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Fixed-Angle Dispersion Relations for Nucleon Compton Scattering. I*

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The Mandelstam representation is used to derive fixed-angle dispersion relations for the twelve scalar amplitudes describing the process $\gamma+N\rightarrow\gamma+N$. The electromagnetic interaction is calculated to order e^2. The strong interactions are estimated by including one- and two-pion exchange on the left-hand cut. These depend on the $\pi^0\rightarrow 2\gamma$ decay lifetime and on the $T=0$ $\pi\pi$ phase shift and the total cross section for photoproduction of pions on pions, respectively. The right-hand cut is estimated by allowing the exchange of a nucleon and a pion-nucleon pair, which depend on the amplitude for meson photoproduction on nucleons. The low-energy limit theorem provides an important tool for estimating the subtractions required in the dispersion relations.

It is hoped that the representation will be accurate for barycentric photon energies up to approximately 300 Mev.

I. INTRODUCTION

THE possibility of much improved accuracy in photon scattering experiments,[1] especially in the energy region just below the pion nucleon 3,3 resonance, and the failure of present theories[2] to account fully for the experimental data suggest that it would be useful to analyze the problem of photon scattering on nucleons in some detail. To this end we have used the Mandelstam representation to set up fixed-angle dispersion relations for the photon-nucleon scattering amplitudes. We work to second order in the electromagnetic coupling constant, so that the jump on the right-hand cut of the dispersion relations will not contain directly the Compton scattering amplitudes. This being so, there is no advantage in writing dispersion relations for the partial wave or multipole amplitudes, since one does not end up with integral equations. On the other hand, the position of the cuts with respect to the region under investigation is strongly dependent on the scattering angle so that a comparison of the energy dependence of the amplitudes at different angles will provide us with some knowledge of those portions of the cuts which we cannot evaluate directly.

On the left-hand cut we shall include the effects of one- and two-pion exchange. The importance of the former, which leads to a pole whose residue is closely related to the $\pi^0\rightarrow 2\gamma$ decay lifetime, has been realized for some time.[2] A disagreement as to the relative sign of this term, is, we believe, finally settled.

The two-pion contribution, on the other hand, has been completely ignored up to the present. Because of the boson nature of the photons it turns out that only an isotopic spin-zero two-pion state is possible. Not much is known about the $T=0$ π-π interaction though Hamilton *et al.*[3] (in their study of π-N scattering) find evidence for a very strong interaction. On the other hand, recent experiments,[4] which yield a measure of the π-π c.m. correlation in pion production, do not show any evidence of a $T=0$ resonance or strong interaction. It is therefore of added interest to see whether a suitably chosen s-wave interaction is able to improve the agreement between theory and experiment in nucleon Compton scattering.

On the right-hand cut we consider one-nucleon and pion-nucleon exchange. The former leads to Born terms analogous to the usual second order Compton scattering amplitude. For the latter we require a knowledge of the meson photoproduction process, which we hope to take from experiment. At present the data are not quite good enough to allow an accurate multipole analysis but we hope that this situation will soon improve.

The spins of the nucleon and the photon add enormously to the complexity of the problem, so we shall be content, in the present paper, to set up the formalism and leave the actual numerical calculation for a future publication.

Section II contains the kinematics of the problem, and the spin and isospin decomposition of the scattering amplitude. The helicity expansions for the $\gamma N\rightarrow\gamma N$ and $N\bar{N}\rightarrow\gamma\gamma$ processes are given, and expressions for various experimentally measurable quantities are noted.

In Sec. III we introduce the Mandelstam representation and use it to set up fixed-angle dispersion relations for the $\gamma N\rightarrow\gamma N$ process.

Section IV uses the unitarity condition to calculate the weight functions required in the dispersion relations,

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[1] G. Bernadini, A. O. Hanson, A. C. Odian, T. Yamagata, L. B. Auerbach, and I. Filosofo, Nuovo cimento **18**, 1203 (1960).

[2] M. Jacob and J. Mathews, Phys. Rev. **117**, 854 (1960).

[3] J. Hamilton, P. Menotti, T. D. Spearman, and W. S. Woolcock, University College, London (to be published).

[4] A. Abrashian, N. E. Booth, and K. M. Crowse, Phys. Rev. Letters **7**, 35 (1961). For references see V. DeAlfaro and B. Vitale, Phys. Rev. Letters **7**, 72 (1961).

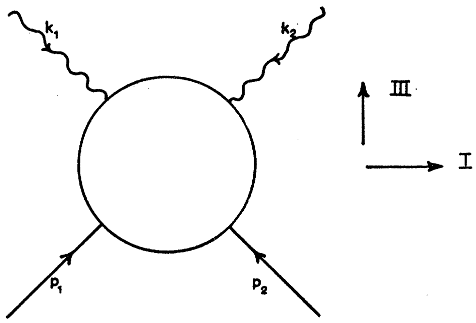

FIG. 1. Channels I and III of the two-nucleon, two-photon problem.

in terms of the amplitudes for the processes

$$\gamma+N\rightarrow N,$$
$$\gamma+N\rightarrow \pi+N,$$
$$N+\bar{N}\rightarrow \pi^0\rightarrow \gamma+\gamma,$$
$$N+\bar{N}\rightarrow 2\pi\rightarrow \gamma+\gamma.$$

In Sec. V the low-energy limit theorem[5] for photon scattering is used to estimate the subtractions required in the dispersion relations.

Section VI contains the conclusions, and in the Appendix can be found some of the less palatable aspects of the unitarity calculations.

II. KINEMATICS

a. Processes Considered

We shall consider simultaneously the three processes

$$\gamma_1+N_1\rightarrow\gamma_2+N_2 \quad \text{(Channel I)}$$
$$\gamma_2+N_1\rightarrow\gamma_1+N_2 \quad \text{(Channel II)}$$
$$N_1+\bar{N}_2\rightarrow\gamma_1+\gamma_2 \quad \text{(Channel III)},$$

where N_1, N_2, γ_1, γ_2 have incoming 4-momenta p_1, p_2, k_1, k_2, respectively (Fig. 1.) Conservation of energy-momentum implies that all scalars may be expressed in terms of the three variables[6]:

$$s=-(p_1+k_1)^2,\quad \bar{s}=-(p_1+k_2)^2,\quad t=-(p_1+p_2)^2, \tag{2.1}$$

which satisfy the mass-shell condition $s+\bar{s}+t=2m^2$ (m being the nucleon mass).

For the barycentric system of channel I, we have

$$s=W^2=[(p^2+m^2)^{\frac{1}{2}}+p]^2,\quad \bar{s}=-2p^2(1+\cos\theta)+[(p^2+m^2)^{\frac{1}{2}}-p]^2,\quad t=-2p^2(1-\cos\theta), \tag{2.2}$$

where W is the total barycentric energy, p and θ are the magnitude of the barycentric 3-momentum and scattering angle, respectively. Note that

$$p=(s-m^2)/2\sqrt{s},\quad \cos\theta=\frac{(s-m^2)^2+2st}{(s-m^2)^2}. \tag{2.3}$$

In the barycentric system of channel III we have similarly

$$s=-2\kappa^2-m^2-2\kappa\cos\psi(\kappa^2+m^2)^{\frac{1}{2}},\quad \bar{s}=-2\kappa^2-m^2+2\kappa\cos\psi(\kappa^2+m^2)^{\frac{1}{2}},\quad t=4k^2=4(\kappa^2+m^2), \tag{2.4}$$

where $\boldsymbol{\kappa}$ is the momentum of the initial nucleon and $\mathbf{k}$ that of the final photon, ψ is the scattering angle, and

$$\cos\psi=(\bar{s}-s)/[t(t-4m^2)]^{\frac{1}{2}}. \tag{2.5}$$

b. Decomposition of the S Matrix

The S matrix for the process may be written in the form[7]:

$$S=1+iR, \tag{2.6}$$

where R is related to the Feynman amplitude F by

$$R=m(2\pi)^{-2}(4p_{10}p_{20}k_{10}k_{20})^{-\frac{1}{2}}\delta(p_1+p_2+k_1+k_2)F. \tag{2.7}$$

Since F is bilinear in the photon polarization vectors we may write, in channel I:

$$\langle\gamma_2N_2|F|\gamma_1N_1\rangle=\epsilon_{2\nu}{}^{\dagger}\bar{u}_2(-p_2)F_{\mu\nu}u_1(p_1)\epsilon_{1\mu}, \tag{2.8}$$

where ϵ_1, ϵ_2 are the polarization vectors of γ_1, γ_2 and u_1, u_2 are the initial and final Dirac spinors which satisfy

$$(i\gamma\cdot p_j+m)u_j=0,\quad j=1,2 \text{ (not summed)}.$$

Neglecting for the moment the charge degrees of freedom, an inspection of the possible spin and polarization states reveals that six independent amplitudes are necessary to describe the process. The form of these as required by the principles of Lorentz and gauge invariance, parity invariance, and charge-conjugation invariance has been fully analyzed by Prange.[7] He shows that $F_{\mu\nu}$ may be written in the form:

$$\begin{aligned}F_{\mu\nu}=&A_1(s,t,\bar{s})\frac{P_\mu'P_\nu'}{P'^2}+A_2(s,t,\bar{s})\frac{N_\mu N_\nu}{N^2}\\&+A_3(s,t,\bar{s})\frac{(P_\mu'N_\nu-P_\nu'N_\mu)i\gamma_5}{(P'^2N^2)^{\frac{1}{2}}}\\&+A_4(s,t,\bar{s})\frac{P_\mu'P_\nu'i\gamma\cdot K}{P'^2}+A_5(s,t,\bar{s})\frac{N_\mu N_\nu i\gamma\cdot K}{N^2}\\&+A_6(s,t,\bar{s})\frac{(P_\mu'N_\nu-P_\nu'N_\mu)i\gamma_5 i\gamma\cdot K}{(P'^2N^2)^{\frac{1}{2}}}.\end{aligned} \tag{2.9}$$

[5] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954); F. E. Low, Phys. Rev. **96**, 1428 (1954).

[6] In our metric, $a\cdot b=\mathbf{a}\cdot\mathbf{b}-a_0b_0$. We are also working in the system of units $\hbar=c=1$.

[7] R. E. Prange, Phys. Rev. **110**, 240 (1958).

We write this for brevity as

$$F_{\mu\nu}=\sum_{i=1}^{6} A_i(s,t,\bar{s})F_{\mu\nu}{}^{(i)}. \qquad (2.10)$$

Here $P_\mu'=P_\mu-(P\cdot K)/K^2K_\mu$, $P_\mu=\frac{1}{2}(p_{1\mu}-p_{2\mu})$, $K_\mu=\frac{1}{2}(k_{1\mu}-k_{2\mu})$, $N_\mu=\epsilon_{\mu\nu\rho\sigma}P_\nu'K_\rho Q_\sigma$, $Q_\mu=k_{1\mu}+k_{2\mu}$, and we *define*[8] $(P'^2N^2)^{\frac{1}{2}}=\frac{1}{2}(m^4-s\bar{s})$ in *all* channels.

Prange, in fact, did not normalize his momentum variables, as we have done, but we find that this is necessary in order that the A_i should have the correct analyticity properties.[9]

The charge degrees of freedom can be incorporated by writing each A_i in the form

$$A_i=A_i{}^SI+A_i{}^V\tau_3. \qquad (2.11)$$

Thus the amplitudes for γ-proton and γ-neutron scattering are given, respectively, by

$$\begin{aligned} A_i{}^p&=A_i{}^S+A_i{}^V,\\ A_i{}^n&=A_j{}^S-A_i{}^V. \end{aligned} \qquad (2.12)$$

c. Crossing Properties

Generalized crossing symmetry tells us that the amplitudes for the processes in channels II and III are given by

$$\begin{aligned} \langle\gamma_1N_2|F|\gamma_2N_1\rangle&=\epsilon_{1\nu}{}^\dagger\bar{u}_2(-p_2)F_{\mu\nu}u_1(p_1)\epsilon_{2\mu},\\ \langle\gamma_1\gamma_2|F|N_1\bar{N}_2\rangle&=\epsilon_{2\nu}{}^\dagger\bar{v}_2(p_2)F_{\mu\nu}u_1(p_1)\epsilon_{1\mu}{}^\dagger, \end{aligned} \qquad (2.13)$$

where the $F_{\mu\nu}$ are the same functions but evaluated in each case in the region of the variables corresponding to the particular physical process involved. Since the amplitude is invariant in going from channel I to channel II, i.e., under the interchange of initial and final photons, we find that

$$A_i(s,t,\bar{s})=\eta_iA_i(\bar{s},t,s)$$

with

$$\begin{aligned} \eta_i&=+1 \quad \text{for} \quad i=1,2,3,6,\\ &=-1 \quad \text{for} \quad i=4,5. \end{aligned} \qquad (2.14)$$

d. Experimental Quantities

In channel I, the differential cross section may be written in the form:

$$d\sigma/d\Omega=\sum|\langle\gamma_2N_2|T|\gamma_1N_1\rangle|^2, \qquad (2.15)$$

where the scattering amplitude T is related to F by

$$T=(m/4\pi W)F, \qquad (2.16)$$

and $\sum$ represents an average over initial and a sum over final spins and polarizations.

