i and f is proportional to $\sum_{s'}\sum_s p(s)|T_{fs',is}|^2$, where s and s' are indices indicating initial and final state spins, and $p(s)$ is the probability for finding the system initially in the spin state s. The multiple scattering expansion, Eq. (2.12), can be applied to each of the $T_{fs',is}$ separately; the calculation of the double scattering corrections to each one will now involve not only an integration over the momentum coordinate of the intermediate state but a sum over its spin coordinate as well, but otherwise will proceed just as above. Thus Eq. (3.18) becomes a set of equations of the form

$$T_{2fs',is}=[(2\pi)^7\hbar^2/\mu]\sum_{s''}[H(d)(M_{ps''s}M_{ns's''} +M_{ns''s}M_{ns's''})+\cdots], \quad (3.21)$$

where, for example, $M_{ps''s}$ represents the matrix element of f_p between the spin states s and s''.

It was decided to obtain the p-p and n-p matrix elements entering into Eq. (3.21) from the phase shifts of Gammel and Thaler.[4,7] Fortunately, Prenowitz and Palmieri at the Harvard Cyclotron Laboratory had available a program for the IBM 704 which produced the matrix elements $M_{s's}$, given a set of phase shifts, using the formulas of Stapp.[8] They very kindly made a series of runs with the Gammel-Thaler phase shifts which provided a grid of values of the matrix elements as functions of energy and angle. The range of energies and angles over which the various matrix elements are to be averaged is unchanged by the introduction of spin, and is still indicated approximately by Table I.

The Stapp formulas give the matrix elements in a representation where the spin is quantized along the z axis, taken to be parallel to the direction of motion of the incident particle. For our purposes it is convenient to transform the matrix M to obtain its matrix elements in a representation where the spin is quantized along the y axis, the normal to the scattering plane. We will designate the matrix elements in the latter representation by M_{ij}'. The use of the matrix elements M' is convenient, both because the incident beam is polarized along the y axis, and because the plane of scattering can be reasonably taken to be the same for the two collisions in a double-scattering process, (this is once again true because of the small target momentum) while the direction of relative motion is not the same in the two collisions.

It is necessary to make use of three different sets of eight spin states in doing the calculation. We designate the ith member of the three states by v_{1i}, v_{2i}, and v_{3i}, respectively. The states v_{1i} are labeled by the eigenvalues of the y component of the spin of particle 1, and by the eigenvalues for the square and y component of the total spin of particles 2 and 3, i.e., particles 2 and 3 are in either a relative triplet or singlet state; the other two sets of states are defined similarly in the obvious way. The necessity for the use of the three sets of states comes from the fact that the matrix elements of the operator t_2 are known between the states v_{3i}, those of t_3 between the states v_{2i}, and those of t_d between the v_{1i}'s. If we are interested in the scattering of an incident beam

[6] J. L. Gammel, R. S. Christian, and R. M. Thaler, Phys. Rev. **105**, 311 (1957).

[7] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957).

[8] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

of polarized protons off the proton in the deuteron, the initial states over which we must average are the three v_{1i}'s which correspond to finding the incident proton with the appropriate spin direction, while the remaining particles are in the three possible triplet states corresponding to the three possible orientations of the deuteron. We take the final states over which to sum to be the states v_{3i}; this is convenient both because the single scattering matrix elements (those of t_2) to these states can be easily found, and also because the spatial symmetry or antisymmetry of the final state, i.e., the value of p in Eq. (3.9), is known. In obtaining the correction terms, then, we must evaluate such spin sums as

$$\langle v_{3f}|t_d t_2|v_{1n}\rangle=\langle v_{3f}|v_{1i}\rangle\langle v_{1i}|t_d|v_{1j}\rangle\langle v_{1j}|v_{3k}\rangle \times\langle v_{3k}|t_2|v_{3l}\rangle\langle v_{3l}|v_{1n}\rangle. \quad (3.22)$$

Once one has written this sum out in terms of the matrix elements, it is easy to obtain the corresponding result for the other products of two-particle scattering operators by making use of symmetry properties of the coefficients $\langle v_{ij}|v_{kl}\rangle$ for different values of i and k.

The corrections in experimentally observed quantities due to double scattering may now be obtained. To find the ratio of the predicted cross section for scattering to a particular final state, with double-scattering effects included, to that predicted by the S.I.A., one need only find the ratio of $\sum_{s'}\sum_s p(s)|T_{fs',is}|^2$ with

$$T_{fs',is}=t_{2fs',is}+T_{2fs',is},$$

where, as before, T_2 represents the sum of the double-scattering contributions, to the value of the sum with only the contribution from t_2 included. (We are once again considering final states containing two fast protons, so that the contribution from the t_3 term in the multiple-scattering expansion may be neglected.)

We may compute the correction to the left-right asymmetry, which gives the polarization, in the following way. The asymmetry, P, is defined by

$$P=(\sigma_R-\sigma_L)/(\sigma_R+\sigma_L)\equiv(\sigma_R-\sigma_L)/2\sigma, \quad (3.23)$$

where σ_R and σ_L represent the differential cross sections for right and left scattering. In terms of the matrix elements M', σ_R is easily shown to be, in the case that the incident beam consists of particles polarized so that the y component of their spin is $+\frac{1}{2}$,

$$\sigma_R=\tfrac{1}{2}|M_{11}'|_R^2+\tfrac{1}{2}|M_{-11}'|_R^2 +\tfrac{1}{4}|M_{00}'|_R^2+\tfrac{1}{4}|M_{ss}'|_R^2, \quad (3.24)$$

where we have followed the usual subscript notation of Stapp[8] to indicate the spin states to which the matrix elements refer. [M_{01}' turns out to vanish identically, which accounts for its absence in Eq. (3.24).] σ_L is of course given by a similar formula. M_{00}' and M_{ss}' turn out to have no left-right dependence, so that

$$P=(\tfrac{1}{2}|M_{11}'|_R^2-\tfrac{1}{2}|M_{11}'|_L^2 +\tfrac{1}{2}|M_{-11}'|_R^2-\tfrac{1}{2}|M_{-11}'|_L^2)/2\sigma. \quad (3.25)$$

