Analysis of Multichannel Reactions

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A method for analysis of multichannel reaction data is presented. We consider a system of n strongly coupled two-body channels in a particular "partial wave." The energy dependence of the cross sections is treated in the effective range approximation; the corrections to the scattering amplitudes due to mass differences between isotopic spin multiplets and Coulomb scattering are included. "Uncoupled" amplitudes, which would describe the scattering in the absence of influences of one of the channels, are introduced. Relations between these uncoupled phases and the actual amplitudes are derived (in a nonperturbative manner). The use of the uncoupled phase procedure, in conjunction with an effective-range analysis, to test certain theories of strong interactions is discussed.

I. INTRODUCTION

WE present in this paper a method for analysis of multichannel reaction data. A system of n strongly coupled two-body channels in a particular "partial wave" (i.e., state of total angular momentum J, z component J_z, parity, and isotopic spin I) is considered. The following properties of this system are treated: (a) the energy dependence of the cross sections in the effective range approximation; the scattering amplitudes as corrected for the virtual electromagnetic effects of (b) mass differences within isotopic spin multiplets; (c) Coulomb scattering; and (d) the relation of "uncoupled" amplitudes, which would describe the scattering in absence of influences of one of the channels, to the actual amplitudes.

In a previous paper,[1] **A**, we developed the multichannel effective range theory, remarkably analogous in form to the familiar one-channel s-wave theory. Appropriate amplitudes M_{ij} (where in the one-channel s-wave case, $M=k\cot\delta$, and in general M is essentially the inverse of the K matrix) can be accurately represented by expressions linear in k^2 where the coefficients of k^2 for the diagonal terms M_{ii} are interpretable in terms of the range of forces and can thus be roughly estimated *a priori*, whereas those for elements M_{ij} connecting different channels are small. In Sec. II, we review the relevant details of this theory.

Even though we assume that the strong interactions conserve isotopic spin, the electromagnetic interaction can cause a mass difference to occur between an isotopic spin doublet. The phenomenon is treated as a kinematical one, i.e., the M_{ij} are assumed to be unaffected by the mass splitting. On the other hand the Coulomb potential in a channel induces changes in the M matrix. These effects have been previously considered by Jackson and Wyld[2] and Dalitz and Tuan.[3] In Sec. III we present simple derivations which reproduce and generalize their results.

There is, however, a serious difficulty which will usually occur in a phenomenological analysis of multichannel reactions using the effective range formalism. In general, data will be available only with certain channels as initial states due, e.g., to the instability of some of the particles. We consider the case where just one initial channel is accessible experimentally. The complex (energy dependent) scattering length, $a(k)$, in this channel, directly characterizes much of the observations. The many amplitudes M_{ij} are, however, not simply related to $a(k)$ [correspondingly, $a(k)$ in general has a complicated energy dependence for multichannel systems]. Thus, while only one channel may be available as an initial state, knowledge of the M_{ij}'s, needed for the full effective range analysis, implies knowledge of *all* the scattering amplitudes. In the presence of *extensive accurate* data involving only one channel as the initial state the energy dependence specified by the effective range formalism could be used to gain knowledge of the M_{ij}'s.

This difficulty of a purely phenomenological approach leads us to study, in Sec. IV, the relation between actual scattering amplitudes and "uncoupled" amplitudes about which one may make theoretical proposals (for example, global symmetry of the pion-baryon interactions). It must be emphasized that the effective range theory has no direct connection or dependence on this messy theoretical problem of relating "uncoupled" to actual phases. Although the concept of "uncoupled" phase shifts is quite useful, the definition of these phases is ambiguous. We tried,[4] in **B**, to develop a workable and meaningful theory. Assume that there is a finite energy region in which certain strongly

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[1] M. Ross and G. Shaw, Ann. Phys. **13**, 147 (1961). We shall refer to this paper as **A**.

[2] J. Jackson and H. Wyld, Phys. Rev. Letters **2**, 355 (1959); Nuovo cimento **13**, 85 (1959).

[3] R. Dalitz and S. Tuan, Ann. Phys. **10**, 307 (1960).

[4] M. Ross and G. Shaw, Ann. Phys. **9**, 391 (1960). We shall refer to this paper as **B**.

coupled two-body channels are significant. We considered a theory explicitly involving only these certain channels, which need not all be open. In principle, there are interactions (in general not simple potentials) between these n channels alone which yield agreement with experiment in this energy region. The uncoupled phases are taken to be those which would exist if the couplings to one of the channels (for example, the "new" channel, i.e., the one with the highest threshold) were to vanish, the interactions among the other $n-1$ ("old") channels remaining unchanged. In **B**, it was then shown how to obtain relations for the actual amplitudes in terms of these (assumed) uncoupled phases in the old channels and the (measured) scattering length and production ratios from the new to the old channels (if there is more than one old channel). These relations for the actual phases are fundamentally nonperturbative approximations. They are based on approximations having some resemblance to those made in effective range theory: It is necessary to equate certain integrals over similar wave functions which are identical outside the range of forces. In **B**, the relation for the two-channel problem was tested in a computer experiment involving equal-range square-well potentials and was found to be *very accurate* for this case. In Sec. IV we outline the derivation of the uncoupled phase procedure; the results are a generalization of those presented in **B**.