In terms of the A_i, the cross section is given by

$$\begin{aligned} \frac{d\sigma}{d\Omega}=\frac{m^2}{64\pi^2W^2}\Bigg[&\left(2+\frac{p^2}{m^2}[1-\cos\theta]\right)(|A_1|^2+|A_2|^2)\\ &+\frac{2p^2}{m^2}(1-\cos\theta)|A_3|^2-\frac{2Wp^2}{m^2}(p[1-\cos\theta]-W)\\ &\times(|A_4|^2+|A_5|^2)+\frac{2W^2p^2(1+\cos\theta)}{m^2}|A_6|^2\\ &+\frac{2m^2-2W^2+2p^2(1-\cos\theta)}{m}\operatorname{Re}(A_1A_4{}^*+A_2A_5{}^*)\Bigg]. \end{aligned} \qquad (2.17a)$$

It is also feasible to measure the recoil nucleon polarization P (in the direction perpendicular to the plane of scattering). For unpolarized incoming beam and target we have:

$$P\frac{d\sigma}{d\Omega}=\frac{Ep^2\sin\theta}{32mW\pi^2}\operatorname{Im}(A_1A_4{}^*+A_2A_5{}^*). \qquad (2.17b)$$

More exhaustive calculations of such polarization formulas have recently been published by Frolov.[10]

e. Spin and Helicity Amplitudes

It is often useful to consider the scattering amplitude as a matrix taken between Pauli spinors rather than Dirac spinors. We can then write, in Channel I[(2)] (suppressing spinors),

$$\begin{aligned} \langle\gamma_2N_2|T|\gamma_1N_1\rangle=&g_1\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2+g_2\hat{p}'\cdot\boldsymbol{\varepsilon}_1\hat{p}\cdot\boldsymbol{\varepsilon}_2\\ &+g_3i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\times\boldsymbol{\varepsilon}_2+g_4\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2i\boldsymbol{\sigma}\cdot\hat{p}\times\hat{p}'\\ &+g_5(\hat{p}'\cdot\boldsymbol{\varepsilon}_1i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_2\times\hat{p}-\hat{p}\cdot\boldsymbol{\varepsilon}_2i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\times\hat{p}')\\ &\quad+g_6(\hat{p}'\cdot\boldsymbol{\varepsilon}_1i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_2\times\hat{p}'-\hat{p}\cdot\boldsymbol{\varepsilon}_2i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\times\hat{p}), \end{aligned} \qquad (2.18)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are initial and final barycentric nucleon 3-momenta, respectively. By writing an explicit representation for the Dirac spinors in terms of the Pauli spinors, we can express the g_i in terms of the A_i, giving:

$$8\pi g_1=\frac{1}{W}(C_1A_2+C_2A_5),$$

$$8\pi g_2=\frac{1}{W\sin^2\theta}[C_1(A_1+\cos\theta A_2)+C_2(A_4+\cos\theta A_5)],$$

$$8\pi g_3=\frac{E-m}{W}[(A_1+\cos\theta A_2)+(m+W)(A_4+\cos\theta A_5)]+\frac{2p}{W}(A_3+WA_6),$$

$$8\pi g_4=\frac{E-m}{W}[A_2+(m+W)A_5],$$

[8] This is to ensure that $F^{(3)}$ and $F^{(6)}$ have the correct generalized crossing properties.

[9] A. C. Hearn, Nuovo cimento **21**, 333 (1961).

[10] G. V. Frolov, Soviet Physics JETP **12**, 1277 (1961).

$$8\pi g_5=-\frac{E-m}{W\sin^2\theta}\times\cos\theta[(A_1+\cos\theta A_2)+(m+W)(A_4+\cos\theta A_5)]-\frac{p}{W\sin^2\theta}[(1+\cos\theta)A_3-W(1-\cos\theta)A_6],$$

$$8\pi g_6=\frac{E-m}{W\sin^2\theta}[(A_1+\cos\theta A_2)+(m+W)(A_4+\cos\theta A_5)]+\frac{p}{W\sin^2\theta}[(1+\cos\theta)A_3+W(1-\cos\theta)A_6]. \quad (2.19)$$

Here E is the barycentric nucleon energy ($E^2=p^2+m^2$), and $C_1=(E+m)-(E-m)\cos\theta$, $C_2=(E+m)(m-W)-(E-m)(m+W)\cos\theta$.

Previous work[2,11] on this problem has analyzed the process in terms of such Pauli spinor forms. A simple description is however possible in terms of the Jacob and Wick helicity states.[12] The process is completely specified in channel I by six independent amplitudes of the type

$$\langle\lambda_{N2}\lambda_2|T|\lambda_{N1}\lambda_1\rangle,$$

where $|\lambda_{Ni}\lambda_i\rangle$ represents a state with nucleon helicity λ_{Ni} and photon helicity λ_i. These six amplitudes can be expanded in terms of partial wave helicity amplitudes as follows:

$$\begin{aligned}
\Phi_1&=\langle\tfrac{1}{2}1|T|\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{1}{2},-\frac{1}{2}}{}^J d_{-\frac{1}{2},-\frac{1}{2}}{}^J(\theta),\\
\Phi_2&=\langle-\tfrac{1}{2}\,-1|T|\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{1}{2},\frac{1}{2}}{}^J d_{-\frac{1}{2},\frac{1}{2}}{}^J(\theta),\\
\Phi_3&=\langle\tfrac{1}{2}\,-1|T|\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{1}{2},\frac{3}{2}}{}^J d_{-\frac{1}{2},\frac{3}{2}}{}^J(\theta),\\
\Phi_4&=\langle-\tfrac{1}{2}1|T|\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{1}{2},-\frac{3}{2}}{}^J d_{-\frac{1}{2},-\frac{3}{2}}{}^J(\theta),\\
\Phi_5&=\langle-\tfrac{1}{2}1|T|-\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{3}{2},-\frac{3}{2}}{}^J d_{-\frac{3}{2},-\frac{3}{2}}{}^J(\theta),\\
\Phi_6&=\langle\tfrac{1}{2}\,-1|T|-\tfrac{1}{2}1\rangle=\frac{1}{2p}\sum_J(2J+1)\Phi_{-\frac{3}{2},\frac{3}{2}}{}^J d_{-\frac{3}{2},\frac{3}{2}}{}^J(\theta).
\end{aligned}\quad(2.20)$$

The properties of the reduced rotation matrices $d_{\lambda\lambda'}{}^J(\theta)$ are given in reference 10. The partial wave helicity amplitude $\Phi_{\lambda\lambda'}{}^J$ is a subamplitude of the R operator, and describes a transition from a state of helicity $\lambda=(\lambda_{N1}-\lambda_1)$ and total angular momentum J to a state of helicity λ' and total angular momentum J.

By writing explicit representations for the spinors involved, we may relate the Φ_i to the A_i, giving

$$\begin{aligned}
8\pi W\Phi_1&=\cos(\theta/2)[m(A_2-A_1)-Wp(A_5-A_4)-2pWA_6],\\
8\pi W\Phi_2&=-\sin(\theta/2)[E(A_1+A_2)-mp(A_4+A_5)-2pA_3],\\
8\pi W\Phi_3&=\cos(\theta/2)[m(A_1+A_2)-Wp(A_4+A_5)],\\
8\pi W\Phi_4&=-\sin(\theta/2)[E(A_2-A_1)-mp(A_5-A_4)],\\
8\pi W\Phi_5&=\cos(\theta/2)[m(A_2-A_1)-Wp(A_5-A_4)+2pWA_6],\\
8\pi W\Phi_6&=\sin(\theta/2)[E(A_1+A_2)-mp(A_4+A_5)+2pA_3],
\end{aligned}\quad(2.21)$$

and, alternatively,

$$\begin{aligned}
A_1&=\frac{2\pi}{p}\left[\frac{m}{\cos(\theta/2)}(\Phi_1-2\Phi_3+\Phi_5)+\frac{W}{\sin(\theta/2)}(\Phi_6+2\Phi_4-\Phi_2)\right],\\
A_2&=\frac{2\pi}{p}\left[-\frac{m}{\cos(\theta/2)}(\Phi_1+2\Phi_3+\Phi_5)+\frac{W}{\sin(\theta/2)}(\Phi_6-2\Phi_4-\Phi_2)\right],\\
A_3&=\frac{2\pi W}{p\sin(\theta/2)}(\Phi_2+\Phi_6),\\
A_4&=\frac{2\pi}{p^2}\left[\frac{E}{\cos(\theta/2)}(\Phi_1-2\Phi_3+\Phi_5)+\frac{m}{\sin(\theta/2)}(\Phi_6+2\Phi_4-\Phi_2)\right],\\
A_5&=\frac{2\pi}{p^2}\left[-\frac{E}{\cos(\theta/2)}(\Phi_1+2\Phi_3+\Phi_5)+\frac{m}{\sin(\theta/2)}(\Phi_6-2\Phi_4-\Phi_2)\right].\\
A_6&=\frac{2\pi}{p\cos(\theta/2)}(\Phi_5-\Phi_1).
\end{aligned}\quad(2.22)$$

[11] L. I. Lapidus and Chou Kuang-Chao, Soviet Physics JETP, **10**, 1213 (1960) and **11**, 147 (1960), and Dubna (to be published).
[12] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

A similar helicity analysis may be made in channel III. There the process may be described by the amplitudes

$$
\begin{aligned}
\mathfrak{M}_1&=\langle\mathbf{k};1\;-1|R|\boldsymbol{\kappa};\tfrac{1}{2}\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{02}{}^Jd_{02}{}^J(\psi),\\
\mathfrak{M}_2&=\langle\mathbf{k};11|R|\boldsymbol{\kappa};\tfrac{1}{2}\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{00+}{}^Jd_{00}{}^J(\psi),\\
\mathfrak{M}_3&=\langle\mathbf{k};-1\;-1|R|\boldsymbol{\kappa};\tfrac{1}{2}\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{00-}{}^Jd_{00}{}^J(\psi),\\
\mathfrak{M}_4&=\langle\mathbf{k};1\;-1|R|\boldsymbol{\kappa};\tfrac{1}{2}\;-\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{12}{}^Jd_{12}{}^J(\psi),\\
\mathfrak{M}_5&=\langle\mathbf{k};-11|R|\boldsymbol{\kappa};\tfrac{1}{2}\;-\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{1\,-2}{}^Jd_{1\,-2}{}^J(\psi),\\
\mathfrak{M}_6&=\langle\mathbf{k};11|R|\boldsymbol{\kappa};\tfrac{1}{2}\;-\tfrac{1}{2}\rangle=\frac{1}{2\pi(k^3\kappa)^{\frac{1}{2}}}\sum_J(2J+1)R_{10+}{}^Jd_{10}{}^J(\psi),
\end{aligned}
\qquad(2.23)
$$

and the relation between the $\mathfrak{M}_i$ and the A_i is

$$
\begin{aligned}
16\pi^2k^2\mathfrak{M}_1&=\kappa(A_2-A_1)+km\cos\psi(A_4-A_5),\\
16\pi^2k^2\mathfrak{M}_2&=\kappa(A_2+A_1)-km\cos\psi(A_4+A_5)+2kA_3,\\
16\pi^2k^2\mathfrak{M}_3&=\kappa(A_2+A_1)-km\cos\psi(A_4+A_5)-2kA_3,\\
16\pi^2k\mathfrak{M}_4&=\sin\psi[(A_4-A_5)k-2\kappa A_6],\\
16\pi^2k\mathfrak{M}_5&=\sin\psi[(A_4-A_5)k+2\kappa A_6],\\
16\pi^2\mathfrak{M}_6&=-\sin\psi(A_4+A_5).
\end{aligned}
\qquad(2.24)
$$

III. THE MANDELSTAM REPRESENTATION

We assume now that each of the six functions $A_i(s,t,\bar{s})$ satisfies a Mandelstam representation of the form

$$
\begin{aligned}
A_i(s,t,\bar{s})&\\
=R_i&\left(\frac{1}{s-m^2}+\frac{\eta_i}{\bar{s}-m^2}\right)+\frac{r_i}{t-\mu_0{}^2}\\
&+\frac{1}{\pi^2}\int_{(m+\mu)^2}ds'\int_{(m+\mu)^2}d\bar{s}'\frac{\chi_i(\bar{s}',s')}{(s'-s)(\bar{s}'-\bar{s})}\\
&+\frac{1}{\pi^2}\int_{(m+\mu)^2}ds'\int_{4\mu^2}dt'\frac{\rho_i(s',t')}{t'-t}\left(\frac{1}{s'-s}+\frac{\eta_i}{s'-\bar{s}}\right),
\end{aligned}
\qquad(3.1)
$$

where η_i is defined in (2.14) and where

$$\chi_i(\bar{s}',s')=\eta_i\chi_i(s',\bar{s}'). \qquad(3.2)$$

Since we are working to order e^2 in the electromagnetic coupling constant we have neglected in (3.1) cuts arising from photon exchange. With this restriction it can be shown, in perturbation theory, that the A_i have the correct analytic properties.[9]

The question of subtractions in (3.1) will be deferred until Sec. V.