Equation (3.25) also gives the asymmetry that would be observed in a scattering experiment from deuterons if the S.I.A. were valid. Suppose that the effect of the inclusion of double-scattering corrections is to change the cross section for scattering into a final state where the two protons are in a triplet state with y component of total spin $+1$ to $\frac{1}{2}\alpha|M_{11}'|_R^2$ and $\frac{1}{2}(\alpha+d\alpha)|M_{11}'|_L^2$ for right and left scattering. Let us similarly define β and $d\beta$ for the case of scattering to a final state with z component of spin -1, and say that the double-scattering corrections change σ to $\gamma\sigma$. Then the change in the polarization will be

$$\Delta P=\{[(\alpha/\gamma)-1]\tfrac{1}{2}(|M_{11}'|_R^2-|M_{11}'|_L^2)+[(\beta/\gamma)-1]\tfrac{1}{2}(|M_{-11}'|_R^2-|M_{-11}'|_L^2)\}/2\sigma -[\tfrac{1}{2}(d\alpha/\gamma)|M_{11}'|_L^2+\tfrac{1}{2}(d\beta/\gamma)|M_{-11}'|_L^2]/2\sigma. \quad (3.26)$$

As may be seen from Eq. (3.26), the corrections affect the asymmetry parameter P in two ways. In the first place, the corrections may simply be different for right and left scattering, as represented by the second term on the right of (3.26), which involves $d\alpha$ and $d\beta$. Secondly, the corrections may tend to make those terms in the cross section which have an asymmetry either larger or smaller relative to the entire cross section. This effect appears in the first term of (3.26), which vanishes if the ratios α/γ and β/γ are both equal to 1.

The effect of the corrections on the other nucleon-nucleon scattering parameters can be similarly determined. As an example, suppose one wishes to measure the parameter[8] R for n-p scattering by means of a triple scattering experiment in which the second scattering was that of a beam of polarized protons off of a deuteron target leading to a final state containing a fast proton and a fast neutron, as has been done by Thorndike *et al.*[9] In this case, the polarization of the incident proton is in the scattering (xz) plane, along the x axis, rather than along the y axis, and we are interested in the polarization of the scattered proton in the scattering plane and perpendicular to its direction of motion, i.e., in the direction of the vector $\mathbf{u}=(\mathbf{k}_1\times\mathbf{k}_1')\times\mathbf{k}_1'$. The main additional step which must be made in order to obtain R is to compute the matrix elements of T (which in this case will be approximated by t_3+T_2) and the n-p scattering amplitude M_n in a representation where the initial and final state spins are quantized along the x axis and the direction of the vector $\mathbf{u}$, respectively; we designate these by M_{ij}'' and T_{ij}''; they can be obtained straightforwardly in terms of the matrix elements calculated above in the representation

[9] R. A. Hoffman, J. Lefrancois, E. H. Thorndike, and Richard Wilson, Bull. Am. Phys. Soc. **6**, 288 (1961), and Phys. Rev. **125**, 973 (1962).

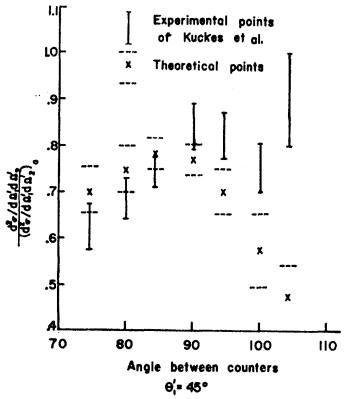

FIG. 1. Experimental values, obtained by Kuckes, Wilson, and Cooper, reference 3, and theoretical values, with the inclusion of double scattering corrections, of the ratio of $d^2\sigma/d\Omega_1'd\Omega_2'$ to the value $(d^2\sigma/d\Omega_1'd\Omega_2')_0$ predicted by the impulse approximation for the case of quasi-elastic p-d scattering at 145 Mev, for the ease in which one of the protons emerges at an angle of 45° in the laboratory system, plotted as a function of the angle between the emerging protons. The errors on the theoretical points indicate the uncertainty in the calculation of the correction terms, as described in the text.

with both spins quantized along the y axis. Let us designate by σ_+ and σ_- the partial cross sections for scattering into final states in which the spin of particle 1 is parallel or antiparallel, respectively, to **u** in the case of free n-p scattering. R_0, the correct value of R, is then given by

$$R_0=(\sigma_+-\sigma_-)/(\sigma_++\sigma_-)\equiv(\sigma_+-\sigma_-)/\sigma'. \quad (3.27)$$

Let us call the values of the partial cross sections obtained in an experiment with a deuteron target $(1-\alpha_+)\sigma_+$ and $(1-\alpha_-)\sigma_-$. Then the value of R which will be obtained in the deuteron experiment will be

$$R=\frac{(1-\alpha_+)\sigma_+-(1-\alpha_-)\sigma_-}{(1-\alpha_+)\sigma_++(1-\alpha_-)\sigma_-}$$
$$\equiv\frac{(1-\alpha_+)\sigma_+-(1-\alpha_-)\sigma_-}{(1-\alpha)\sigma'}. \quad (3.28)$$

Hence R will differ from R_0 by

$$R_0-R\equiv\Delta R=\left(1-\frac{1-\alpha_+}{1-\alpha}\right)R_0-\frac{(\alpha_+-\alpha_-)\sigma_-}{(1-\alpha)\sigma'}. \quad (3.29)$$

The approximate value of α_+ and α_-, and hence of ΔR, with double scattering effects included can be calculated once the matrix elements T_{ij}'' are known.

IV. RESULTS OF MULTIPLE SCATTERING CORRECTIONS

A. Proton-Proton Case

We now apply the theory of the preceding section to the case of proton-deuteron scattering at 145 Mev. We consider first the case of quasi-elastic p-p scattering (i.e., events in which two fast protons emerge, so that the neutron is viewed as the "spectator" particle) so that we may compare the predictions of our calculation with the experimental results of KWC.