The uncoupled-phase method serves two important purposes. In addition to permitting us to confront certain theoretical models of strong interactions with experiment, it enables us to determine all the M parameters, and thus make the full effective range analysis. Thus for practical purposes we can combine the two approximations: The uncoupled-phase method can provide certain parameters at a given energy which are needed to utilize the effective range form for the energy dependence of the scattering amplitude. In Sec. V, a brief discussion of this procedure, and comparison with other approximations, is given for the two-channel problem.

The formalism presented in this paper is applied to the s-wave $\bar{K}N$ system in the following paper.

II. MULTICHANNEL EFFECTIVE RANGE THEORY

Consider a system of coupled two-body channels. Let there be n channels. We consider each partial wave separately. Let γ be a set of eigenvalues of the constants of the motion Γ, such as J, J_z, parity, and I (the corrections induced by the electromagnetic field which does not conserve isotopic spin are treated in Sec. III). Consider the state vectors describing scattering at an energy E where all the channels are open.[5] The effective range theory rests entirely on the form of these state vectors in the position representation beyond the "range of forces." In a given partial wave γ, this form is well known. We write the state vector as a $1\times n$ column matrix whose elements, labeled by j, correspond to the wave function in that channel. For the outgoing wave boundary condition with incident beam in channel i, we have, outside the range of forces,[6]

$$\psi_i^+(r)\rightarrow\|(\rho_j/\rho_i)^{\frac{1}{2}}[(\delta_{ij}+iT_{ij})j_{l_j}(k_jr)+T_{ij}n_{l_j}(k_jr)]\mathcal{Y}_j\|, \quad (2.1)$$

for the "principal value" boundary condition,

$$\psi_i^p(r)\rightarrow\|(\rho_j/\rho_i)^{\frac{1}{2}}[\delta_{ij}j_{l_i}(k_ir)+K_{ij}n_{l_j}(k_jr)]\mathcal{Y}_j\|, \quad (2.2)$$

with

$$\rho_i=k_i\omega_i,$$

the product of momentum k_i and reduced energy ω_i in the center-of-mass system in channel i, and $\mathcal{Y}_i$ are normalized eigenfunctions of the operators Γ with eigenvalues γ. Note that the orbital angular momentum l is one of the channel properties (not necessarily in γ).

The normalization of our T (or scattering) matrix elements and K (or reaction) matrix elements in (2.1) and (2.2) is perhaps most conveniently given in terms of the usual Hamiltonian theory, although no explicit reference is made to the interactions in deriving effective range theory. Our normalization is such that[7]

$$T_{ij}=-2(\rho_i\rho_j)^{\frac{1}{2}}\langle\phi_i,H_I\psi_j^+(r)\rangle, \quad (2.3)$$

$$K_{ij}=-2(\rho_i\rho_j)^{\frac{1}{2}}\langle\phi_i,H_I\psi_j^p(r)\rangle, \quad (2.4)$$

where H_I is the interaction and ϕ_i is the γth partial wave of a plane wave incident in channel i:

$$\phi_i=\|\delta_{ij}j_{l_i}(k_ir)\mathcal{Y}_i\|. \quad (2.5)$$

The relation between the $n\times n$ matrices T and K is

$$T=K(1-iK)^{-1}. \quad (2.6)$$

K is Hermitian (when all channels are open) guaranteeing the unitarity of the S matrix, and it is symmetric as a result of time reversal invariance. A diagonal element of T has the usual form in terms of the (complex) phase shift δ_i:

$$T_{ii}=(e^{2i\delta_i}-1)/2i. \quad (2.7)$$

The partial cross section from channel i to channel j is given by

$$\sigma_{ij}=\frac{4\pi}{k_i^2}\frac{2J+1}{2S_i+1}|T_{ij}|^2, \quad (2.8)$$

where S_i is the spin of the initial (unpolarized) channel. We emphasize that all of these relations apply to a single partial wave.

Neither of the scattering states ψ^+, ψ^p is directly convenient to derive a useful effective range theory.

[5] Our results can all be extended into an energy region in which not all of the n channels are open.

[6] The symbol $\rightarrow$ will be used to indicate equality outside the range of forces.

[7] We use units $\hbar=c=1$.

For this purpose we consider the scattering state ψ_i^M which approaches outside the range of interaction

$$\psi_i^M(r) \to \left\| \left(\frac{\omega_j}{\omega_i}\right)^{\frac{1}{2}} [k_j^{-l_j} M_{ij} j_{l_j}(k_j r) + \delta_{ij} k_i^{l_i+1} n_{l_i}(k_i r)] \mathcal{Y}_j \right\|. \quad (2.9)$$

ψ_i^M has the essential property that outside the range of forces, there is an irregular function only in the channel i. (It is shown in **A** that because of this property, *only* the matrix M has a simple effective range expansion.) The $n\times n$ matrix of coefficients M_{ij} is given by

$$M = k^{l+\frac{1}{2}} K^{-1} k^{l+\frac{1}{2}}, \quad (2.10)$$

where $(k^{l+\frac{1}{2}})_{ij} = \delta_{ij} k_i^{l_i+\frac{1}{2}}$. M (essentially the inverse of the K matrix) is an even function of all the channel momenta and thus remains real and symmetric when all the channels are not open. (Note that we still use the $n\times n$ matrices which explicitly refer to all the n channels even when some are closed. See Sec. VI.) The relation between M and T is