For brevity we shall write

$$P_i=R_i\left(\frac{1}{s-m^2}+\frac{\eta_i}{\bar{s}-m^2}\right)+\frac{r_i}{t-\mu_0{}^2}, \qquad(3.3)$$

for the pole terms in (3.1).

The fact that the photon has zero mass causes the "pole" terms to have certain unusual features. These will be discussed in Sec. IV.

The unitarity condition will allow us to determine the imaginary parts of the A_i in the various channels. From (3.1) we see that

$$
\begin{aligned}
A_i{}^{\rm I}(s,t,\bar{s})&\equiv {\rm Im}A_i(s,t,\bar{s})\quad\text{for}\quad s\geqslant(m+\mu)^2,\\
&=\frac{1}{\pi}\int_{(m+\mu)^2}d\bar{s}^1\frac{\chi_i(\bar{s}',s)}{\bar{s}'-\bar{s}}+\frac{1}{\pi}\int_{4\mu^2}dt'\frac{\rho_i(s,t')}{t'-t},
\end{aligned}
\qquad(3.4)
$$

and

$$
\begin{aligned}
A_i{}^{\rm III}(s,t,\bar{s})&\equiv {\rm Im}A_i(s,t,\bar{s})\quad\text{for}\quad t\geqslant 4\mu^2,\\
&=\frac{1}{\pi}\int_{(m+\mu)^2}ds'\frac{\rho_i(s',t)}{s'-s}+\frac{\eta_i}{\pi}\int d\bar{s}'\frac{\rho_i(\bar{s}',t)}{\bar{s}'-\bar{s}}.
\end{aligned}
\qquad(3.5)
$$

From these we have the general identification

$$A_i{}^{\rm I}(x,y)=\frac{1}{\pi}\int_{(m+\mu)^2}d\bar{s}'\frac{\chi_i(\bar{s}',x)}{\bar{s}'-\bar{s}(x)}+\frac{1}{\pi}\int_{4\mu^2}dt'\frac{\rho_i(x,t')}{t'-y}\quad\text{for}\quad x\geqslant(m+\mu)^2, \qquad(3.6)$$

and

$$A_i{}^{\rm III}(x,y)=\frac{1}{\pi}\int_{(m+\mu)^2}ds'\,\rho_i(s',y)\left(\frac{1}{s'-x}+\frac{\eta_i'}{s'-\bar{s}(x)}\right)\quad\text{for}\quad y\geqslant 4\mu^2. \qquad(3.7)$$

By $\bar{s}(x)$ we mean $\bar{s}$ expressed as a function of s with $s\rightarrow x$, i.e.,

$$\bar{s}(x)=\frac{1}{x}[m^4-\tfrac{1}{2}(x-m^2)^2(1+\cos\theta)], \qquad (3.8)$$

and we shall analogously use

$$t(x)=-\frac{(x-m^2)^2}{2x}(1-\cos\theta). \qquad (3.9)$$

Note that crossing symmetry implies

$$A_i^{\text{II}}(x,y)=\eta_i A_i^{\text{I}}(\bar{s}[x],y). \qquad (3.10)$$

We wish now to separate in (3.1) the terms contributing solely to the right-hand cut. There are many ways to do this, the most direct being to expand the double denominators in (3.1) into partial fractions. After some manipulation we obtain the representation at fixed angle θ, in the form[13]

$$\begin{aligned}
&A_i(s,\cos\theta)\\
&=P_i+\frac{1}{\pi}\int_{(m+\mu)^2} ds'\frac{A_i^{\text{I}}(s',t(s'))}{s'-s}\\
&\quad+\frac{2\eta_i}{\pi}\int_{(m+\mu)^2} d\bar{s}'\frac{1}{\bar{s}_+(\bar{s}')-\bar{s}_-(\bar{s}')}\\
&\quad\times\left\{\frac{\bar{s}_+(\bar{s}')}{s(1+\cos\theta)-\bar{s}_+(\bar{s}')}A_i^{\text{I}}\left[\bar{s}',t\left(\frac{\bar{s}_+(\bar{s}')}{1+\cos\theta}\right)\right]\right.\\
&\quad\left.-\frac{\bar{s}_-(\bar{s}')}{s(1+\cos\theta)-\bar{s}_-(\bar{s}')}A_i^{\text{I}}\left[\bar{s}',t\left(\frac{\bar{s}_-(\bar{s}')}{1+\cos\theta}\right)\right]\right\}\\
&\quad+\frac{2}{\pi}\int_{4\mu^2} dt'\frac{1}{t_+(t')-t_-(t')}\left\{\frac{t_+(t')}{s(1-\cos\theta)-t_+(t')}\right.\\
&\quad\times A_i^{\text{III}}\left[\frac{t_+(t')}{1-\cos\theta},t'\right]-\frac{t_-(t')}{s(1-\cos\theta)-t_-(t')}\\
&\quad\left.\times A_i^{\text{III}}\left[\frac{t_-(t')}{1-\cos\theta},t'\right]\right\}, \qquad (3.11)
\end{aligned}$$

where

$$t_\pm(t')=m^2(1-\cos\theta)-t' \pm\{[m^2(1-\cos\theta)-t']^2-m^4(1-\cos\theta)^2\}^{\frac{1}{2}},$$

and

$$\bar{s}_\pm(\bar{s}')=m^2(1+\cos\theta)-\bar{s}' \pm\{[m^2(1+\cos\theta)-\bar{s}']^2+m^4\sin^2\theta\}^{\frac{1}{2}}. \qquad (3.12)$$

Since we shall only be able to evaluate the integrals in (3.11) for a very small part of their integration ranges, i.e., for s', $\bar{s}'\leqslant(m+2\mu)^2$ corresponding to the exchange of a meson-nucleon pair and for $t'\leqslant 9\mu^2$ corresponding two-pion exchange we must, as usual, rely upon the philosophy of the importance of the nearest singularities. To see the situation more clearly we may consider all the integrations in (3.11) transformed to the s plane. Figure 2 shows the integration contours in the s plane for various values of θ. The position of the pole terms is also indicated. The right-hand cut, $s\geqslant(m+\mu)^2$ comes from the first integral in (3.11). The second integral contributes along the cuts

$$-\infty\leqslant s\leqslant \mathcal{S}_-(\cos\theta),$$

and

$$0\leqslant s\leqslant \mathcal{S}_+(\cos\theta),$$

where

$$\mathcal{S}_\pm(\cos\theta)=\frac{1}{1+\cos\theta}\bar{s}_\pm[(m+\mu)^2]. \qquad (3.13)$$

In the third integral, if

$$4\mu^2>2m^2(1-\cos\theta),$$

then the cut runs along

$$-\infty\leqslant s\leqslant \mathcal{T}_-(\cos\theta),$$

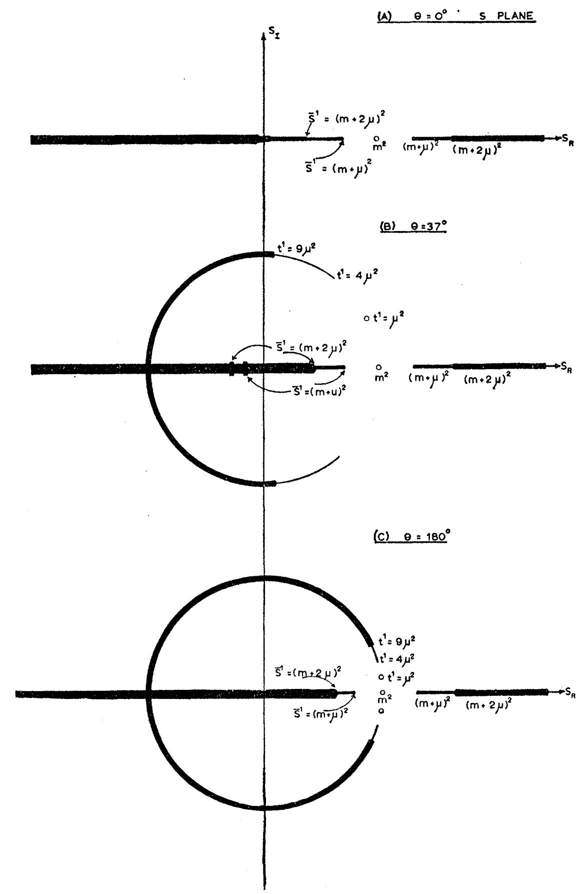

FIG. 2. The singularities of the scattering amplitudes plotted in the s plane for various values of θ.

[13] There appear in (3.11) a number of spurious singularities. These have arisen purely from the partial fraction decomposition and always combine to give finite results.

and

$$\mathcal{T}_+(\cos\theta) \leqslant s \leqslant 0,$$

with

$$\mathcal{T}_\pm(\cos\theta) = \frac{1}{1-\cos\theta} t_\pm(4\mu^2), \tag{3.14}$$

but if $4\mu^2 < 2m^2(1-\cos\theta)$, then the contour lies along

$$-\infty \leqslant s \leqslant 0$$

and along the circle $|s| = m^2$ up to the points

$$\begin{aligned} \mathrm{Re}(s) &= m^2 - 4\mu^2/(1-\cos\theta), \\ \mathrm{Im}(s) &= \pm[m^4 - (\mathrm{Re}(s))^2]^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

It is seen that the effect of the two-pion exchange on the circle increases towards backward angles. However the pion-pion interaction also makes itself felt in the other cuts through its effect on meson photoproduction, so that if we were to utilize for A_i^{I} its analytical expression as derived from the theory of meson photoproduction[14] it would contain parameters describing the $J=1$, $T=1$ π-π interaction and possible $J=1$, $T=0$ 3π effects. Since we already have to contend, on the circle cut, with parameters describing a $T=0$ π-π interaction the total number of parameters in the theory would be too large to allow a reasonable comparison with the present experimental situation.

We shall therefore assume in the following that the A_i^{I} can be obtained from experiment,[15] so that the π-π interaction appears *explicitly* only on the circle cut.

If we look at the second integral of Eq. (3.11) we see that we need to know $A_i^{\mathrm{I}}(\bar{s}',t)$ for values $\bar{s}' \geqslant (m+\mu)^2$ and $t = t[\bar{s}_\pm(\bar{s}')/(1+\cos\theta)]$. Examination shows that for the (+) sign the values of t, for all $\bar{s}'$ required, correspond to physical values of the scattering angle in channel II, so that a partial wave expansion is permissible. The (−) values on the other hand correspond to unphysical channel II scattering. In any case, the (−) values contribute only to the left of the origin in the s plane, and may be disregarded in terms of the "nearest singularity" philosophy.

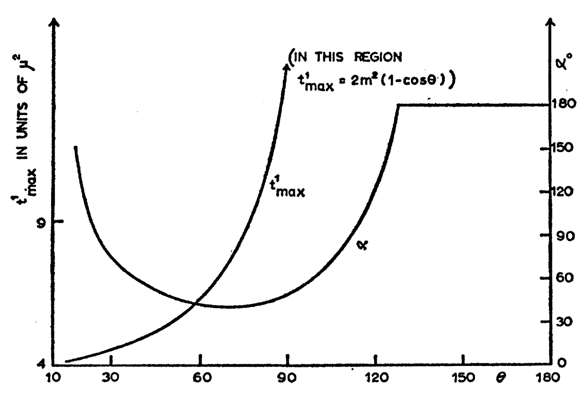

FIG. 3. Curves showing the range of validity of the Legendre polynomial expansion of the $N\bar{N} \rightarrow 2\gamma$ amplitude, for various θ. The maximum values of $\alpha = -i\log(s/m^2)$ for which the expansion converges, and their corresponding t' value are shown.