The quantity measured in the KWC experiment is $d^3\sigma/d\Omega_1'd\Omega_2'dE_1'$, i.e., the cross section for scattering of one proton into the element of solid angle $d\Omega_1'$ and energy dE_1', with the other proton scattered into $d\Omega_2'$; because of the constraints imposed by energy and momentum conservation $d\Omega_1'$, $d\Omega_2'$, and dE_1' completely specify the state of the system following the scattering; we will abbreviate this cross section hereafter by $d^3\sigma$. Because of the statistical error which arises in considering the number of particles scattered with a given E_1', and also because of the uncertainty in energy measurement, it is easier to consider the result for the cross section integrated over E_1', which we designate by $d^2\sigma/d\Omega_1'd\Omega_2'$, abbreviated hereafter by $d^2\sigma$.

Let us designate by $(d^2\sigma/d\Omega_1'd\Omega_2')_0$, or $d^2\sigma_0$, the S.I.A. value for $d^2\sigma$, i.e., the value obtained by keeping only the term t_2 in Eq. (2.12). $d^2\sigma_0$ can be computed entirely from the experimentally measured free p-p cross section; hence, in contrast to the correction terms, it is not dependent on the assumption of a particular set of p-p phase shifts. In Fig. 1 we show the results of KWC on the ratio of the measured values of $d^2\sigma$ to $d^2\sigma_0$ for cases in which one of the protons emerged at an angle θ_1' of 45° with the incident beam direction (in the laboratory system) plotted as a function of the included angle between the counters, i.e., the angle between the outgoing protons. If the target proton is at rest, and no energy is imparted to the neutron in the collision, then the angle between the two protons will be about 87.5° (slightly less than 90° because of the deuteron binding energy); thus the average initial momentum of the target protons (and thus of the emerging neutron) becomes higher the more the included angle differs from 87.5° if the S.I.A. is approximately correct. As will be seen, the experimental points are uniformly less than those predicted by the S.I.A., and in general the ratio becomes smaller as the included angle becomes different from 87.5°. Figure 2

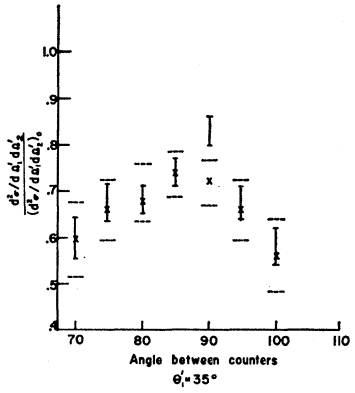

FIG. 2. Same notation as in Fig. 1, but for the case in which one of the protons emerges at an angle of 35° in the laboratory system.

presents the same sort of experimental information for the case where $\theta_1'=35°$.

Figures 1 and 2 also show the ratio of the theoretical values of $d^2\sigma$, with the effects of the four correction terms discussed in the previous section included, to $d^2\sigma_0$. The corrected values were obtained as follows. The values of k_1', k_2', and k_3' were obtained for the given values of θ' and the included angle and several values of E_1', using the kinematic tables available in KWC. Once these are known, the ratio of $d^3\sigma$ with the effects of the corrections included to its S.I.A. value for the particular value of E_1' in question can be obtained using the theory of Sec. III. The change in values of the ratio, for a given θ_1', is due mainly to the kinematic factors, especially the factor $a(k_3')$ which appears in the first order matrix element, Eq. (3.6). Hence the same set of average values for the nucleon-nucleon scattering matrix elements was used throughout the calculation for a fixed θ_1', so that the very tedious summation over intermediate spin states of products of matrix elements was done only once for a given angle. The ratio of corrected to uncorrected values of $d^2\sigma$ was then obtained by averaging the ratios of $d^3\sigma$ over energy, with the weighting factors in the average being obtained from the experimental cross sections at the appropriate values of E_1'.

Figures 1 and 2 indicate the estimated error in the computation of the corrected theoretical results, arising from the approximations made in calculating the double-scattering terms, as well as the experimental error. The principal source of error is the variation of the two particle scattering amplitudes with angle and energy, thus making it difficult to know the correct average values to substitute for them. This is especially true of the n-p amplitude in the vicinity of 70 Mev, which varies rather rapidly with both energy and angle. The uncertainty thus introduced was estimated by means of the grid of values for the scattering amplitudes obtained from the Prenowitz-Palmieri program. Other sources of error in the calculation, which turned out to be rather less important, are uncertainties in the deuteron wave function and the noncoplanarity of the intermediate state momenta. The deuteron wave function is uncertain primarily for large values of the momentum, a range which does not contribute strongly to the double scattering integrals, while, as mentioned earlier, the noncoplanarity is not important because of the small momentum of the target particles. The net result is that the error in the double scattering contribution to the matrix element of T is of the order of 20%, producing the indicated uncertainties in the ratio of corrected to uncorrected cross sections. It will be seen that the agreement between the experimental and corrected theoretical points is quite good. The only exceptions are the point with $\theta_1'=45°$ and large angular separations between the counters. The reason for this discrepancy is not clear. It is difficult to imagine theoretical effects which would mainfest themselves in an effect of this magnitude at these angles and not elsewhere. As KWC point out, there is some experimental problem at these angles, due to the possibility of detecting elastic scattering events.