$$T = k^{l+\frac{1}{2}} (M - ik^{2l+1})^{-1} k^{l+\frac{1}{2}}. \quad (2.11)$$

If there is only one channel,

$$M = k^{2l+1} \cot\delta. \quad (2.12)$$

Before going on to the effective-range formalism, we list some useful properties of M and K. It is useful to introduce a complex scattering "length" [having the dimensions of length raised to the $(2l_n+1)$ power] as function of energy $a(k)$, in the nth channel.[8] We define the scattering length by

$$T_{nn} = -k_n^{2l_n+1} [1/a(k) + ik_n^{2l_n+1}]^{-1}, \quad (2.13)$$

so that for a one-channel system

$$-1/a(k) = M, \quad (2.14)$$

(note the minus sign). We find that

$$-1/a(k) = M_{nn} + \mathfrak{M}_{nn}^{-1} \sum_{i\neq n} M_{ni} \mathfrak{M}_{in}, \quad (2.15)$$

and

$$\det(M - ik^{2l+1}) = \sum_{i=1}^{n} M_{ni} \mathfrak{M}_{in} = -\mathfrak{M}_{nn} [1/a(k) + ik_n^{2l_n+1}], \quad (2.16)$$

where $\mathfrak{M}_{ij}$ is the i, j minor of $(M - ik^{2l+1})$. Similarly for K:

$$-a(k) = K_{nn} + (\mathfrak{M}_{nn}')^{-1} \sum_{i\neq n} K_{ni} \mathfrak{M}_{in}', \quad (2.17)$$

$$\det(1 - iK) = \mathfrak{M}_{nn}' [1 + ia(k) k_n^{2l_n+1}], \quad (2.18)$$

where $\mathfrak{M}_{ij}'$ is the i, j minor of $(1-iK)$. Finally, the production amplitude from channel n to i has a simple dependence on a, besides being a function of those elements of M excluding M_{nn} and one M_{in}, $i\neq n$ (or a similar set of elements of K). That is, for i (and j)$\neq n$,

$$T_{ni} = [k_n^{2l_n+1} \operatorname{Im} 1/a(k)]^{\frac{1}{2}} [1/a(k) + ik_n^{2l_n+1}]^{-1} \times f_{ni}(M_{ij}, M_{nj}/M_{ni}), \quad (2.19)$$

and

$$\sum_{i\neq n} |f_{ni}|^2 = 1. \quad (2.20)$$

The role of $a(k)$ in T_{ij} (i, $j\neq n$) is complicated.

According to the effective range theory developed in **A**, M has the following energy dependence near any energy E_0:

$$M = M(E_0) + \tfrac{1}{2} R(k^2 - k^2(E_0)), \quad (2.21)$$

where R is a real approximately energy-independent and *diagonal* matrix, and the diagonal elements are roughly interpreted in terms of the "range of forces." We have

$$M_{ij} = M_{ij}(E_0) + \tfrac{1}{2} \delta_{ij} R_{ii} (k_i^2 - k_i^2(E_0)), \quad (2.22)$$

where

$$R_{ii} = C_{l_i} R_i^{-2l_i+1}, \quad C_0 = 1, \quad C_1 = -3, \quad (2.23)$$

with R_i a measure of the range of forces in the ith channel. We emphase that R was found to be approximately diagonal even for R_{ii} greatly different from R_{jj} and for $l_i \neq l_j$. The derivation of the results (2.21)–(2.23) depended only on the form (2.9) for ψ_i^M outside the range of forces. It did *not* involve the use of nonrelativistic kinematics or the introduction of a Schrödinger equation. The accuracy and applicability of the effective range formalism is discussed at some length in **A** (especially Sec. VI). See also Sec. VI of this paper.

It is seen from (2.15) that the energy dependence of $1/a(k)$ is not in general simple. In fact, the "zero-range" approximation, i.e., taking the matrix $R\equiv 0$, is not equivalent to constant scattering length; there is explicit dependence in (2.15) on the momenta k_i, $i\neq n$.

Below the threshold of the ith channel, all relations are extended by

$$k_i \to i\kappa_i, \quad \kappa > 0. \quad (2.24)$$

The cusp behavior[9] of the cross sections in the vicinity of a threshold is contained in the above formalism. Near the ith threshold, and in the partial wave γ in which channel i is in an s state, the cross sections σ_{jk} where $j\neq i$ will contain terms proportional to $|k_i|$ and will exhibit infinite derivatives as a function of energy E at $k_i = 0$ [since $(dk_i/dE)_{k_i=0} = \infty$].

III. CHARGE-DEPENDENT EFFECTS

In this section we consider two effects which the electromagnetic field can have in altering the formalism

[8] Note that the channel n singled out does not have to be the channel with the highest threshold or even the highest open threshold.

[9] G. Breit, Phys. Rev. **107**, 1612 (1958); R. Newton, Phys. Rev. **114**, 1611 (1959).

of the Sec. II which treated the strong interactions which conserve total isotopic spin. The phenomena of a mass splitting between isotopic-spin multiplets and Coulomb scattering have been previously treated by Jackson and Wyld[2] and Dalitz and Tuan.[3] We present simple derivations which reproduce and generalize their results.