[14] M. Gourdin, D. Lurié, and A. Martin, CERN (to be published). J. S. Ball, Phys. Rev. 124, 2014 (1961).

[15] This is discussed fully in Sec. IV.

Similarly, for the third integral of (3.11) one requires $A_i^{\mathrm{III}}(s,t')$ for values $t' \geqslant 4\mu^2$ and $s = t_\pm(t')/(1-\cos\theta)$. Let us describe points lying on the circle cut by $s = m^2e^{i\alpha}$. Then Fig. 3 shows the maximum values of α, and its corresponding t' value, for which the Legendre expansion channel III converges, for various θ. The fact that the region of convergence is extremely small for low values of θ is not very serious because, as discussed earlier, the distance of the 2π cut from the physical region increases with decreasing θ so that our inability to feed in the 2π effects should not be important at these angles.

Also even where the Legendre expansion does converge for fairly large values of t' we can in any case only handle the 2π effects accurately up to $t = 9\mu^2$, the 3π threshold. We hope therefore that the restrictions resulting from the Legendre expansion convergence requirements do not seriously weaken the accuracy with which we are able to estimate the circle-cut effects.

In the next few sections we evaluate explicit expressions for the A_i^{I} and A_i^{III} by means of which the $A_i(s, \cos\theta)$ can be calculated in (3.11).

IV. THE UNITARITY CONDITION

(1) General

The usual condition for the unitarity of the S matrix,

$$S^\dagger S = I, \tag{4.1}$$

when written in terms of the R operator leads to the relation

$$i\langle\alpha|R^\dagger - R|\beta\rangle = \textstyle\sum_\nu \langle\alpha|R^\dagger|\nu\rangle\langle\nu|R|\beta\rangle, \tag{4.2}$$

where the sum is over all permissible physical states ν having the same energy-momentum as α or β.

Since the invariants $F_{\mu\nu}^{(i)}$ are self-adjoint, Eq. (4.2) with the help of Eq. (2.9) may be written as

$$\sum_{i=1}^{6} \mathrm{Im}A_i(s,t,\bar{s})\langle\alpha|F_{\mu\nu}^{(i)}|\beta\rangle = \frac{4\pi^2 E_B E_F}{m} \sum_\nu \langle\alpha|R^\dagger|\nu\rangle\langle\nu|R|\beta\rangle, \tag{4.3}$$

where E_B, E_F are the c.m. energies of the photon and nucleon in the states α or β.

(2) Channel I

For this channel we expand the right-hand side of (4.3) in terms of the mass exchange and the electromagnetic coupling keeping as intermediate states only the one-nucleon (Born) term and the pion-nucleon term, so that

$$\sum_i A_i^{\rm I}(s,t,\bar{s})\langle\gamma_2N_2|F_{\mu\nu}{}^{(i)}|\gamma_1N_1\rangle \propto\sum\langle\gamma_2N_2|R^\dagger|N\rangle\langle N|R|\gamma_1N_1\rangle +\sum\langle\gamma_2N_2|R^\dagger|\pi N\rangle\langle\pi N|R|\gamma_1N_1\rangle+\cdots. \quad (4.4)$$

By expressing the right-hand side of (4.4) in terms of the invariants $F_{\mu\nu}{}^{(i)}$ we are able to identify the contributions to the $A_i^{\rm I}(s,t,\bar{s})$.

Let us consider first the Born terms. The matrix element $\langle N|R|\gamma_1N_1\rangle$ is essentially the photon-nucleon vertex and can be written

$$\langle p''|R|p_1k_1\rangle=-\frac{m}{2(\pi p_0''p_{10}k_{10})^{\frac{1}{2}}}\bar{u}(p'') \times[F_1(\tilde{q}^2)\gamma_\nu-F_2(\tilde{q}^2)\sigma_{\nu\mu}\tilde{q}_\mu]u(p_1)\epsilon_{1\nu}, \quad (4.5)$$

where F_1 and F_2 are the usual nucleon form factors normalized so that

$$F_1\equiv F_1(0)=\frac{e}{2}(I+\tau_3)$$

$$F_2\equiv F_2(0)=\frac{\mu_p'+\mu_n}{2}I+\frac{\mu_p'-\mu_n}{2}\tau_3, \quad (4.6)$$

with μ_p', μ_n the anomalous proton and neutron magnetic moments, and where $\tilde{q}=p''-p_1$ is the momentum transfer in the intermediate channel.

Substituting Eq. (4.5) into Eq. (4.4), carrying out the sum over intermediate spin states and the integration over the momentum of the intermediate state, and expanding into the invariants $F_{\mu\nu}{}^{(i)}$, we obtain for the Born terms

$$A_i^{\rm IB}=-\pi\beta_i(m/W)\delta(s-m^2), \quad (4.7)$$

with

$$\beta_1=2mF_1^2,\qquad \beta_2=0,$$
$$\beta_3=mF_1(F_1+2mF_2),\qquad \beta_4=-F_1^2,$$
$$\beta_5=(F_1+2mF_2)^2,\qquad \beta_6=-F_1(F_1+2mF_2).$$

It follows then that the R_i of (3.3) are given by

$$R_i=\beta_i.$$

In terms of e and μ, these are:

$$\begin{aligned} R_1&=me^2(I+\tau_3),\\ R_2&=0,\\ R_3&=m^2e\mu_p(I+\tau_3),\\ R_4&=-\tfrac{1}{2}e^2(I+\tau_3),\\ R_5&=2m^2[\mu_p^2(I+\tau_3)+\mu_n^2(I-\tau_3)],\\ R_6&=-em\mu_p(I+\tau_3),\end{aligned} \quad (4.8)$$

where $\mu_p=\mu_p'+e/2m$ is the full proton magnetic moment.

However, as was mentioned earlier, the vanishing of the photon mass causes certain unusual features to appear in the Born terms. So long as we consider $A_i(s,t)$ as a function of the *independent* variables s and t, with $t\neq0$, then all the above is valid and there are poles in the $A_i(s,t)$ at

$$s=m^2 \quad\text{and}\quad \bar{s}=m^2,\quad \text{i.e.,}\quad s=m^2-t. \quad (4.9)$$

For $t=0$, however, only A_4 and A_5 have poles[16] at $s=m^2$ while the amplitudes with even crossing symmetry, $A_{1,2,3,6}$ are finite there.

On the other hand, if one is interested in the A_i as functions of s and $\cos\theta$, then for *all* finite $\cos\theta$, $s\to m^2$ implies $\bar{s}\to m^2$ so that for $A_{1,2,3,6}$ the poles at $s=m^2$ and $\bar{s}=m^2$ cancel out.

It follows then that the Channels I and II contribution to P_i of (3.11) is given by

$$P_i=\frac{(\cos\theta-1)R_i}{2m^2+(1+\cos\theta)(s-m^2)} \quad \text{for}\quad i=1,2,3,6,$$

and

$$P_i=R_i\left[\frac{2}{s-m^2}+\frac{(1-\cos\theta)}{2m^2+(1+\cos\theta)(s-m^2)}\right] \quad \text{for}\quad i=4,5. \quad (4.10)$$

It is interesting to note that if one uses the reduction formula to calculate $\langle\gamma_2N_2|R^\dagger-R|\gamma_1N_1\rangle$, then the cancellation of the poles at $s=m^2$ occurs automatically in the unitarity expression, for even-crossing amplitudes. For one has then that

$$\langle p_2k_2|R^\dagger-R|p_1k_1\rangle =\sum_n\langle p_2k_2|R^\dagger|p''\rangle\langle p''|R|p_1k_1\rangle\delta^4(p''-p_1-k_1) -\sum_n\langle p_2,-k_1|R^\dagger|p''\rangle\langle p''|R|p_1,-k_2\rangle\delta^4(p''-p_1+k_2). \quad (4.11)$$

The second term, which arises from the commutator of the current operators, is usually zero in the region where the first term contributes, since the regions in which the arguments of the two δ functions vanish, are mutually exclusive. The vanishing of the photon mass, however, causes these two regions to touch at $s=m^2$, for finite $\cos\theta$, and if one then calculates the Born term contribution to the $\mathrm{Im}A_i(s,\cos\theta)$ from (4.11) one is left with just the δ function of $2m^2+(1+\cos\theta)(s-m^2)$ for $i=1, 2, 3, 6$, and (4.10) follows immediately.

In order to calculate the second term of (4.4) we have to know the matrix elements for pion photoproduction.

The intermediate channel process $\gamma_1+N_1\to\pi+N$ may be described in terms of four independent helicity

[16] The invariants $F^{(4)}$ and $F^{(5)}$ vanish at this point, and so the quantities $A_4F^{(4)}$ and $A_5F^{(5)}$ remain bounded.

amplitudes, namely

$$\psi_1=\langle\tfrac{1}{2}|\psi|\tfrac{1}{2}1\rangle=\frac{1}{2(pq)^{\frac{1}{2}}}\sum_J(2J+1)\psi_{-\frac{1}{2},\frac{1}{2}}{}^J d_{-\frac{1}{2}\frac{1}{2}}{}^J(\theta),$$

$$\psi_2=\langle-\tfrac{1}{2}|\psi|\tfrac{1}{2}1\rangle=\frac{1}{2(pq)^{\frac{1}{2}}}\sum_J(2J+1)\psi_{-\frac{1}{2},-\frac{1}{2}}{}^J d_{-\frac{1}{2}-\frac{1}{2}}{}^J(\theta),$$

$$\psi_3=\langle\tfrac{1}{2}|\psi|-\tfrac{1}{2}1\rangle=\frac{1}{2(pq)^{\frac{1}{2}}}\sum_J(2J+1)\psi_{-\frac{3}{2},\frac{1}{2}}{}^J d_{-\frac{3}{2}\frac{1}{2}}{}^J(\theta),$$

$$\psi_4=\langle-\tfrac{1}{2}|\psi|-\tfrac{1}{2}1\rangle=\frac{1}{2(pq)^{\frac{1}{2}}}\sum_J(2J+1)\psi_{-\frac{3}{2},-\frac{1}{2}}{}^J d_{-\frac{3}{2}-\frac{1}{2}}{}^J(\theta),\quad(4.12)$$

where q is the magnitude of the intermediate channel barycentric 3-momentum, and the scattering amplitude $\langle\pi N|\psi|\gamma N\rangle$ is related to the cross section for photoproduction by

$$d\sigma/d\Omega=(q/p)\sum|\langle\pi N|\psi|\gamma N\rangle|^2.\quad(4.13)$$

The evaluation of the unitarity equation for the two-particle intermediate state is simplified immensely by the fact that only the low partial waves from the intermediate channel process are large for low energies. If we therefore expand both sides of the unitarity equation in terms of the relevant helicity subamplitudes rather than in terms of the invariant amplitudes $F_{\mu\nu}{}^{(i)}$, the orthogonality properties of the d functions may be used to integrate the equation immediately, yielding

$$\text{Im}\Phi_{\lambda\lambda'}{}^J=\tfrac{1}{2}\sum_{\lambda_N=\pm\frac{1}{2}}\psi_{\lambda\lambda_N}{}^J\psi_{\lambda'\lambda_N}{}^{J*}.\quad(4.14)$$

If we now retain only the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ partial waves, which should be sufficient for energies up to 300 Mev at least, then the imaginary part of each Φ_i may be written in the form:

$$\text{Im}\Phi_i=\frac{1}{2p}\left[(\psi_{\lambda,\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\lambda',\frac{1}{2}}{}^{\frac{1}{2}*}+\psi_{\lambda,-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\lambda',-\frac{1}{2}}{}^{\frac{1}{2}*})d_{\lambda\lambda'}{}^{\frac{1}{2}}(\theta)+2(\psi_{\lambda,\frac{1}{2}}{}^{\frac{3}{2}}\psi_{\lambda',\frac{1}{2}}{}^{\frac{3}{2}*}+\psi_{\lambda,-\frac{1}{2}}{}^{\frac{3}{2}}\psi_{\lambda',-\frac{1}{2}}{}^{\frac{3}{2}*})d_{\lambda\lambda'}{}^{\frac{3}{2}}(\theta)\right].\quad(4.15)$$