It may be of interest to consider qualitatively why the double-scattering corrections have the effects shown. Let us consider the matrix element of T between given initial and final spin states. The single-scattering contribution, which we can call T_{1ij}, is just proportional to the corresponding matrix element of the p-p scattering matrix, i.e.,

$$T_{1ij}=K_1M_{pij}, \tag{4.1}$$

where K_1 is a constant determined by Eq. (3.6). If we direct our attention for the moment to the double-scattering terms $C1$ and $C3$ [Eq. (3.7)], we find that the spin summation appearing in them also contains a term proportional to M_{pij} evaluated at nearly the same energy and angle as in the single scattering term; it turns out also that these are the largest terms in the summation over intermediate state spins. Hence, considering only the contribution of these terms in $C1$ and $C3$ to the double scattering, we can approximate the double scattering contribution to T_{ij} by

$$T_{2ij}=K_2M_{nii}M_{pij}, \tag{4.2}$$

where K_2 is another constant determined by formula (3.21). Hence

$$T_{2ij}/T_{1ij}=K_2M_{nii}/K_1. \tag{4.3}$$

As we have remarked before, the main contribution to K_2 comes from the delta-function part of the double-scattering integrals; this is because the principal value integrands are rather symmetric about their singularities. If we consider only the delta-function parts, then K_2/K_1 turns out to be positive imaginary. Since the double-scattering contributions are small, we are interested mainly in the interference term between single and double scattering, which will be destructive in the event that the imaginary part of M_{nii} is greater than zero. This is, however, assured by unitarity in the forward direction, and the second scattering is confined to an angular region about the forward direction where the sign of M_{nii} persists. Thus, because of unitarity, these correction terms cause a removal of particles from the outgoing beam, rather than adding particles to it. As for the other correction terms, the scattering amplitudes describing the first collision in the correction term $C2$ are close enough in phase to the single scattering term that $C2$ also gives rise to destructive interference for similar reasons; $C4$ is very small, both because the quantity h appearing in the denominator is rather large and because both of the scattering amplitudes are evaluated at high energies and are small. Physically, $C4$, or at least those terms in $C4$ in which the matrix element for the first collision is diagonal in the spin, corresponds to the removal of particles from the incident beam by the neutron, i.e., to the "shadow effect" which has been discussed in

connection with slightly different experiments by Glauber.[10] Again unitarity tells us that particles are removed from the incident beam, and hence $C4$ will interfere destructively with the single scattering. Hence the tendency for the interference between single and double scattering to be destructive, so that the inclusion of double scattering leads to a predicted reduction in the cross section, is related to the unitarity of the two particle scattering matrices. It also depends, however, on the specific details of these matrices, i.e., on which particular terms in the summation over intermediate state spins furnish the main double scattering contributions, and on how fast the phase of the matrix elements varies as one goes away from the forward direction.

The reason the reduction in cross section is more pronounced for large or small angular separations between the outgoing protons arises from the fact, mentioned earlier, that single scattering at these angles requires, on the average, a more rapidly moving target particle and hence is suppressed by the deuteron wave function, i.e., by the factor $a(k_3')$ in Eq. (3.6). The double scattering terms, however, involve an average over the initial momentum of the target particle, and hence are quite insensitive to the particular kinematics of the final state. Hence the double scattering becomes relatively more important for final states in which the included angle differs widely from 87°.

Lastly, we remark on the qualitative variation as θ_1' becomes different from 45°. There are two reasons for variations in the importance of the correction terms. In the first place, all of the terms in $C1$, $C2$, and $C3$ are propertional either to the quantities $1/d$ or $1/a$, defined in Eq. (3.8), and hence essentially to either $1/\sin\theta_1'$ or $1/\cos\theta_1'$, respectively. Hence, half of the terms will increase and half will decrease as θ_1' becomes smaller; the increase, however, will be faster than the decrease. Moreover, the energies at which the n-p matrix elements describing the second scattering are evaluated are roughly proportional to d^2 or a^2. Since the n-p cross section increases quite rapidly at energies below 70 Mev, once again this means the terms involving d increase rapidly as θ_1' becomes small, while those involving a decrease more slowly. (Physically, of course, this is saying that as θ_1' becomes small the energy of the recoil proton becomes small, and its final state interaction with the neutron becomes important.) Similarly, for large angles the terms involving a increase rapidly. Since the identity of the two protons guarantees that at 45° the terms involving d and a are of the same size, this means that the effect of the double scattering corrections will be a minimum at 45°. The effect of this can be noticed by comparing Figs. 1 and 2, for it will be seen that both the experimental and corrected theoretical points are somewhat lower for the 35° case. To obtain the corrections at $\theta_1'=20°$ requires a knowledge of the n-p scattering amplitude at an incident laboratory energy of 15 Mev. Approximating this amplitude by means of the values given by Blatt and Weisskopf[11] on the basis of effective-range theory yields the result that at this angle the double-scattering contributions to T have increased from about 10% at 45° to about 25%, and hence the maximum value of $d^2\sigma$ at 20° would be about 60% of the value predicted by the simple impulse approximation.

KWC also measured the left-right asymmetry in quasi-elastic p-p scattering. In contrast with the cross-section results, they found the experimental asymmetry to be in excellent agreement with the predictions of the S.I.A. Their measurements cover a range of values of θ_1' from 35° to 60°, with the angle between the counters near 87°. The experimental observations are in accord with the fact that the double-scattering corrections to the asymmetry, over the range of angles used, change the predicted asymmetry by less than 3% of its own value. In this case both terms on the right of Eq. (3.26) for ΔP, the change in the asymmetry due to the inclusion of the correction terms, turn out to be small. The effect of the corrections tends to be the same in all spin states, so that α/γ and β/γ are both nearly 1, while the asymmetry in the correction terms themselves must be small for scattering angles near 45° because of the indistinguishability of the two protons in the final state.

As pointed out before, the uncertainties shown in Figs. 1 and 2 for the ratio of corrected to S.I.A. value of the cross section include only the estimated errors in the calculation of the double-scattering terms. A rough estimate was also made of the triple-scattering terms in order to get some idea of the error introduced by neglecting the higher order terms in Eq. (2.12). To do this, suppose we consider a typical triple-scattering term, e.g., $t_2gt_dgt_2$. We note that we can write this as

$$t_2gt_dgt_2=t_2gC1\equiv T_3, \tag{4.4}$$

where $C1$ is defined by Eq. (3.7). The matrix element between initial and final states of T_3 can now be calculated by the methods of Sec. III. Keeping only the delta function part, one obtains the result

$$T=\tfrac{1}{4}|k_1'-k_2'|\langle f_p(\theta'-\theta'')C1(\theta'')\rangle_{\rm av}, \tag{4.5}$$

where $C1(\theta'')$ is the matrix element of the correction term $C1$ to an intermediate state in which the total energy of the two protons is the same as in the final state, but in which they have undergone scattering through an angle θ'' in their center-of-mass system, while the scattering angle in the actual final state is θ'. It is difficult to estimate the result of the average over θ'' because of the rather wide variation of the matrix elements; a rough numerical estimate of these quantities was attempted, however. The result is that the total contribution of the triple-scattering terms is about 75%

[10] R. Glauber, Phys. Rev. **100**, 242 (1955).