First we treat the mass splitting induced by the virtual electromagnetic field. In the absence of mass splitting, consider the channels A and B which are members of a charge multiplet so that[8]

$$\begin{aligned}|A\rangle&=s|n,\alpha\rangle+r|n,\beta\rangle,\\ |B\rangle&=-r|n,\alpha\rangle+s|n,\beta\rangle,\end{aligned}\tag{3.1}$$

where $|n,\alpha\rangle$ and $|n,\beta\rangle$ are eigenfunctions of the operators Γ with $I=\alpha$ and $I=\beta$ (the other eigenvalues being the same), and $s^2+r^2=1$. There are n_α coupled channels with $I=\alpha$ and n_β with $I=\beta$. However, the virtual electromagnetic field (which does not conserve isotopic spin) may produce a small mass difference between channels A and B so that $k_A\neq k_B$. This phenomenon is treated as purely a kinematical one, more specifically, we assume that the M^I are unaffected. Then to find any T-matrix element one evaluates a big $(n_\alpha+n_\beta)\times(n_\alpha+n_\beta)$ matrix instead of a $n_\alpha\times n_\alpha$ plus a $n_\beta\times n_\beta$.

If we are only concerned with T-matrix elements from channel A, then simple relations may be obtained. The elastic T_{AA} and charge exchange T_{AB} elements are found by using non-Hermitian M^I to describe channel-n reactions alone: $M^I=-1/a_I$ where a is the complex energy-dependent scattering length for channel n. The M matrix for channels A and B alone is

$$-M=\begin{pmatrix} r^2a_\alpha^{-1}+s^2a_\beta^{-1} & rs(a_\beta^{-1}-a_\alpha^{-1})\\ rs(a_\beta^{-1}-a_\alpha^{-1}) & s^2a_\alpha^{-1}+r^2a_\beta^{-1}\end{pmatrix},\tag{3.2}$$

from which we obtain

$$T_{AA}=-k_A^{2l_n+1}(r^2a_\beta+s^2a_\alpha+ik_B^{2l_n+1}a_\alpha a_\beta)D,\tag{3.3}$$

$$T_{AB}=rs(k_Ak_B)^{l_n+\frac{1}{2}}(a_\alpha-a_\beta)D,\tag{3.4}$$

where

$$D^{-1}=1+ia_\beta(s^2k_B^{2l_n+1}+r^2k_A^{2l_n+1})+ia_\alpha(r^2k_B^{2l_n+1}+s^2k_A^{2l_n+1})-(k_Ak_B)^{2l_n+1}a_\alpha a_\beta.\tag{3.5}$$

Consider now the actual $(n_\alpha+n_\beta)\times(n_\alpha+n_\beta)$ M matrix which can be written as

$$M=\begin{pmatrix} N^\alpha & 0 & -rU^\alpha & sU^\alpha\\ 0 & N^\beta & sU^\beta & rU^\beta\\ -r\tilde{U}^\alpha & s\tilde{U}^\beta & (r^2M^\alpha+s^2M^\beta)_{nn} & rs(M^\beta-M^\alpha)_{nn}\\ s\tilde{U}^\alpha & r\tilde{U}^\beta & rs(M^\beta-M^\alpha)_{nn} & (s^2M^\alpha+r^2M^\beta)_{nn}\end{pmatrix},\tag{3.6}$$

where N^α is an $(n_\alpha-1)\times(n_\alpha-1)$ matrix and U^α is a $(n_\alpha-1)$ column vector (with $\tilde{U}^\alpha$ its transpose). Then

$$T_{AA}=k_A^{2l_n+1}[\det(M-ik^{2l+1})]^{-1}\times[-ik_B^{2l_n+1}\det(N-ik^{2l+1})^\alpha\det(N-ik^{2l+1})^\beta+\text{terms independent of } k_A,\ k_B],\tag{3.7}$$

$$T_{Ai}=k_A^{l_n+\frac{1}{2}}k_i^{l_i+\frac{1}{2}}[\det(M-ik^{2l+1})]^{-1}\times[-ik_B^{2l_n+1}\det(N-ik^{2l+1})^\beta g((N^\alpha)_{ij})+\text{terms independent of } k_A,\ k_B],\tag{3.8}$$

for channel $i(\neq n)$ having $I=\alpha$. Then simply by noting the two forms (3.3) and (3.7) for T_{AA} and using the fact that the β dependence of T_{Ai} must drop out as $k_B\to k_A$ we have

$$T_{Ai}=(1+ia_\beta k_B^{2l_n+1})DF(\alpha),$$

where $F(\alpha)$ is independent of k_B. Thus the inelastic matrix element from A to channel $i(\neq n)$ having $I=\alpha$

$$T_{Ai}=s(1+ia_\beta k_B^{2l_n+1})(1+ia_\alpha k_A^{2l_n+1})DT_{ni}^{\alpha},\tag{3.9}$$

or for $I=\beta$

$$T_{Ai}=r(1+ia_\alpha k_B^{2l_n+1})(1+ia_\beta k_A^{2l_n+1})DT_{ni}^{\beta},$$

(where the T^I are the isotopic-spin-conserving scattering matrices).

The other phenomenon we treat is the case when there are "charged" channels, i.e., channels with both particles charged. Then, in addition to the total (includes all partial waves) scattering amplitude for channel i with particles having charge $Z_{i1}e$ and $Z_{i2}e$,

$$\frac{\csc^2(\theta/2)}{2k_iB_i}\exp\left(\frac{2i}{k_iB_i}\ln[\sin(\theta/2)]\right),\tag{3.10}$$

where

$$B_i=-(Z_{i1}Z_{i2}e^2\omega_i)^{-1},\tag{3.11}$$

the "strong interaction" partial wave T amplitudes are altered to some new values T^c. We assume that there is a radius R_i outside of which the strong interaction can be neglected, whereas for $r<R_i$ the Coulomb interaction can be neglected.