So far, we have neglected the isotopic spin dependence of the ψ_i. The analysis of Watson[17] shows that an amplitude for photoproduction may be written in the form:

$$\psi_{\lambda\lambda_N}{}^J=\psi_{\lambda\lambda_N}{}^{J(+)}\mathcal{G}_\beta{}^{(+)}+\psi_{\lambda\lambda_N}{}^{J(-)}\mathcal{G}_\beta{}^{(-)}+\psi_{\lambda\lambda_N}{}^{J(0)}\mathcal{G}_\beta{}^{(0)},\quad(4.16)$$

with β the isotopic spin index of the pion, and

$$\mathcal{G}_\beta{}^{(+)}=\delta_{\beta3},\quad \mathcal{G}_\beta{}^{(-)}=\tfrac{1}{2}[\tau_\beta,\tau_3],\quad \mathcal{G}_\beta{}^{(0)}=\tau_\beta.$$

Hence

$$\psi_{\lambda\lambda_N}{}^J\psi_{\lambda'\lambda_N}{}^{J*}=(\psi_{\lambda\lambda_N}{}^{J(+)}\psi_{\lambda'\lambda_N}{}^{J(+)*}+2\psi_{\lambda\lambda_N}{}^{J(-)}\psi_{\lambda'\lambda_N}{}^{J(-)*}+3\psi_{\lambda\lambda_N}{}^{J(0)}\psi_{\lambda'\lambda_N}{}^{J(0)*})I+(\psi_{\lambda\lambda_N}{}^{J(+)}\psi_{\lambda'\lambda_N}{}^{J(0)*}+\psi_{\lambda\lambda_N}{}^{J(0)}\psi_{\lambda'\lambda_N}{}^{J(+)*}-2\psi_{\lambda\lambda_N}{}^{J(-)}\psi_{\lambda'\lambda_N}{}^{J(0)*}-2\psi_{\lambda\lambda_N}{}^{J(0)}\psi_{\lambda'\lambda_N}{}^{J(-)*})\tau_3.\quad(4.17)$$

Finally we can relate the helicity partial wave amplitudes to the multipole amplitudes $E_{l\pm}$, $M_{l\pm}$ defined by Chew, Goldberger, Low, and Nambu.[18] A straightforward comparison reveals that

$$\begin{aligned}\psi_{-\frac{1}{2},\frac{1}{2}}{}^J&=(pq/2)^{\frac{1}{2}}[l(M_{l+}-E_{(l+1)-})+(l+2)(E_{l+}+M_{(l+1)-})],\\ \psi_{-\frac{1}{2},-\frac{1}{2}}{}^J&=(pq/2)^{\frac{1}{2}}[(l+2)(E_{l+}-M_{(l+1)-})+l(M_{l+}+E_{(l+1)-})],\\ \psi_{-\frac{3}{2},\frac{1}{2}}{}^J&=(pq/2)^{\frac{1}{2}}[l(l+2)]^{\frac{1}{2}}[-E_{l+}+M_{l+}-E_{(l+1)-}-M_{(l+1)-}],\\ \psi_{-\frac{3}{2},-\frac{1}{2}}{}^J&=(pq/2)^{\frac{1}{2}}[l(l+2)]^{\frac{1}{2}}[E_{l+}-M_{l+}-E_{(l+1)-}-M_{(l+1)-}],\end{aligned}\quad(4.18)$$

where $J=l+\frac{1}{2}$.

Our program of calculation in this channel is now straightforward. For suitable values of the multipole amplitudes we use (4.18) and (4.17) to compute the values of the $\psi_i{}^J$, and then by (4.15) the values of $\text{Im}\Phi_i$. By Eq. (2.22) we now find $A_i{}^{\text{I}}$ which can be substituted into the first and second integrals of Eq. (3.11).

(3) Channel III

In this channel we are dealing with the process of nucleon antinucleon annihilation into two photons, and on the right-hand side of Eq. (4.3) we keep only the one- and two-pion intermediate states, so that

$$\sum_{i=1}^{6}A_i{}^{\text{III}}(s,t,\bar{s})\langle\gamma_1\gamma_2|F_{\mu\nu}{}^{(i)}|N_1\bar{N}_2\rangle\propto\sum\langle\gamma_1\gamma_2|R^+|\pi\rangle\langle\pi|R|N_1\bar{N}_2\rangle+\sum\langle\gamma_1\gamma_2|R^\dagger|\pi_1\pi_2\rangle\langle\pi_1\pi_2|R|N_1\bar{N}_2\rangle+\cdots,\quad(4.19)$$

where again the right-hand side must be expanded into the invariants $F_{\mu\nu}{}^{(i)}$ so as to identify the contributions to the $A_i{}^{\text{III}}$.

Consider first the one-pion terms. This depends on the pion-nucleon vertex $\langle\pi|R|N_1\bar{N}_2\rangle$, and the matrix element $\langle\gamma_1\gamma_2|R^\dagger|\pi\rangle$ for $\pi^0\to2\gamma$ decay.

For these we may write

$$\langle\pi(q)|R|N_1\bar{N}_2\rangle=\frac{-img(t)}{2\pi^{\frac{1}{2}}(p_{10}p_{20}q_0)^{\frac{1}{2}}}\bar{v}(p_2)\gamma_5u(p_1),\quad(4.20)$$

with $g^2(\mu^2)/4\pi=g^2/4\pi\sim14$, and

$$\langle\gamma_1\gamma_2|R^\dagger|\pi(q)\rangle=\frac{F(t)}{8\pi^{\frac{1}{2}}(k_{10}k_{20}q_0)^{\frac{1}{2}}}\epsilon_{1\nu}{}^\dagger\epsilon_{2\mu}{}^\dagger\epsilon_{\mu\nu\rho\sigma}(k_1+k_2)_\rho(k_1-k_2)_\sigma,\quad(4.21)$$

[17] K. M. Watson, Phys. Rev. **95**, 228 (1954).

[18] G. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

where $F(t)$ is the function introduced by Goldberger and Treiman[19] in their study of neutral pion decay. It is normalized so that

$$F(\mu^2)\equiv F=-8(\pi/\mu_0^3\tau)^{\frac{1}{2}}, \qquad (4.22)$$

where τ is the $\pi^0\rightarrow 2\gamma$ decay lifetime.

Putting (4.20) and (4.21) into (4.19) and carrying out the integration over the intermediate state momentum, one obtains for the one-pion contributions to A_i^{III}:

$$A_i^{\text{III}B}=0 \quad \text{for} \quad i\neq 3,$$
$$A_3^{\text{III}B}=-\frac{\pi\mu_0^2 qF}{2}\delta(\mu_0^2-t). \qquad (4.23)$$

From (4.22) and (4.23) one sees that the r_i of Eq. (3.3) are given by (including isotopic spin)

$$r_i=0 \quad \text{for} \quad i\neq 3,$$
$$r_3=-4g(\pi\mu_0/\tau)^{\frac{1}{2}}\tau_3. \qquad (4.24)$$

There has recently been some controversy[2,11] about the relative sign of the one-π and one-nucleon Born terms. Our determination, Eqs. (4.24), (5.6), (4.8), (5.5), and (2.17) shows that the inclusion of the one-π term tends to *increase* the differential cross section in channel I in agreement with Lapidus and Chou Kuang-Chao.[20]

However, our Born terms[21] differ from the usually quoted results in that they contain as numerators only *residues* of functions taken at the pole. The difference depends purely on the question of the subtractions required in the dispersion relation and is discussed under the section on the low-energy limit.

Let us now evaluate the two-π contribution to (4.19). We shall work in the c.m. of channel III, in which the pions have momentum $\mathbf{q}$, $-\mathbf{q}$, and energy $\omega=(\mathbf{q}^2+\mu^2)^{\frac{1}{2}}$. The kinematics of the photons and nucleons are given in (2.4).

The sum over intermediate states in (4.19) can be simplified to yield

$$\Sigma\langle\gamma_1\gamma_2|R^\dagger|2\pi\rangle\langle 2\pi|R|N\bar{N}\rangle$$
$$=\tfrac{1}{8}[t(t-4\mu^2)]^{\frac{1}{2}}\int\langle q|R|\gamma_1\gamma_2\rangle^*\langle q|R|N\bar{N}\rangle d\Omega_{\mathbf{q}}, \qquad (4.25)$$

where the integration is over the solid angle of $\mathbf{q}$. An alternative expression is obtained if we substitute for each of the matrix elements their expansions into helicity amplitudes and carry out the angular integration as was done in channel I. We obtain then

$$\sum\langle\gamma_1\gamma_2|R^\dagger|2\pi\rangle\langle 2\pi|R|N\bar{N}\rangle$$
$$=\frac{2}{\pi t^{\frac{1}{2}}}\frac{1}{(k\kappa)^{\frac{1}{2}}}\sum_{J=0}(J+\tfrac{1}{2})R_{\lambda_N\lambda_{\bar{N}}}{}^J R_{\lambda_1\lambda_2}{}^{J*}d_{\lambda\mu}{}^J(\psi), \qquad (4.26)$$

with $\lambda=\lambda_N-\lambda_{\bar{N}}$ and $\mu=\lambda_1-\lambda_2$. The amplitudes $R_{\lambda_N\lambda_{\bar{N}}}{}^J$, $R_{\lambda_1\lambda_2}{}^J$ are the helicity amplitudes,[12] i.e., the matrix elements in a representation in which the energy, total angular momentum, and helicity are diagonal, for the processes

$$N_{\lambda_N}+\bar{N}_{\lambda_{\bar{N}}}\rightarrow\pi(\mathbf{q})+\pi(-\mathbf{q}),$$

and

$$\gamma_{\lambda_1}+\gamma_{\lambda_2}\rightarrow\pi(\mathbf{q})+\pi(-\mathbf{q}),$$

respectively.

In practice it turns out most useful to utilize a combination of Eqs. (4.25) and (4.26). The reason for this is that for the processes involved, e.g., $N\bar{N}\rightarrow 2\pi$ only the lowest few partial waves may be large, suggesting the usefulness of (4.26). On the other hand, there are usually terms, e.g., the Born terms in $N\bar{N}\rightarrow 2\pi$ (and in general the left-hand cut of the process considered). which contribute small amounts to a very large number of waves, and these are most simply handled by (4.25).

Let us assume therefore that we can split R into two parts

$$R=R_l+\mathcal{R}, \qquad (4.27)$$

in which R_l contributes only to the largest waves, i.e., those for which the right-hand cut is essential to satisfy unitarity. Let us suppose that the highest wave which need be included in R_l is $J=J_{\max}$. We have then

$$\sum_{J=0}(J+\tfrac{1}{2})R_{\lambda_N\lambda_{\bar{N}}}{}^J R_{\lambda_1\lambda_2}{}^{J*}d_{\lambda\mu}{}^J(\psi)$$
$$\simeq\sum_{J=0}^{J_{\max}}(J+\tfrac{1}{2})R_{\lambda_N\lambda_{\bar{N}}}{}^J R_{\lambda_1\lambda_2}{}^{J*}d_{\lambda\mu}{}^J(\psi)$$
$$+\sum_{J=J_{\max}}(J+\tfrac{1}{2})\mathcal{R}_{\lambda_N\lambda_{\bar{N}}}{}^J\mathcal{R}_{\lambda_1\lambda_2}{}^{J*}d_{\lambda\mu}{}^J(\psi)$$
$$=\sum_{J=0}^{J_{\max}}(J+\tfrac{1}{2})(R_{\lambda_N\lambda_{\bar{N}}}{}^J R_{\lambda_1\lambda_2}{}^{J*}-\mathcal{R}_{\lambda_N\lambda_{\bar{N}}}{}^J\mathcal{R}_{\lambda_1\lambda_2}{}^{J*})d_{\lambda\mu}{}^J(\psi)$$
$$+\frac{\pi t(k\kappa)^{\frac{1}{2}}(t-4\mu^2)^{\frac{1}{2}}}{16}\int\langle\mathbf{q}|\mathcal{R}|\gamma_1\gamma_2\rangle^*$$
$$\times\langle\mathbf{q}|\mathcal{R}|N\bar{N}\rangle d\Omega_{\mathbf{q}}. \qquad (4.28)$$

Since not enough is known about the $N\bar{N}\rightarrow 2\pi$ and $\gamma+\gamma\rightarrow 2\pi$ processes to determine $J_{\max}$, we shall tentatively assume that only the s wave is large (the p wave is forbidden), though d and higher waves can easily be incorporated.

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[19] M. L. Goldberger and S. B. Treiman, Nuovo cimento **9**, 451 (1958).