[11] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 176.

TABLE II. Double-scattering corrections to n-p scattering parameters. Column 2 gives the ratio of the predicted value of the cross section $d^3\sigma/dE_1'd\Omega_1'd\Omega_2'$ with double-scattering effects included, for the case in which the spectator proton is at rest and the center-of-mass scattering angle of the incident proton is that given in column 1, to the value predicted by the simple impulse approximation. Succeeding columns give the values of the parameters P, A, and R predicted by the impulse approximation, and the predicted change in these parameters due to double-scattering effects. Uncertainties shown are those arising from the computation. All results are based on the phase shifts of reference 7.

Angle	$d^3\sigma/d^3\sigma_0$	P	ΔP	R	ΔR	A	ΔA
90°	0.87±0.03	−0.05	−0.02±0.02	−0.02	−0.01±0.02	0.8	0.03±0.03
40°	0.7 ±0.05	0.52	0.05±0.03	0.29	−0.2 ±0.05	0.05	0.17±0.06

of that due to double scattering and of the opposite sign. The opposite sign is not surprising, in view of the fact that triple-scattering terms are, from Eq. (4.4), seen to be related to the double-scattering terms in much the same way as the latter are related to single scattering. Although the estimates are very rough, the general conclusion that the triple scattering is of the same general magnitude as double scattering is probably correct. The good agreement between the double-scattering theory and experiment thus appears to be due to a cancellation between the triple and higher order scattering contributions. It appears that the effects of successive orders of multiple scattering will tend to be in opposite directions, so that there will be a tendency for such cancellation to occur.

B. Neutron-Proton Case

Since one way of obtaining information about the neutron-proton interaction is by the analysis of quasi-elastic n-p scattering from a deuterium target using the S.I.A., it is of interest to estimate the difference, due to neglect of the higher order terms in the multiple-scattering expansion, in the n-p parameters obtained in this way from their true values as measured in a free n-p scattering experiment. We have therefore calculated the double-scattering corrections to the cross section $d^3\sigma/dE_1'd\Omega_1'd\Omega_2'$ and to the polarization and triple scattering parameters R and A for a proton energy of 145 Mev and center-of-mass scattering angles of the proton of 90° and 40°, corresponding to the experiment of Kuckes and Wilson[12] and Thorndike *et al.*[9] This can, of course, only be done by assuming a set of n-p scattering amplitudes and we have again made use of those obtained from the Gammel-Thaler phase shifts. It seems likely that these will give qualitatively correct information concerning the nature of the double-scattering corrections. If the data are used in a phase shift analysis it would of course be possible to use the amplitudes which result from the analysis to recalculate the corrections; it is, however, by no means clear that the accuracy with which the corrections can be calculated would warrant this procedure.

The results are shown in Table II, which gives the effect of the inclusion of the four double-scattering terms on the values of the cross section, the polarization, and parameters R and A for two different scattering angles. At 40° the energy of the recoil neutron is sufficiently small that the effects of both of the single-scattering terms, t_2 as well as t_3, have been included in the calculation. It might be noted that the ratio of cross sections in column 2 is also approximately the ratio of the integrated cross sections $d^2\sigma/d\Omega_1'd\Omega_2'$ in the case that the angle between the emerging neutron and fast proton is near 87°. The uncertainties shown in Table II are the estimated uncertainties arising in the calculation of the double scattering terms from causes similar to those discussed in the p-p case. There is the additional uncertainty due to the higher order terms in the multiple-scattering expansion. Although these appeared to have little effect on the p-p results, this seems to be due, as mentioned above, to a cancellation among successive orders of multiple scattering. This cancellation may have been fortuitous; there appears to be no way in which one can be assured that the cancellation will occur as completely in the n-p case. A rough estimate of the triple-scattering terms in the n-p case indicates that they are again quite appreciable, and, as in the p-p case, tend to have the opposite sign from the double-scattering terms. One can say, in view of the fact that successive orders of multiple scattering tend to have opposite signs, that the double-scattering corrections, as tabulated in Table II, probably represent the maximum corrections to the impulse approximation.

V. CALCULATION OF PROTON-PROTON SCATTERING MATRIX OFF THE ENERGY SHELL

We next consider the second type of correction to the simple impulse approximation, that which arises from the fact that the two-nucleon scattering operators in the single-scattering terms connect states in which the two particles have different kinetic energies. (Of course there will be similar corrections to the double-scattering terms, but the effect of these corrections will be small and we shall neglect them.) We shall confine ourselves to the study of these effects in the quasi-elastic p-p experiments of KWC. The p-p scattering matrix elements which enter the single-scattering terms have the form

$$M_{fi}=-\mu\langle\mathbf{k}',f|t_p(e)|\mathbf{k},i\rangle/2\pi\hbar^2, \qquad (5.1)$$

[12] A. F. Kuckes and Richard Wilson, Phys. Rev. **121**, 1226 (1961).

where μ is the reduced mass of the two protons, $\mathbf{k}$, i, $\mathbf{k}'$, and f refer to the initial and final relative wave vectors and total spins of the two protons, and $e=\hbar^2k'^2/2M$. Also

$$\hbar^2k^2/4M-\hbar k'^2/4M=\Delta E \tag{5.2}$$

where ΔE is defined in Eq. (2.13). The use of the measured free p-p cross section in calculating the single-scattering contribution corresponds to evaluating the matrix elements at $\Delta E=0$; we wish, therefore, to study their dependence on ΔE.