Consider the situation when the charged channels i are in s states. Then just inside $r=R_i$, ψ_i^M has the free asymptotic form (2.9):

$$\psi_i^M=\|(\omega_j/\omega_i)^{\frac{1}{2}}[k_j^{-l_j}M_{ij}j_{l_j}(k_jr)+\delta_{ij}k_i^{l_i+1}n_{l_i}(k_ir)]\mathcal{Y}_j\|,\tag{3.12}$$

whereas just outside $r=R_i$, ψ_i^M has the Coulomb-corrected form

$$\psi_i^M=C_i\|(\omega_j/\omega_i)^{\frac{1}{2}}[(M_{ij}{}^cF_0/k_jr+\delta_{ij}G_0/r)\theta_{cj}+(k^{-l_j}M_{ij}{}^cj_{l_j}+\delta_{ij}k_i^{l_i+1}n_{l_i})(1-\theta_{cj})]\mathcal{Y}_j\|,\tag{3.13}$$

where C_i is a normalization constant, θ_{cj} is 1 if channel j is in a charged s state and zero otherwise, and F_0 and G_0 are the $l=0$ regular and irregular Coulomb wave functions. Matching the wave functions (3.12) and (3.13) and their derivatives with respect to k_ir, denoted

by primes, we have the Coulomb-corrected relations

$$M_{ii}^{c}=\{M_{ii}[G_0-G_0'\tan(k_iR_i)] -k_i[G_0'+G_0\tan(k_iR_i)]\} \times\{M_{ii}[F_0'\tan(k_iR_i)-F_0] -k_i[F_0\tan(k_iR_i)-F_0']\}^{-1}\theta_{ci} +M_{ii}(1-\theta_{ci}), \quad (3.14)$$

$$M_{ij}^{c}=[C_i^{-1}\theta_{ci}+(1-\theta_{ci})][C_j^{-1}\theta_{cj}+(1-\theta_{cj})]M_{ij}, \quad (3.15)$$

$$C_i^{-1}=\sin(k_iR_i)/F_0(k_iR_i). \quad (3.16)$$

Expanding in terms of the small parameter R_i/B_i, we have for $k_iR_i\ll 1$ (the Coulomb corrections are appreciable only for small k_i),

$$F_0=c_ik_iR_i(1-R_iB_i^{-1}+\cdots), \quad F_0'=c_i(1-2R_iB_i^{-1}+\cdots), \quad (3.17)$$

$$G_0=c_i^{-1}[1-2R_iB_i^{-1}(\xi_i-1)+\cdots], \quad G_0'=2\xi_i(k_iB_ic_i)^{-1}+\cdots, \quad (3.18)$$

with

$$c_i^2=2\pi k_i^{-1}B_i^{-1}[1-\exp(-2\pi k_i^{-1}B_i^{-1})]^{-1} \quad (3.19)$$

and

$$\xi_i=\ln(2k_iR_i)+0.577 +(k_iB_i)^{-2}\sum_{m=1}^{\infty}\{m[m^2+(k_iB_i)^{-2}]\}^{-1}. \quad (3.20)$$

Putting these expansions back into (3.14)–(3.16), we obtain[10]

$$M_{ii}^{c}=c_i^{-2}(M_{ii}+2B_i^{-1}\xi_i)\theta_{ci}+M_{ii}(1-\theta_{ci}), \quad (3.21)$$

$$M_{ij}^{c}=[c_i^{-1}\theta_{ci}+(1-\theta_{ci})][c_j^{-1}\theta_{cj}+(1-\theta_{cj})]M_{ij}. \quad (3.22)$$

The strong-interaction scattering amplitudes corrected for charged s-wave charged channels is

$$T^{c}=ck^{l+\frac{1}{2}}(M-ik^{2l+1}\mathbf{c})^{-1}k^{l+\frac{1}{2}}c, \quad (3.23)$$

where the diagonal matrices c and $\mathbf{c}$ are given by

$$c_{ii}=c_i, \quad \mathbf{c}_{ii}=c_i^2+i2\xi_i(B_ik_i)^{-1}, \quad (3.24)$$

for channel i charged and $l_i=0$, and

$$c_{ii}=\mathbf{c}_{ii}=1, \quad (3.25)$$

for channel i uncharged. The corrections induced by charged channels with $l_i>0$ may be obtained from expressions similar to Eqs. (3.14)–(3.16).

IV. THE UNCOUPLED PHASE METHOD

The desire to confront certain special theories of strong interactions, such as global symmetry of the pion-baryon interactions, with experiment led us to develop in **B** the method of "uncoupled phases." We want to make predictions on the basis of simplified interactions that would exist if other interactions were not present. The concept of turning off interactions is complicated and not exact. Indeed, articles on basic interactions do not usually discuss this difficult point, e.g., renormalization of the pion-hyperon interactions due to the presence of the kaon interaction.[11] Our definition of the amplitude association with the "uncoupled" interaction may not correspond to other definitions. In **B**, we considered our procedure in some detail in the case of simple potentials. There the definition of uncoupled amplitudes was relatively unambiguous and we found reasonably accurate relations, as checked by a numerical program, between them and observable amplitudes.