[20] This is, of course, assuming that the addition of the dispersion integral estimates to the channel I and II Born terms does not change their over-all sign. This has been verified by previous calculations.[2,11] We are, of course, also assuming that the Goldberger-Treiman determination of the sign gF is correct.

[21] This applies also to the one-nucleon Born terms of channel I and II.

With this assumption, then, we have finally

$$\sum \langle\gamma_1\gamma_2|R^\dagger|2\pi\rangle\langle 2\pi|R|N\bar{N}\rangle$$
$$=\frac{1}{\pi t^{\frac{1}{2}}(k\kappa)^{\frac{1}{2}}}(R_{\lambda_N\lambda_{\bar{N}}}{}^0 R_{\lambda_1\lambda_2}{}^{0*}-\mathcal{R}_{\lambda_N\lambda_{\bar{N}}}{}^0\mathcal{R}_{\lambda_1\lambda_2}{}^{0*})d_{\lambda\mu}{}^0(\psi)$$
$$+\tfrac{1}{8}[t(t-4\mu^2)]^{\frac{1}{2}}\int\langle\mathbf{q}|\mathcal{R}|\gamma_1\gamma_2\rangle^*\times\langle\mathbf{q}|\mathcal{R}|N\bar{N}\rangle d\Omega_{\mathbf{q}}. \quad (4.29)$$

We must now introduce some information about the $N\bar{N}\rightarrow 2\pi$ and $\gamma+\gamma\rightarrow 2\pi$ matrix elements.

In our normalization we have for the first process

$$\langle 2\pi|R|N\bar{N}\rangle=(-m/8\pi^2k^2)\langle 2\pi|\tau|N\bar{N}\rangle, \quad (4.30)$$

with

$$\langle\pi_\alpha(-\mathbf{q})\pi_\beta(\mathbf{q})|\tau|\bar{N}(-\boldsymbol{\kappa})N(\boldsymbol{\kappa})\rangle =\bar{v}_{\lambda_{\bar{N}}}(-\boldsymbol{\kappa})(-A_{\alpha\beta}-i\gamma\cdot\mathbf{q}B_{\alpha\beta})u_{\lambda_N}(\boldsymbol{\kappa}). \quad (4.31)$$

The functions A, B are the usual pion-nucleon scalar amplitudes[22] and are functions of the scalars

$$t=4k^2=4(\kappa^2+m^2),\quad s_\kappa=-(\boldsymbol{\kappa}-\mathbf{q})^2,\quad \bar{s}_\kappa=-(\boldsymbol{\kappa}+\mathbf{q})^2, \quad (4.32)$$

which are, respectively, the squares of the c.m. energy and momentum transfers in the $N\bar{N}\rightarrow 2\pi$ process.

The isotopic spin decomposition is

$$A_{\alpha\beta}=A^{(+)}\delta_{\alpha\beta}+\tfrac{1}{2}A^{(-)}[\tau_\alpha,\tau_\beta]. \quad (4.33)$$

For the process $\gamma_1+\gamma_2\rightarrow 2\pi$ we refer to the work of Martin and Gourdin.[23]

The transition amplitude may be written

$$\langle\pi_\alpha(-\mathbf{q})\pi_\beta(\mathbf{q})|R|\gamma_1(-\mathbf{k})\gamma_2(\mathbf{k})\rangle =\frac{1}{48\pi^2k^2}(D_aI_a+D_bI_b)_{\alpha\beta}. \quad (4.34)$$

D_a and D_b are scalar functions of the variables t and

$$s_1=-(\mathbf{k}+\mathbf{q})^2,\quad s_2=-(\mathbf{k}-\mathbf{q})^2, \quad (4.35)$$

and the invariants I_a, I_b contain the spin dependence of the amplitude and are given by

$$I_a=\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2,\quad I_b=-8k^2\boldsymbol{\varepsilon}_1\cdot\mathbf{q}\boldsymbol{\varepsilon}_2\cdot\mathbf{q}. \quad (4.36)$$

The isospin decomposition of the D_a, D_b is

$$D_{\alpha\beta}=2D^{(0)}\delta_{\alpha\beta}+D^{(2)}(\delta_{\alpha\beta}-3\delta_{\alpha 3}\delta_{\beta 3}),$$

where $D_{a,b}{}^{(0)}$, $D_{a,b}{}^{(2)}$ are the functions introduced in reference (23).

It is shown there that the two pions may only be in a $T=0$ or $T=2$ isotopic spin state. Since the nucleon-antinucleon pair can only have $T=0$ or 1 and isospin is conserved in the $N\bar{N}\rightarrow 2\pi$ process, it follows that the process

$$2\gamma\rightarrow 2\pi\rightarrow N\bar{N}$$

takes place *purely through the $T=0$ channel.*

We shall therefore require only the $T=0$ amplitudes $A^{(+)}$, $B^{(+)}$, $D_a{}^{(0)}$, $D_b{}^{(0)}$.

Carrying out the isospin sum implied in (4.29) we get, finally,

$$\sum\langle\gamma_1\gamma_2|R^\dagger|2\pi\rangle\langle 2\pi|R|N\bar{N}\rangle$$
$$=\frac{\sqrt{3}}{\pi t^{\frac{1}{2}}(k\kappa)^{\frac{1}{2}}}[R_{\lambda_N\lambda_{\bar{N}}}{}^{0(+)}R_{\lambda_1\lambda_2}{}^{0(0)*}-\mathcal{R}_{\lambda_N\lambda_{\bar{N}}}{}^{0(+)}\mathcal{R}_{\lambda_1\lambda_2}{}^{0(0*)}]d_{\lambda\mu}{}^0(\psi)$$
$$+\frac{\sqrt{3}[t(t-4\mu^2)]^{\frac{1}{2}}}{8}\int\langle\mathbf{q}|\mathcal{R}^{(0)}|\gamma_1\gamma_2\rangle^*\times\langle\mathbf{q}|\mathcal{R}^{(+)}|N\bar{N}\rangle d\Omega_{\mathbf{q}}, \quad (4.37)$$

and the contribution is only to the isoscalar part of the A_i.

For the functions $A^{(+)}$, $B^{(+)}$ we use the representations[24]

$$A^{(+)}(s_\kappa,\bar{s}_\kappa,t)=\frac{1}{\pi}\int_{(m+\mu)^2}\sigma_A{}^{(+)}(s',t)\left(\frac{1}{s'-s_\kappa}+\frac{1}{s'-\bar{s}_\kappa}\right)ds'+\frac{1}{\pi}\int_{4\mu^2}\frac{v_A{}^{(+)}(t',s_\kappa-\bar{s}_\kappa)}{t'-t}dt',$$
$$B^{(+)}(s_\kappa,\bar{s}_\kappa,t)=\frac{1}{\pi}\int_0\bar{\sigma}_B{}^{(+)}(s',t)\left(\frac{1}{s'-s_\kappa}-\frac{1}{s'-\bar{s}_\kappa}\right)ds'+\frac{1}{\pi}\int_{4\mu^2}\frac{v_B{}^{(+)}(t',s_\kappa-\bar{s}_\kappa)}{t'-t}dt', \quad (4.38)$$

where

$$\bar{\sigma}_B{}^{(+)}(s',t)=\pi g^2\delta(s'-m^2)+\sigma_B{}^{(+)}(s',t).$$

The representation (4.38) contains pole terms, and the cuts in s_κ and $\bar{s}_\kappa$ incorporating the πN rescattering corrections as discussed by Chew, Goldberger, Low, and Nambu.[22] These are dominated by the 3,3 resonance. The cuts in t have been considered by Bowcock, Cottingham, and Lurié.[24] They show that the dependence of $v_{A,B}$ on s_κ, $\bar{s}_\kappa$ is probably weak since the cuts in s_κ, $\bar{s}_\kappa$ of $v_{A,B}$ only begin at $(m+2\mu)^2$. In other words, for reasonable energies the last terms of (4.38) will con-

[22] G. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[23] M. Gourdin and A. Martin, Nuovo cimento **17**, 224 (1960).

[24] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento **16**, 918 (1960).

tribute appreciably only to the low waves of the $N\bar{N}\rightarrow 2\pi$ process.

In accordance with the scheme of (4.27) we therefore define $A=A_l+\mathcal{A}$, etc., with

$$\mathcal{A}^{(+)}(s_\kappa,\bar{s}_\kappa,t)=\frac{1}{\pi}\int_{(m+\mu)^2}\sigma_A^{(+)}(s',t)\left(\frac{1}{s'-s_\kappa}+\frac{1}{s'-\bar{s}_\kappa}\right)ds',$$
$$\mathcal{B}^{(+)}(s_\kappa,\bar{s}_\kappa,t)=\frac{1}{\pi}\int_0\bar{\sigma}_B^{(+)}(s',t)\left(\frac{1}{s'-s_\kappa}-\frac{1}{s'-\bar{s}_\kappa}\right)ds'. \quad (4.39)$$

For the functions $D_{a,b}^{T=0}$ we use the representations given in reference 23.

$$D_a^{(0)}(s_1,s_2,t)=2e^2+[2s_1t+(s_1-\mu^2)^2]\frac{1}{\pi}\int\frac{\sigma_T(x)dx}{(x-\mu^2)(x-s_1)}+(s_1\rightarrow s_2)+\frac{t}{\pi}\int_0\frac{F_0^s(q'^2)h_0^{*s}(q'^2)dq'^2}{(q'^2+\mu^2)[4(q'^2+\mu^2)-t]}, \quad (4.40)$$

$$D_b^{(0)}(s_1,s_2,t)=\frac{2e^2}{t}\frac{1}{\mu^2-s_1}+\frac{1}{\pi}\int_{4\mu^2}\frac{\sigma_T(x)dx}{(\mu^2-x)(x-s_1)}+(s_1\rightarrow s_2).$$

Besides the Born terms, Eq. (4.40) contains also the effect of two-pion intermediate states in the reactions

$$\gamma+\pi\rightarrow 2\pi\rightarrow\gamma+\pi,$$

and

$$\gamma+\gamma\rightarrow 2\pi\rightarrow 2\pi,$$

but cuts arising from three-pion and higher mass states are assumed to yield a negligible dependence on the variables concerned.

The function σ_T is just the total cross section for the process $\gamma+\pi\rightarrow 2\pi$, and the simplicity of the weight function arises from the assumption that this process occurs mainly in the $T=1$, $J=1$ state.

h_0^S is the s-wave, $T=0$, pion-pion scattering amplitude

$$h_0^S(q^2)=\exp(i\delta_0^S)\sin\delta_0^S, \quad (4.41)$$

and F_0^S is proportional to the $T=0$, s-wave amplitude for $\gamma+\gamma\rightarrow 2\pi$.

For a full discussion of the approximations involved in (4.40) the reader is referred to reference 22. For our purposes we now split $D_a^{(0)}$, $D_b^{(0)}$ according to (4.27) and define

$$\mathfrak{D}_a^{(0)}=[2s_1t+(s_1-\mu^2)^2]\times\frac{1}{\pi}\int_{4\mu^2}\frac{\sigma_T(x)dx}{(x-\mu^2)(x-s_1)}+(s_1\rightarrow s_2), \quad (4.42)$$
$$\mathfrak{D}_b^{(0)}=\frac{2e^2}{t}\frac{1}{\mu^2-s_1}+\frac{1}{\pi}\int\frac{\sigma_T(x)dx}{(\mu^2-x)(x-s_1)}+(s_1\rightarrow s_2).$$

We are now in a position to evaluate the integral in (4.37). Substituting (4.38, 4.30, 4.42, and 4.34) we have[25,26]

$$\int\langle\mathbf{q}|\mathcal{R}|\gamma_1\gamma_2\rangle^*\langle\mathbf{q}|\mathcal{R}|N\bar{N}\rangle=-\frac{m}{64\pi^4k^4}\bar{v}_{\lambda_N}(-\boldsymbol{\kappa})\int d\Omega_q(\mathfrak{D}_a^{0*}I_a^*+\mathfrak{D}_b^{0*}I_b^*)\times[-\mathcal{A}^{(+)}-i\gamma\cdot\mathbf{q}\mathcal{B}^{(+)}]u_{\lambda_N}(\boldsymbol{\kappa}). \quad (4.43)$$

It is a straightforward but tedious matter to develop the right-hand side into the invariants $F_{\mu\nu}^{(i)}$. Substituting (4.43) into (4.37) and (4.19), we may then identify those parts of the A_i^{III} arising from the $\mathcal{R}$ part of the matrix elements. Let us call these $A_i^{\text{III}\mathcal{R}}$.