In Eq. (5.1) we can replace $t_p(e)$ acting to the left on a plane-wave state of energy e by V acting to the left on the state $\psi_{\mathbf{k}',f}^{(-)}$, where V is the proton-proton potential, and $\psi_{\mathbf{k}',f}^{(-)}$ is an exact eigenstate of the two-proton Hamiltonian consisting of a plane wave and incoming spherical waves. Hence we need to evaluate $\langle\psi_{\mathbf{k}',f}^{(-)}|V|\mathbf{k},i\rangle$. This could be done perfectly well, but as one is more accustomed to dealing with outgoing wave solutions, we can make use of the time-reversal transformation to replace this by $\langle-\mathbf{k},-i|V|\psi_{-\mathbf{k}',-f}^{(+)}\rangle$. Hence M_{fi} is given by

$$M_{fi}=(-\mu/2\pi\hbar^2)\int d\mathbf{r}\sum_{s''}\langle-\mathbf{k},-i|\mathbf{r},-i\rangle \times\langle\mathbf{r},-i|V|\mathbf{r},s''\rangle\langle\mathbf{r},s''|\psi_{-\mathbf{k}',-f}^{(+)}\rangle. \tag{5.3}$$

If we expand the wave functions in (5.3) in terms of radial functions multiplying spin-angle functions which are eigenstates of the total angular momentum, we can carry out the angular integration and spin summation, and are then left with a sum of radial integrals multiplied by somewhat involved expressions depending on i, f, and the scattering angle. A typical one of these radial integrals will have the form

$$I(\beta)=\int d\rho\; j_l(\beta\rho)F(\rho)R_{l'j}(\rho), \tag{5.4}$$

where $\rho=k'r$, $\beta k'=k$, $R_{l'j}$ is the radial function in $\psi^{(+)}$ which multiplies the spin-angle function for total angular momentum j and orbital angular momentum l'; the form of $F(\rho)$ depends on the radial dependence of the potential. l and l' are either equal, or else they may differ by two because of the presence of a tensor force.

The evaluation of the matrix elements was carried out by means of a program written for the IBM 704 computer. The computations were performed at the MIT computation center. The program solved the Schrödinger equation with the Gammel-Thaler potential to obtain the radial function $R_{l'j}$, performed the integrals of the form (5.4) numerically for the value $\beta=1$, and also evaluated $dI/d\beta$. It then evaluated the spin- and angle-dependent functions, and computed M_{fi} and $dM_{fi}/d\beta$ for $\beta=1$ and a given value of the scattering angle for the six combinations of f and i which yield independent matrix elements; details of the calculation are given in reference 13. The parameter β is given by

$$\beta=\left(\frac{\frac{1}{2}e+\Delta E}{\frac{1}{2}e}\right)^{\frac{1}{2}}\approx 1+\frac{\Delta E}{e}. \tag{5.5}$$

For $\beta\approx 1$, we have therefore

$$dM_{fi}=M_{fi}(\beta)-M_{fi}(1)\approx(dM_{fi}/d\beta)|_{\beta=1}(\Delta E/e). \tag{5.6}$$

The detailed calculations were for $e=140$ Mev and a center-of-mass scattering angle of 90°, appropriate to the analysis of the data presented in Fig. 1. For $\theta_1'=45°$, the average values of $\Delta E/e$ are 0.05 when the included angle between the protons is 75°, 0.02 for an included angle of 90°, and 0.08 for an included angle of 105°. The values obtained for the matrix elements and their derivatives were, using the notation of reference 8,

$$\begin{aligned} M_{ss}&=-0.071+0.063i, & dM_{ss}/d\beta&=+0.056-0.05i,\\ M_{01}&=-1.20\ -0.118i, & dM_{01}/d\beta&=+0.240-0.302i,\\ M_{10}&=+0.337+0.185i, & dM_{10}/d\beta&=+0.961-0.013i. \end{aligned} \tag{5.7}$$

(The remaining p-p matrix elements and their derivatives are identically zero at 90°.)

Unfortunately, we feel sure only of the qualitative correctness of these results. The phase shifts obtained by our program, while they had the same general magnitude and variation with angular momentum as those reported by Gammel and Thaler, differed in detail from the Gammel-Thaler values by amounts of the order of 30%. This is reflected in similar differences between the values of the matrix elements given in (5.7) and those obtained from the Palmieri-Prenowitz program. Regrettably a change in the type of computer at MIT made our program obsolete before the source of this discrepancy could be tracked down. Errors of this magnitude will not affect our general conclusions.

The effect of using the off-the-energy-shell values of the matrix elements as computed from Eqs. (5.6) and (5.7) is to lower the value of the theoretical curve in Fig. 1 by about 0.01 at the maximum, 0.03 at 75°, and 0.04 at 105°. These changes are obviously much less than the uncertainty in either the experimental or theoretical results, even neglecting the unknown effect of higher order multiple scattering.

These results are again dependent on the specific form of the Gammel-Thaler potential. It had been hoped to investigate the sensitivity of the variation with β to the specific form of the potential, but this was also largely prevented by the change in the MIT com-

[13] Allen Everett, Ph.D. thesis, Harvard University (unpublished).

puter. The only change in the potential whose effect we had a chance to study was a decrease in the radius of the repulsive core by about two-thirds; this resulted in an increase in the cross section by about a factor of 2, and changed the ratio of the cross section evaluated at $\beta=1.08$ to that for $\beta=1$ from 0.96 to 1.09.

Since the foregoing results are dependent on Eq. (5.6), one run was made with a small change in the program in order to compute directly the value of the matrix elements at $\beta=1.1$ in order to ascertain whether it was in fact reasonable to represent them by the first two terms in their power series expansion in β. This yielded the result that the change in the cross section was about 1.5 times the change predicted by Eq. (5.6). This effect is clearly of no consequence in the present case because of the experimental and theoretical uncertainties, especially since most of the experimental data involve values of β which differ from 1 by 0.05 or less; it does suggest, however, that the range of β over which Eq. (5.6) is adequate is quite limited.