We now outline a derivation of our method. Consider a system of n strongly coupled, two-body channels for which experiments can only be initiated in the nth channel.[8] Thus we want to construct observable amplitudes, which contain $\frac{1}{2}(n^2+n)$ real numbers, from n real empirical parameters [i.e., the complex scattering length and the production ratios from the nth channel into the other channels and the other $\frac{1}{2}(n^2-n)$ real numbers that would describe all the reactions if the nth channel did not exist]. The guide we have in mind is the model of an $n\times n$ potential matrix H_I (with elements H_{ij}) coupling the n channels which would yield agreement with experiment. We work at a fixed energy and so make no restrictive assumption at all about these potentials. Consider an energy where all n channels are open.[5] In the partial wave γ we consider the K-matrix elements (2.4):

$$K_{ij}=-2(\rho_i\rho_j)^{\frac{1}{2}}\langle\phi_i,H_I\psi_j^{p}\rangle, \quad (4.1)$$

where the $1\times n$ column matrix

$$\psi_i^{p}=\|(\rho_j/\rho_i)^{\frac{1}{2}}[\delta_{ij}j_{l_i}(k_ir)+K_{ij}g_{ij}]\mathfrak{Y}_j\|, \quad (4.2)$$

and outside the range of forces

$$g_{ij}\to n_{l_j}(k_jr). \quad (4.3)$$

Inserting the explicit forms (2.5) and (4.2) for ϕ_i and ψ_j^{p} into (4.1), we have n^2 equations for the K_{ij} as linear inhomogeneous equations in the K_{ij}'s themselves:

$$-K_{ij}=I_{ij}{}^{i}J_{ij}{}^{i}+\sum_{k=1}^{n}J_{ik}{}^{i}K_{kj}. \quad (4.4)$$

In the matrix potential model, the coefficients J_{ij} are radial integrals over the potential H_{ij} coupling channels i and j with radial wave functions normalized *independent of the interactions* outside the range of forces:

$$J_{ij}{}^{k}\propto\int j_{l_i}(k_ir)H_{ij}g_{kj}(k_jr)r^2dr,$$

[10] Both the derivation using the state vector ψ^{M} and the resulting expressions for the Coulomb-corrected M matrix are simpler than the corresponding results for the K matrix found, in reference 3, by matching the principal-value wave function ψ^{p}.

[11] See M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065 (1961).

and

$$I_{ij}{}^{j} \propto \int j_{l_i}(k_i r) H_{ij} j_{l_j}(k_j r) r^2 dr / J_{ij}{}^{j}.$$

We define uncoupled quantities (printed in boldface) as those that would exist if there were no coupling to the nth channel, the interactions among the $n-1$ other channels being unchanged. Then we have the relations

$$-\mathbf{K}_{ij} = (\mathbf{I}_{ij}{}^{j}\mathbf{J}_{ij}{}^{j} + \sum_{k=1}^{n-1} \mathbf{J}_{ik}{}^{j}\mathbf{K}_{kj})(1-\delta_{in})(1-\delta_{jn}). \quad (4.5)$$

Now we adopt the approximations (a) that the coefficients $J_{ik}{}^{j}$ are insensitive to details of the g_{jk} well inside the range of forces so that we drop both the boldface and superscript notation for J (and I), and (b) in addition I_{ij} is independent of i. Indeed we expect that

$$I_{in} \approx I_n \equiv j_{l_n}(k_n R_n)/n_{l_n}(k_n R_n), \quad (4.6)$$

for some suitable range R_n. These approximations appear to be accurate for simple potentials. See results of computer experiment in **B**.

Using these approximations, we have from (4.4) and (4.5), for all j *but* $i \neq n$

$$-(K-\mathbf{K})_{ij} = I_n J_{in}\delta_{jn} + \sum_{k=1}^{n} J_{ik}(K-\mathbf{K})_{kj}, \quad (4.7)$$

which may be written as

$$\sum_{k=1}^{n} (\delta_{ik}+J_{ik})\mathfrak{K}_{kj} = 0, \quad (4.8)$$

where

$$\mathfrak{K}_{kj} = K_{kj} - \mathbf{K}_{kj}(1-\delta_{jn})(1-\delta_{kn}) + I_n \delta_{kn}\delta_{jn}. \quad (4.9)$$

In order that a solution of (4.8) exist, we readily establish that all 2×2 submatrices (formed by elimination of $n-2$ rows and $n-2$ columns) of the matrix $\mathfrak{K}$ must have zero determinant. There are $\frac{1}{2}(n^2-n)$ independent relations of this kind, for example, the $n-1$ relations

$$\det\begin{pmatrix} K_{ii}-\mathbf{K}_{ii} & K_{in} \\ K_{in} & K_{nn}+I_n \end{pmatrix} = 0, \quad (4.10)$$

for $i \neq n$, and the $\frac{1}{2}(n-1)(n-2)$ relations

$$\det\begin{pmatrix} K_{ii}-\mathbf{K}_{ii} & K_{ij}-\mathbf{K}_{ij} \\ K_{ij}-\mathbf{K}_{ij} & K_{jj}-\mathbf{K}_{jj} \end{pmatrix} = 0, \quad (4.11)$$

for i, $j \neq n$. The number of independent relations is equal to the number of uncoupled amplitudes $\mathbf{K}_{ij}$.

Given the uncoupled amplitudes $\mathbf{K}_{ij}$ we can then construct the actual K matrix if we know the experimentally-measurable $(n-2)$ production ratios T_{nj}/T_{ni} and complex scattering length $a(k)$. The procedure then is to calculate M from K (at one energy) and use effective range theory to describe the cross sections over a large energy region.