We have then

$$\begin{aligned}A_1^{\text{III}\mathcal{R}}&=(\alpha_1+\alpha_2)N, & A_4^{\text{III}\mathcal{R}}&=(\alpha_3+\alpha_4)N,\\ A_2^{\text{III}\mathcal{R}}&=-\alpha_1N, & A_5^{\text{III}\mathcal{R}}&=-\alpha_3N,\\ A_3^{\text{III}\mathcal{R}}&=0, & A_6^{\text{III}\mathcal{R}}&=-\alpha_5N,\end{aligned} \quad (4.44)$$

where $N=-(t-4\mu^2)^{\frac{1}{2}}/16\pi^2t^{\frac{1}{2}}$ and the $\alpha_i(s,t)$ are complicated functions expressed as integrals over the $\sigma_T(x)$ and $\sigma_{A,B}(s')$ weight functions. These are given in the appendix.

Finally, we must calculate the expression in square brackets in (4.37).

For the $N\bar{N}\rightarrow 2\pi$ helicity amplitudes we have,[27] for $J=0$,

$$\mathcal{R}_{-+}^{0(+)}=0,$$

$$\mathcal{R}_{++}^{0(+)}=-\frac{1}{8\pi^2k}\left(\frac{q}{\kappa}\right)^{\frac{1}{2}}\left(\frac{\kappa}{q}\int_{(m+\mu)^2}\sigma_A^{\dagger}(s',t)Q_0(\beta')ds'+m\int_0\bar{\sigma}_B^{(+)}(s',t)Q_1(\beta')ds'\right), \quad (4.45)$$

with

$$\beta'=(s'+\kappa^2+q^2)/2\kappa q,$$

[25] Since we are below the threshold for the physical process $N\bar{N}\rightarrow 2\pi$, some care must be exercised in dealing with the operation of complex conjugation. See Eq. (3.19) of reference 26.

[26] D. Amati, E. Leader, and B. Vitale, Nuovo cimento **17**, 68 (1960).

[27] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

and Q_l are the Legendre functions of the second kind.[28] Also we have

$$R_{++}^{J(+)}=\frac{1}{k}(\kappa q)^J(q/\kappa)^{\frac{1}{2}}f_+^{J(+)},$$
$$R_{-+}^{J(+)}=(\kappa q)^J(q/\kappa)^{\frac{1}{2}}f_-^{J(+)}, \tag{4.46}$$

where the $f_{\pm}^{J(\pm)}$ are the Frazer-Fulco helicity amplitudes,[27] for the $N\bar{N}\rightarrow 2\pi$ process.

The helicity amplitudes for the $\gamma\gamma\rightarrow 2\pi$ process are obtained by saturating the expression (4.34) with different photon helicities and then inverting the helicity expansion of the left-hand side of (4.34). We obtain then for $J=0$, $T=0$,

$$\mathcal{R}_{1,-1}^{0(0)}=\mathcal{R}_{-1,1}^{0(0)}=0,$$
$$\mathcal{R}_{1,1}^{0(0)}=\mathcal{R}_{-1,-1}^{0(0)}$$
$$=-\frac{1}{2\sqrt{3}\pi}\left(\frac{k}{q}\right)^{\frac{1}{2}}\left\{\frac{2}{\pi}\int_{4\mu^2}\frac{\sigma_T(x)}{x-\mu^2}\times\left[\left(\mu^2-\frac{t}{2}+k\beta''(t-4\mu^2)^{\frac{1}{2}}\right)Q_0(\beta'')-k(t-4\mu^2)^{\frac{1}{2}}\right]dx\right.$$
$$\left.-\frac{e^2}{t}(t-4\mu^2)\left[\frac{k}{q}+\left(1-\frac{k^2}{q^2}\right)Q_0(k/q)\right]\right\}, \tag{4.47}$$

with

$$\beta''=(x+2k^2-\mu^2)/2kq.$$

The amplitudes $R_{\lambda_1\lambda_2}^{J(T)}$ for the $\gamma+\gamma\rightarrow 2\pi$ process are related to the partial wave amplitudes F_T^{2l} and f_T^{2l} introduced in reference (23).

$$R_{1,1}^{J(0)}=R_{-1,-1}^{J(0)}=-\frac{1}{\sqrt{3}4\pi(2J+1)}\left(\frac{q}{k}\right)^{\frac{1}{2}}F_0^J \quad (J=2l,\, l\geqslant 0),$$
$$R_{1,-1}^{J(0)}=R_{-1,1}^{J(0)}=-\frac{1}{\sqrt{3}24\pi}\left(\frac{q}{k}\right)^{\frac{1}{2}}q^2t[(J-1)J(J+1)(J+2)]^{\frac{1}{2}}f_0^J \quad (J=2l,\, l\geqslant 1). \tag{4.48}$$

Clearly for $J=0$ we require to know only F_0^0. This has been obtained by Martin and Gourdin[29] by solving an integral equation with certain assumptions about the reasonable behavior of the π-π s-wave phase shift $\delta_{T=0}^S$. The solution gives F_0^0 as a function of δ_0^S and σ_T. Eqs. (4.45), (4.46), (4.47), and (4.48) furnish us with the "square bracket" part of (4.37). It remains only to identify the contributions to the individual A_i.

Let us define

$$C_{\lambda_N\lambda_{\bar{N}}}^{\lambda_1\lambda_2}=[R_{\lambda_N\lambda_{\bar{N}}}^{0(+)}R_{\lambda_1\lambda_2}^{0(0)*}-\mathcal{R}_{\lambda_N\lambda_{\bar{N}}}^{0(+)}\mathcal{R}_{\lambda_1\lambda_2}^{0(0)*}]d_{\lambda\mu}^0(\psi). \tag{4.49}$$

By inverting the channel III helicity expansion (2.24) we obtain the contributions to A_i^{III} arising from the "square bracket" part of (4.37):

$$A_1^{\text{III}c}=A_2^{\text{III}c}=\frac{2\pi}{\kappa}\left(\frac{k}{\kappa}\right)^{\frac{1}{2}}\left[\frac{1}{2}(C_{++}^{11}+C_{++}^{-1\,-1})-\frac{4mk}{t}\cot\psi C_{-+}^{11}\right],$$
$$A_3^{\text{III}c}=\frac{\pi}{k}\left(\frac{k}{\kappa}\right)^{\frac{1}{2}}(C_{++}^{11}-C_{++}^{-1\,-1}), \tag{4.50}$$
$$A_4^{\text{III}c}=A_5^{\text{III}c}=\frac{-8\pi}{t\sin\psi}\left(\frac{k}{\kappa}\right)^{\frac{1}{2}}C_{-+}^{11},$$
$$A_6^{\text{III}c}=0.$$

This completes the evaluation of the two-pion exchange contributions to A_i^{III}, i.e., we have

$$A_i^{\text{III}}=[A_i^{\text{III}\mathcal{R}}+A_i^{\text{III}c}]I, \tag{4.51}$$

with $A_i^{\text{III}\mathcal{R}}$ and $A_i^{\text{III}c}$ given by (4.44) and (4.50) and I is the unit operator in the nucleon isospace [see (2.11)].

V. LOW-ENERGY LIMIT

It is well known[5] that the scattering of photons on spin-$\frac{1}{2}$ particles depends, to first order in the photon momentum, on the static charge and magnetic moment of the target. For scattering on nucleons the result may be written

$$F=\frac{F_1^2}{m}\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2-2\left(\frac{F_1}{2m}+F_2\right)^2 pi\boldsymbol{\sigma}\cdot[(\hat{p}\times\boldsymbol{\varepsilon}_2)\times(\hat{p}\times\boldsymbol{\varepsilon}_1)]$$
$$-\frac{F_1}{m}\left(\frac{F_1}{2m}+F_2\right)ip\left[\frac{\boldsymbol{\sigma}\cdot\hat{p}(\hat{p}\times\boldsymbol{\varepsilon}_1)\cdot\boldsymbol{\varepsilon}_2+\boldsymbol{\sigma}\cdot\hat{p}\times\boldsymbol{\varepsilon}_1\hat{p}\cdot\boldsymbol{\varepsilon}_2}{2}-\frac{\boldsymbol{\sigma}\cdot\hat{p}'(\hat{p}'\times\boldsymbol{\varepsilon}_2)\cdot\boldsymbol{\varepsilon}_1+\boldsymbol{\sigma}\cdot\hat{p}'\times\boldsymbol{\varepsilon}_2\hat{p}'\cdot\boldsymbol{\varepsilon}_1}{2}\right]$$
$$-\frac{F_1F_2}{m}pi\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\times\boldsymbol{\varepsilon}_2, \tag{5.1}$$

where F_1 and F_2 have been defined in (4.6).

After some rearrangement of the invariants in (5.1) it is possible to identify the contributions to the g_i of (2.18) and thence via the inverse of (2.19) to calculate the limiting values for the A_i.

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[28] P. M. Morse and H. Feshbach, *Methods of Mathematical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2.

[29] See Eq. (38) of reference 23. Note that F_0^0 is there written F_0^s.

We obtain then for the limit, at fixed angle, as $p \to 0$

$$A_1 \to \frac{F_1^2}{m}(\cos\theta - 1),$$

$$A_2 \to 4F_2(F_1 + mF_2),$$

$$A_3 \to F_1\left(\frac{F_1}{2m} + F_2\right)(\cos\theta - 1) - 2F_2(F_1 + mF_2),$$

$$A_4 \to -\frac{1}{P}\frac{F_1^2}{m}, \tag{5.2}$$

$$A_5 \to \frac{4m}{p}\left(\frac{F_1}{2m} + F_2\right)^2,$$

$$A_6 \to 2\left[F_2^2 - \frac{F_1}{2m}\left(\frac{F_1}{2m} + F_2\right)(\cos\theta - 1)\right].$$

Let us now see what the Born terms give us in this limit.

Remembering that as $p \to 0$

$$s - m^2 = O(p), \tag{5.3}$$

we see from Eqs. (4.10), (4.8), and (3.11) that the Born terms yield

$$A_1 \to \frac{F_1^2}{m}(\cos\theta - 1),$$

$$A_2 \to 0,$$

$$A_3 \to F_1\left(\frac{F_1}{2m} + F_2\right)(\cos\theta - 1),$$

$$A_4 \to -\frac{F_1^2}{mp}, \tag{5.4}$$

$$A_5 \to \frac{4m}{p}\left(\frac{F_1}{2m} + F_2\right)^2,$$

$$A_6 \to -\frac{F_1}{m}\left(\frac{F_1}{2m} + F_2\right)(\cos\theta - 1).$$

Comparing (5.4) and (5.2) we see that only in A_1, A_4, and A_5 is the correct low-energy limit guaranteed by the Born terms. These amplitudes will therefore not require subtractions.

For all the other amplitudes it will be necessary to perform subtractions in s in the fixed-angle dispersion relations.[30]

The final representation for the A_i will then be as follows:

(a) For $i=4$, 5 the representation is given by (3.11) with the P_i given by Eqs. (4.10) and (4.8).

(b) For $i=1$, 2, 3, 6,

$$A_i(s, \cos\theta) = S_i + \frac{s-m^2}{\pi}\int_{(m+\mu)^2} ds' \frac{A_i^{\mathrm{I}}(s', t(s'))}{(s'-m^2)(s'-s)} + \frac{2\eta_i}{\pi}(m^2 - s)(1+\cos\theta)\int_{(m+\mu)^2} ds' \frac{1}{\bar{s}_+(s') - \bar{s}_-(s')}$$
$$\times\left\{\frac{\bar{s}_+(s')A_i^{\mathrm{I}}(s', t(\bar{s}_+(s')/(1+\cos\theta)))}{[m^2(1+\cos\theta) - \bar{s}_+(s')][s(1+\cos\theta) - \bar{s}_+(s')]} - \frac{\bar{s}_-(s')A_i^{\mathrm{I}}(s', t(\bar{s}_-(s')/(1+\cos\theta)))}{[m^2(1+\cos\theta) - \bar{s}_-(s')][s(1+\cos\theta) - \bar{s}_-(s')]}\right\}$$
$$+ \frac{2}{\pi}(m^2 - s)(1-\cos\theta)\int_{4\mu^2} dt' \frac{1}{t_+(t') - t_-(t')}\left\{\frac{t_+(t')A_i^{\mathrm{III}}(t_+(t')/(1-\cos\theta), t')}{[m^2(1-\cos\theta) - t_+(t')][s(1-\cos\theta) - t_+(t')]}\right.$$
$$\left. - \frac{t_-(t')A_i^{\mathrm{III}}(t_-(t')/(1-\cos\theta), t')}{[m^2(1-\cos\theta) - t_-(t')][s(1-\cos\theta) - t_-(t')]}\right\}, \tag{5.5}$$

and the S_i are given by

$$S_i = \frac{(s-m^2)\sin^2\theta R_i}{2m^2[2m^2 + (1+\cos\theta)(s-m^2)]} + A_i(m^2, \cos\theta),$$

for $i=1$, 2, 6 and for $i=3$,

$$S_i = \frac{(s-m^2)\sin^2\theta R_3}{2m^2[2m^2 + (1+\cos\theta)(s-m^2)]} + \frac{(s-m^2)^2(1-\cos\theta)r_3}{\mu^2[2\mu_0^2 s + (s-m^2)^2(1-\cos\theta)]} + A_3(m^2, \cos\theta), \tag{5.6}$$

where the R_i and r_i are given in (4.8) and (4.24) and the $A_i(m^2, \cos\theta)$ are given by the limiting values of (5.2).