VI. CONCLUSIONS

We now summarize the conclusions from the foregoing discussion. We begin by considering the importance of the various terms in the expansion (2.12), first of all as they affect a calculation of a differential cross section for scattering from a deuteron or some other two particle target. In the one case in which we have made a direct comparison of theory and experiment, namely the quasi-elastic *p-d* scattering as measured by KWC, we have found that we could obtain reasonable agreement with the experimental results by taking into account, in addition to the simple impulse approximation terms in (2.12), those terms which correspond to double scattering, and those arising from the final state interaction of the particles in the deuteron which have the form of double scattering. This is evidently true because of cancellation among the higher order multiple-scattering terms, since the triple-scattering terms seem to be of rather comparable magnitude to the double-scattering terms which we have included. This result, therefore, does not seem necessarily to justify the adequacy of the single- plus double-scattering approximation in other situations.

It is easy, from our previous discussion, to make a general estimate of the ratio of the magnitudes of the single- and double-scattering terms. Let f be the scattering amplitude entering the single-scattering process, and suppose we consider one of the double-scattering processes, defined in Eq. (1.7), in which two of the particles undergo an interaction described by the scattering amplitude f', and one of them recoils with a wave vector of magnitude k, assumed to be much larger than μ_1, the typical wave vector of nucleons in a deuteron, and interacts, with amplitude f'', with the remaining particle. Let us also suppose k_3', the wave vector of the "spectator" particle in the final state is essentially 0, so that $a(k_3')\approx 1/\mu_1^2\approx R^2$, where R is the average separation of the particles in a deuteron. Then it is easy to see from Eqs. (3.6) and (3.18), together with the definitions of the vectors d, a, and h in Eq. (3.8), that the ratio of the double-scattering term to the single scattering is given by numerical factors times the parameter r, where

$$r=(f'/f)(f''/R)(1/kR). \tag{6.1}$$

The numerical factors, e.g., the logarithm in (3.13), depend only on the kinematics and turn out to be of order unity, so that an estimate of the importance of any given double-scattering process can be made by finding the parameter r associated with it. Since the numerical factor multiplying r turns out to be positive imaginary, if r is small, one is interested mainly in $\mathrm{Im}(r)$, which determines the size of the interference term between single and double scattering. If one takes spin into account, then the quantity $f'f''$ really stands for a sum of terms resulting from the summation over intermediate spin states. Equation (6.1) tells us that the double scattering will be small if the amplitude f' is small compared to f, if $f''\ll R$, i.e., if the deuteron is rather transparent to the recoil particle, or if $kR\ll 1$, that is, if the wave length of the recoil particle from the first collision is much smaller than the deuteron radius, in which case one gets considerable destructive interference when one averages over the separation of the deuteron particles. It is pointed out in references 1 and 2 that r is the appropriate parameter for estimating the importance of true double-scattering corrections; we note that in the present case it also determines the magnitude of the lowest order corrections due to the binding of the deuteron.

As pointed out in our earlier discussion, the fact that unitarity determines *a priori* the sign of the imaginary part of some of the scattering amplitudes allows one to conclude that some of the double-scattering terms will interfere destructively with the single scattering and hence lead to a reduction in the cross section. Since the remaining terms are, *a priori*, as likely to increase as to decrease the cross section, one would expect that on the average the effect of the double-scattering corrections will be to reduce differential cross sections, as they did in all of the calculations presented in this paper. In the analysis of the KWC data, the imaginary part of the parameter r was about 0.015 for $C1$, $C2$, and $C3$, and about half of that for $C4$. However, the fact that all of the interference terms had the same sign, coupled with the effect of the numerical factors, meant that the net effect of the corrections was to lower the amplitude by about 10% from the S.I.A. value.

As was pointed out, these results can easily be generalized to the case of relativistic kinematics. If one carries out the calculation indicated in the discussion following Eq. (3.20), allowing also for the fact that the mass of the incident particle may be different from the

nucleon mass, one finds that the double-scattering term describing a process in which a particle emerges from an initial collision with a total energy E' and then interacts with the remaining particle whose total energy is E'' should be multiplied by the factor $E_L'(E_c'+E_c'')/2E_c'E_c''$, where the subscripts L and c refer to the laboratory system and the center-of-mass system of the two particles undergoing the second collision. At the energies at which our calculations were done, this changes the correction terms by only about 5% of their own value. Of course a more serious effect at relativistic energies may be the necessity of including in the sum over intermediate states, involved in calculating the correction, states containing a different number of particles than the initial and final states, which is a more difficult problem to deal with by the methods which we have employed.

Unfortunately, as we have noted, the usefulness of our approximation of including single- and double-scattering-like terms is lessened by the fact that the triple-scattering terms, which can only be estimated roughly, appear to be of the same order as the double scattering. It does, however, at least seem likely that the double-scattering approximation will give a relatively reliable estimate of an upper bound to the correction to the simple impulse approximation, in view of the tendency for triple and double scattering to interfere destructively, just as the latter tends to interfere destructively with single scattering.

It is, at first sight, surprising that the ratio of triple to double scattering is not roughly the same as that of double to single. The reason this is not so can be seen from Eq. (4.5), which indicates that the ratio of a triple- to a double-scattering term in the expansion, rather than being governed by the parameter r, is instead of order fk, where f and k represent the relative wave vector and scattering amplitude for the third collision; thus it does not decrease as the deuteron radius becomes larger. Physically, this arises from the fact that, in order for double scattering to occur, one must find all three of the particles involved relatively close to one another, and the liklihood of this obviously decreases as the deuteron becomes bigger. However, once double scattering has occurred, the particles are close together, and the chance of a third collision, involving two of *the same three particles*, no longer depends on the size of the initial target nucleus. This is to be distinguished from the sort of triple-scattering process, which could occur in a larger nucleus, in which an incident particle collides successively with three *different* target particles. The liklihood of this kind of triple scattering would decrease, compared to the liklihood of double scattering, as the radius of the nucleus increased.