V. EXAMPLE: UNCOUPLED PHASE METHOD FOR TWO-CHANNEL PROCESSES

Let us consider the case of two interacting channels, assuming that it is not necessary to take into explicit account the effects of other open or closed channels in order to describe these reactions in a limited energy region. We suppose that just one of the channels is sufficiently stable to be available experimentally as the initial state. Take this to be the new channel. The complex energy-dependent scattering length is the convenient quantity with which to describe the experimental results. At a given energy, one other real parameter is needed to specify all three amplitudes among the two channels. Indeed, this third parameter enters into the phase of the nondiagonal amplitude, T_{21}, and in the diagonal amplitude in the old channel, T_{11}. Additional importance of this parameter arises from the fact that the effective range procedure for specifying the energy dependence of *any* of the amplitudes requires knowledge of all of them at one energy.

We take this third parameter to be the uncoupled phase which would describe the reaction in channel 1 in the absence of coupling to channel 2. Thus all the amplitudes at a given energy are expressed in terms of the scattering length (of channel 2) and uncoupled K-matrix element in channel 1, $\mathbf{K}_{11}$. It is convenient at the same time to compare these results with other approximation schemes having the same aim. For these purposes we exhibit not the actual amplitude T, but K and M.

In terms of the K matrix, we find after some algebra, from (4.10) and (2.17),

$$K_{11} = \frac{\mathrm{Im}a(k) + [\mathrm{Re}a(k)-L]\mathbf{K}_{11}}{-\mathrm{Im}a(k)\mathbf{K}_{11} + \mathrm{Re}a(k) - L}, \quad (5.1)$$

or

$$\tan^{-1}K_{11} = \tan^{-1}\mathbf{K}_{11} + \tan^{-1}[\mathrm{Im}a/(\mathrm{Re}a-L)], \quad (5.2)$$

$$K_{12}{}^2 = -k_2{}^{2l_2+1}\,\mathrm{Im}a(1+K_{11}{}^2), \quad (5.3)$$

$$K_{22} = -k_2{}^{2l_2+1}(\mathrm{Re}a + \mathrm{Im}a K_{11}), \quad (5.4)$$

where we have defined [see (4.6)]

$$L = I_2 k_2{}^{-(2l_2+1)} \approx R_2{}^{2l_2+1}/(2l_2+1)!!(2l_2-1)!!, \quad (5.5)$$

the latter if our energy corresponds to small k_2. A weak-coupling approximation sometimes used instead of the standard but often trivial weak-coupling approximation in T (in which $T_{12}=0$) is "weak coupling in K" in which K_{11} is replaced by $\mathbf{K}_{11}$ in (5.1)–(5.4). In comparison with our uncoupled phase results we see that weak coupling in K will be satisfactory[12] if

$$|\mathrm{Im}a(k)/[\mathrm{Re}a(k)-L]| \ll |\mathbf{K}_{11}|, \quad (5.6)$$

[12] Properly said, there are conditions under which it coincides with the uncoupled phase procedure. The latter is a nonperturbative approximation which we have numerically tested and found satisfactory for a wide range of interaction strengths (see **B**). Thus we are confident that the uncoupled phase method has a much wider range of validity than either of the other approximation schemes discussed here.

and

$$|\mathrm{Im}a(k)/[\mathrm{Re}a(k)-L]|\ll|\mathbf{K}_{11}|^{-1}. \quad (5.7)$$

For practical purposes the condition may be given as

$$|\mathrm{Im}a(k)/[\mathrm{Re}a(k)-L]|\ll 1. \quad (5.8)$$

A subtler approximation has been proposed by Matthews and Salam.[13] To discuss it and to go on to combine the uncoupled phase results with effective range theory, we write the M-matrix elements according to the uncoupled phase procedure:

$$M_{11}{}^{-1}k_1{}^{2l_1+1}=\frac{\mathrm{Im}a^{-1}(\mathrm{Re}a^{-1}-L^{-1})^{-1}+\mathbf{K}_{11}}{1-\mathbf{K}_{11}\,\mathrm{Im}a^{-1}(\mathrm{Re}a^{-1}-L^{-1})^{-1}}, \quad (5.9)$$

or

$$\tan^{-1}M_{11}{}^{-1}k_1{}^{2l_1+1}=\tan^{-1}\mathbf{K}_{11}+\tan^{-1}\mathrm{Im}a^{-1}(\mathrm{Re}a^{-1}-L^{-1})^{-1}, \quad (5.10)$$

$$M_{12}{}^2=k_1{}^{2l_1+1}\,\mathrm{Im}a^{-1}(1+M_{11}{}^2k_1{}^{-(4l_1+2)}), \quad (5.11)$$

$$M_{22}=-\mathrm{Re}a^{-1}+M_{11}k_1{}^{-(2l_1+1)}\,\mathrm{Im}a^{-1}. \quad (5.12)$$

Matthews and Salam propose a "weak-coupling procedure for M":

$$M_{ij}=\mathbf{M}_{ij} \quad \text{for} \quad i,\ j\neq n. \quad (5.13)$$

That is, for the two-channel system, replace M_{11} by $\mathbf{M}_{11}$ ($\equiv k_1{}^{2l_1+1}\mathbf{K}_{11}$) in (5.9)–(5.12). Referring to (5.9), we see that the condition for the weak-coupling M procedure is

$$|\mathrm{Im}a^{-1}/(\mathrm{Re}a^{-1}-L^{-1})|\ll|\mathbf{K}_{11}|,$$
$$|\mathrm{Im}a^{-1}/(\mathrm{Re}a^{-1}-L^{-1})|\ll|\mathbf{K}_{11}|^{-1},$$

or for practical purposes,

$$|\mathrm{Im}a^{-1}/(\mathrm{Re}a^{-1}-L^{-1})|\ll 1. \quad (5.14)$$

Our proposed scheme for analyzing the two-channel reaction data is to evaluate the M_{ij}'s at a given energy; for example, the threshold of channel 2 and to substitute them into the effective range expressions (2.22). Then, with the ranges R_i, we have the scattering amplitudes (2.11) as a function of energy. These are all given in terms of five real constants: Rea, Ima, and $\mathbf{K}_{11}$ evaluated at energy E_0, and R_1 and R_2. The parameter L should be evaluated using R_2 in (5.4). For example, the elastic scattering in the new channel is given by

$$T_{22}=k_2{}^{2l_2+1}\{[M_{22}(E_0)+\tfrac{1}{2}R_{22}(k_2{}^2-k_2{}^2(E_0))-M_{12}{}^2(E_0)[M_{11}(E_0)+\tfrac{1}{2}R_{11}(k_1{}^2-k_1{}^2(E_0))-ik_1{}^{2l_1+1}]^{-1}]-ik_2{}^{2l_2+1}\}^{-1}, \quad (5.15)$$

where the R_{ii} are given in terms of the ranges R_i by (2.23), and the $M_{ij}(E_0)$ are to be taken from (5.9)–(5.12).

In summary, for two channels, with only one channel available experimentally as an initial state, while two real numbers describe the directly available experimental data at a given energy (ignoring possible interference effects with other isotopic spin states), five real numbers occur in the effective range expressions (the two ranges may be roughly specified *a priori*). One notes that the effective range expression energy-dependent scattering length [$-1/a(k)$ is shown in the square brackets of (5.15)] consists of the familiar one-channel result:

$$-1/a(k)=\mathrm{const}+\tfrac{1}{2}R[k_2{}^2-k_2{}^2(E_0)], \quad (5.16)$$

(where R is real and interpretable as a range) *plus* additional energy dependence. If we can establish in a particular case that $M_{12}{}^2$ is sufficiently small, or $|M_{11}|$ large, or that k_1 varies sufficiently slowly with respect to k_2, then the one-channel effective range from (5.16) can be used to express T_{22} and $|T_{12}|$ as functions of energy, with great attendant simplification.

VI. DISCUSSION

In summary, we have treated two separate theoretical aspects of the analysis of multichannel reactions: First, in Secs. II and III we dealt with the form and energy dependence of scattering amplitudes. Second, in Sec. IV we treated the problem of using the phenomenological parameters determined using the formalism of Secs. II and III to gain some theoretical knowledge of the underlying interactions (or alternately assuming some special theoretical model of the interactions, along with some empirical knowledge, predict certain of the phenomenological parameters).

The effective range formalism presented in Sec. II had two striking features of simplicity: (i) The non-diagonal elements of the M matrix are essentially constant even when the diagonal elements have large (and numerically quite different) effective range terms. (ii) The diagonal effective range terms (as in the familiar one-channel problem) are directly interpretable in terms of the range of forces. As discussed in Sec. VI of **A**, there are situations in which features (i) and (ii) do not hold, in particular, when some of the M_{ij} are very large. However, even for these cases, a rather good approximation to the absolute magnitudes of the cross sections can be obtained using our formalism (2.21)–(2.23). (We would like to thank Dr. J. J. de Swart for informing us of the results of his calculations of $\Lambda N-\Sigma N$ reactions in ${}^3s_1$ and ${}^3d_1$ states for several physically interesting potentials. His results bear out the above remark.)

In the one-channel potential model, the effective-range approximation should be valid for a strong potential of "short, well-defined range." The potential must be relatively simple [for example, the one-channel effective-range theory does not allow a zero in the scattering amplitude at finite energy, while it is easy to construct a (non-monotonic) potential leading to a zero]. In some respects, the real many-channel elemen-

[13] See M. Gupta, Nuovo cimento **16**, 737 (1961).

tary particle problem is more straightforward. We may postulate that the basic interactions are very simple and that the complications arise from the different roles played by the many states that enter. This leads to a generalized criterion for validity of effective range theory. Since long-range interactions arise in association with nearby (in energy) new channels are stable states *that are not explicitly accounted for* in our n-channel theory, all channels near the given energy region must be considered explicitly in order that the effective range theory be accurate. Speaking more formally, each distinct physical singularity in the amplitude corresponds to a channel. All channels entering in or near the energy region of interest, assuming their weight to be significant (i.e., that the amplitude to that state at these energies is not small), should be explicitly represented in the effective range theory. Then the effective range form of the amplitude will reasonably represent all nearby singularities. The (energy) region of validity will depend on the distance of the nonphysical singularities and of the physical singularities not explicitly considered. Further discussion of the restrictions on the validity and applicability of the effective range theory is given in **A**.

While the derivation of uncoupled phase method presented in Sec. IV *has no connection or dependence* on the effective range theory, it is an extremely useful tool to use in conjunction with a phenomenological analysis using the formalism of Secs. II and III. Because of its applicability in predicting the zero-range parameters and confronting theoretical models of strong interactions with experiment, it would seem very worthwhile to investigate in greater detail the theoretical basis of the uncoupled phase procedure.

The formalism presented here is applied in the following paper to the s-wave $\bar{K}N$ system.

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