The effect of the corrections on the double- or triple-scattering parameters, e.g., P or R, can be seen by analogy with the treatment in Eqs. (3.27)–(3.29). The value of any of these parameters, let us call it X, obtained by a measurement with a deuterium target may be written analogously to Eq. (3.28) as

$$X=[(1-\alpha_+)\sigma_+-(1-\alpha_-)\sigma_-]/(1-\alpha)\sigma, \qquad (6.2)$$

where the appropriate partial cross sections for the parameter in question are chosen. Thus ΔX, the difference between X and the true value X_0, has the form

$$\Delta X=X_0\left(1-\frac{(1-\alpha_+)}{(1-\alpha)}\right)-\frac{(\alpha_+-\alpha_-)\sigma_-}{(1-\alpha)\sigma}. \qquad (6.3)$$

We assume, for the purposes of estimating the behavior of ΔX as compared with that of the cross section, that α_+, α_-, and α all have somewhat comparable magnitudes, and, for the sake of definiteness, that $\sigma_+ \geqq \sigma_-$. It will be seen that ΔX consists of two terms. The first is simply proportional to X_0, and introduces a relative error which one would estimate to be of the same general magnitude as that introduced in the cross section, i.e., α, which we have already discussed. The exact magnitude depends on the relationship of α_+ and α, and hence on the details of the matrix elements. One can say, however, that as X_0 becomes very large, the percent error introduced by the first term will go to zero, since a large X_0 implies $\sigma_+ \gg \sigma_-$ and hence $\alpha_+ \approx \alpha$. The second term, however, is not proportional to X_0, and hence does not necessarily become small as X_0 becomes small; one would estimate its absolute magnitude to be of the order of α for small values of X_0, again depending on the details of the relationship between α_+ and α. The second term will also vanish as X_0 becomes large, as then $\sigma_- \ll \sigma_+$. Hence, in general, the values of one of these parameters measured in an experiment with deuterons should be quite accurate if the value of the parameter turns out to be very large. On the other hand, if the parameter is small, one might expect absolute errors of the order of α. An exception to these remarks occurs for the KWC experiment in the case of the correction to the left-right asymmetry, where the fact that one has two identical particles coming off at nearly equal angles in the laboratory system implies that the second term in (6.3) must be nearly zero, so that the error in the asymmetry becomes small as the asymmetry becomes small.

Finally, as to the corrections due to the fact that the matrix elements involved are off-the-energy shell, it appears that these corrections are small compared to those due to double scattering. They are, in fact, about comparable with the uncertainty in the double scattering corrections, so that it seems that the possibility of measuring the off-the-energy-shell effects in an experiment of the type we have been discussing is rather remote. Unfortunately, those final states in which the off-the-energy-shell effects are most pronounced are

also those in which the uncertainty in the double-scattering corrections is most important.

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Remarks on the Relativistic Kepler Problem*

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By means of a new representation, the Dirac-Coulomb spherical wave functions are treated in a manner which brings out the close formal similarity between these solutions and the spherical wave solutions for the free-electron problem. The radial functions in the new representation have the same form as the nonrelativistic radial Coulomb functions, but with an irrational orbital "angular momentum," $l(\gamma)$. This representation is utilized to deduce a general recursion relation for radial Coulomb eigenfunctions, and show the existence of the Coulomb helicity operator as a constant of the motion. The advantages and properties of this formulation are discussed briefly.

I. INTRODUCTION AND SUMMARY

THE problem of a Dirac electron in a pure Coulomb field ($\alpha Z/r$), as was first shown by Darwin in 1928, is one of the few problems involving the Dirac equation with external fields, which permits of an "exact" solution—exact, that is, within the restriction to the one-particle theory (unquantized fields) for a point nucleus of large mass.[1] The fundamental importance of this elementary problem, and the necessity for exploring the implications of the solutions need no emphasis.

It is the purpose of the present work to re-examine and rederive the Dirac-Coulomb solutions in a representation not hitherto discussed in the literature. This representation is chosen in order to diagonalize (in Dirac ϱ space) two operators, Γ (Sec. III) and $\mathfrak{K}$ (Sec. IV), of central importance to the Dirac-Coulomb problem. The first of these operators Γ was introduced by Martin and Glauber[2]; the second operator $\mathfrak{K}$ was introduced much earlier by Johnson and Lippmann in a brief note.[3] The operator $\mathfrak{K}$, as we shall show, has the significance of a generalized helicity operator, and is a constant of the motion for the relativistic Kepler problem; the operator Γ is more difficult to categorize briefly, but is connected with a generalization of the operator $\rho_3 K$ (where K is Dirac's operator) and is not a constant of the motion.

The importance of the representation S which diagonalizes the operator Γ lies in the fact that it enables us to treat the Dirac-Coulomb eigenfunctions as the precise analogs to the spherical wave solutions of the free (Dirac) electron. Moreover, in this representation the radial wave functions are surprisingly simple, being of precisely the same form as the radial functions in the nonrelativistic Coulomb problem. The transformation to the representation S makes it evident that the integer orbital angular momentum of the free-electron problem, becomes in the relativistic Coulomb problem a noninteger (irrational) "orbital angular momentum." In neither the integer nor the noninteger case is the orbital angular momentum sharp, yet it is conceptually helpful in understanding the problem.

The plan of the present paper is to discuss (Sec. II) the free-electron (plane wave) problem first, employing techniques which permit generalization to the Dirac-Coulomb problem (Sec. III). A basic result of this

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[1] The classic treatment is that of Arnold Sommerfeld, *Atombau und Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, 1939), Vol. II, Chap. 4, p. 209ff. See also H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York, 1957); and M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

[2] P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958). These authors were concerned with a specific calculation, and primarily with the discrete spectrum, and did not discuss the representation that diagonalized Γ.

[3] M. H. Johnson and B. A. Lippmann, Phys. Rev. **78**, 329(A) (1950). The operator actually introduced by Johnson and Lippmann differs trivially from $\mathfrak{K}$, however. The use of this operator was originally suggested to Martin and Glauber by K. A. Johnson (cf. footnote 6 of reference 2).