CONCLUSION

The formalism given in the previous sections allows us to calculate the energy and angular dependence of the scalar amplitudes $A_i(s, \cos\theta)$ in so far as this dependence is determined by near-lying singularities and branch cuts. Normally one expects that distant singularities will contribute a constant or very slowly vary-

[30] In practice, from a computational point of view, we prefer to make one subtraction also in A_1.

ing energy dependence to the amplitudes. It seems, however, that in photon scattering processes the low-energy limit theorem permits us to fix these constants theoretically. We therefore hope that the representation will yield a reasonable approximation to the $A_i(s,\cos\theta)$ without the necessity of introducing any phenomenological parameters.

The left-hand cut, however, depends on various quantities which are not well determined experimentally, i.e., τ, the π^0 lifetime; the $T=0$, $\pi\pi$ s-wave phase shift; and σ_T, the total cross section for the process $\gamma+\pi\rightarrow\pi+\pi$.

The right-hand cut, too, being dependent on the meson photoproduction process, is at present not very well determined experimentally. However, it should soon be possible to make a fairly reliable estimate of the contribution arising from this.

It is useful to note, too, that the recoil nucleon polarization is independent of both the π^0 lifetime and the $\tau=0$ $\pi\pi$ s-wave interaction, and thus yields a direct measure of the accuracy of the contributions from the right-hand cut.

We hope, therefore, that improved experimental data in the near future will allow a detailed comparison between the dispersion relation predictions and the experimental values for nucleon Compton scattering, and in particular will shed some more light on the details of the $\pi\pi$ interaction.

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APPENDIX

The expansion of (4.43) into the invariants $F_{\mu\nu}^{(i)}$ leads to expressions for the α_i as complicated combinations of integrals over the πN and $\pi\gamma$ weight functions.

Let us define the following weight functions:

$$\begin{aligned} k_0&=\frac{\sigma_A^{(+)}(s',t)\sigma_T(x)}{x-\mu^2},\\ k_{1,2}&=\frac{\bar{\sigma}_B^{(+)}(s',t)\sigma_T(x)}{x-\mu^2},\\ k_{3,4}&=\sigma_A^{+}(s',t)\left\{\frac{\sigma_T(x)}{x-\mu^2}-\frac{2\pi e^2}{t}\delta(x-\mu^2)\right\},\\ k_{5,6,7}&=\sigma_B^{+}(s',t)\left\{\frac{\sigma_T(x)}{x-\mu^2}-\frac{2\pi e^2}{t}\delta(x-\mu^2)\right\}.\end{aligned}\tag{A1}$$

Then it is possible to write for the α_i:

$$\begin{aligned}\alpha_1&=\{-L_0(s)-2tL_4(s)-m[2tL_7(s)+L_2(s)]\}+[s\rightarrow\bar{s}],\\ \alpha_2&=2[s(2m^2-s-t)-m^4]\left[-L_3(s)-mL_5(s)-\frac{8m}{t-4m^2}L_7(s)\right]+[s\rightarrow\bar{s}],\\ \alpha_3&=\{-L_1(s)+(2s-2m^2+t)[2tL_7(s)+L_2(s)]\}-[s\rightarrow\bar{s}],\\ \alpha_4&=2[s(2m^2-s-t)-m^4]\left[-L_6(s)+\frac{2s-2m^2+t}{t}L_5(s)\right]-[s\rightarrow\bar{s}],\\ \alpha_5&=2tL_7,\end{aligned}\tag{A2}$$

where the L_i can be expressed as

$$L_i(s,t)=\frac{1}{\pi^2}\int_0 ds'\int_0 da\,K_i(s',t,x)l_i(s,t,s',x),\tag{A3}$$

and the l_i are just combinations of integrals over the denominators occurring in the representations of $\mathfrak{A}^{(+)}$, $\mathfrak{B}^{(+)}$, $\mathfrak{D}_a^{(0)}$, $\mathfrak{D}_b^{(0)}$.

Let us put

$$\begin{aligned}d_1&=x-s_1=x+(\mathbf{k}+\mathbf{q})^2=\gamma_1+2\mathbf{q}\cdot\mathbf{k},\\ d_2&=s'-\bar{s}_\kappa=s'+(\boldsymbol{\kappa}+\mathbf{q})^2=\gamma_2+2\mathbf{q}\cdot\boldsymbol{\kappa},\end{aligned}\tag{A4}$$

where

$$\begin{aligned}\gamma_1&=x+\tfrac{1}{2}t-\mu^2,\\ \gamma_2&=s'+\tfrac{1}{2}t-\mu^2-m^2,\end{aligned}$$

and let us define the following integrals

$$\begin{aligned}\mathcal{g}_0&=\int\frac{d\Omega_{\mathbf{q}}}{d_1d_2},\\ \mathcal{g}_1&=\int\frac{d\Omega}{d_1},\quad \mathcal{J}_1=\int\frac{d\Omega}{d_2},\\ \mathcal{g}_2&=\int\frac{\mathbf{q}\cdot\boldsymbol{\kappa}d\Omega}{d_1},\quad \mathcal{J}_2=\int\frac{\mathbf{q}\cdot\mathbf{k}d\Omega}{d_2},\\ \mathcal{g}_3&=\int\frac{(\mathbf{q}\cdot\boldsymbol{\kappa})^2d\Omega}{d_1},\quad \mathcal{J}_3=\int\frac{(\mathbf{qk})^2d\Omega}{d_2}.\end{aligned}\tag{A5}$$

We put $X_0=Y_0=\mathcal{g}_0$ and define

$$X_m=\tfrac{1}{2}(\mathcal{g}_m-\gamma_2 X_{m-1}),$$
$$Y_m=\tfrac{1}{2}(\mathcal{g}_m-\gamma_1 Y_{m-1}),\quad m=1,2\cdots. \tag{A6}$$

Finally we define

$$Z_1=\int\frac{\mathbf{q}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\kappa}}{d_1d_2}d\Omega=\frac{1}{2}\left(\pi-\frac{\gamma_1}{2}\mathcal{g}_1-\gamma_2Y_1\right),$$
$$Z_2=\int\frac{(\mathbf{q}\cdot\mathbf{k})^2\mathbf{q}\cdot\boldsymbol{\kappa}}{d_1d_2}d\Omega=-\tfrac{1}{2}(\gamma_2\mathcal{g}_2+\gamma_1Z_1), \tag{A7}$$
$$Z_3=\int\frac{\mathbf{q}\cdot\mathbf{k}(\mathbf{q}\cdot\boldsymbol{\kappa})^2}{d_1d_2}d\Omega=-\tfrac{1}{2}(\gamma_1\mathcal{g}_2+\gamma_2Z_i),$$

and

$$\zeta_1=\mu^4+2(t-\mu^2)x+x^2,$$
$$\zeta_2=\mu^2-x-\tfrac{3}{2}t. \tag{A8}$$

In terms of these functions, the $l_i(s,t,s',x)$ are given by

$$l_0=\zeta_1\mathcal{g}_0+\zeta_2\mathcal{g}_1+2\mathcal{g}_2,$$

$$l_1=\frac{4}{t}[\zeta_1Y_1+\zeta_2\mathcal{g}_2+2\mathcal{g}_3],$$

$$l_2=\frac{1}{\mathcal{P}'^2}[\zeta_1X_1+\pi\zeta_2-\gamma_2(\tfrac{1}{2}\zeta_2\mathcal{g}_1+\mathcal{g}_2)-\tfrac{1}{4}(2m^2-s-t)l_1],$$

$$l_3=\frac{1}{\mathcal{P}'^2}\left[(m^2-\tfrac{1}{4}t)X_0+\frac{2}{\mathcal{P}'^2}X_2+\frac{4}{t}(1+2u)Y_2-4vZ_1\right],$$

$$l_4=\left(\frac{t}{4}-m^2\right)X_0-\frac{1}{\mathcal{P}'^2}X_2-\frac{4}{t}(1+u)Y_2+2vZ_1, \tag{A9}$$

$$l_5=\frac{1}{\mathcal{P}'^2}\left\{\frac{1}{\mathcal{P}'^2}\left[3\left(m^2-\frac{t}{4}\right)X_1+\frac{4}{\mathcal{P}'^2}X_3-12vZ_3+\frac{12}{t}(1+4u)Z_2\right]+v\left[3\left(\frac{t}{4}-m^2\right)Y_1+\frac{t}{4}(4u-3)Y_3\right]\right\},$$

$$l_6=\frac{4}{\mathcal{P}'^2t}\left[\left(m^2-\frac{t}{4}\right)Y_1+\frac{4}{t}(1+2u)Y_3-4vZ_2+\frac{2}{\mathcal{P}'^2}Z_3\right],$$

$$l_7=\frac{1}{\mathcal{P}'^2}\left[\left(\frac{t}{4}-m^2\right)X_1-\frac{1}{\mathcal{P}'^2}X_3-\frac{4}{t}(1+3u)Z_2+3vZ_3\right]+v\left[\left(m^2-\frac{t}{4}\right)Y_1+\frac{4}{t}(1-u)Y_3\right],$$

with

$$\mathcal{P}'^2=-(s-m^2)^2/t-s,$$
$$v=(2m^2-2s-t)/t\mathcal{P}'^2, \tag{A10}$$
$$u=\tfrac{1}{4}(2m^2-2s-t)v.$$

The actual integrals in (A5) are not difficult to evaluate and we list their explicit expressions: We define

$$a(\gamma_1,k)=\frac{1}{k(m^2-\frac{1}{4}t)^{\frac{1}{2}}}\arctan\left(\frac{2k}{\gamma_1}(m^2-\tfrac{1}{4}t)^{\frac{1}{2}}\right). \tag{A11}$$

Then

$$\begin{aligned}
\mathcal{G}_1(s,t)&=2\pi a(\gamma_1,k),\\
\mathcal{G}_2(s,t)&=(\pi/t)(2s+t-2m^2)[2-\gamma_1 a(\gamma_1,k)],\\
\mathcal{G}_3(s,t)&=(\pi/t^2)\{[t\kappa^2(q^2t-\gamma_1^2)+\tfrac{1}{4}(2s+t-2m^2)^2(3\gamma_1^2-q^2t)]a(\gamma_1,k)-(\gamma_1/2t)[\tfrac{3}{4}(2s+t-2m^2)^2-\kappa^2t]\},\\
\mathfrak{J}_1(s,t)&=2\pi a(\gamma_2,k),\\
\mathfrak{J}_2(s,t)&=\frac{\pi}{t-4m^2}(2s+t-2m^2)[2-\gamma_2 a(\gamma_2,k)],\\
\mathfrak{J}_3(s,t)&=\frac{\pi}{(t-4m^2)^2}\left\{\left[4t\kappa^2\left(q^2\kappa^2-\frac{\gamma_2^2}{4}\right)+\tfrac{1}{4}(2s+t-2m^2)^2(3\gamma_2^2-4q^2\kappa^2)\right]a(\gamma_2,k)-\frac{\gamma_2}{2(t-4m^2)}[\tfrac{3}{4}(2s+t-2m^2)^2-\kappa^2t]\right\}.
\end{aligned}\tag{A12}$$

Lastly, $\mathcal{G}_0(s,t)$ is given as follows: put

$$\begin{aligned}
w_1&=(t/4)[(s-x-s')^2-4xs']-\mu^2s^2+s[2\mu^2m^2-(x-\mu^2)(s'-\mu^2-m^2)]+m^2[(\mu^2-x)(x-s')-xm^2],\\
w_2&=(t/2-\mu^2+x)(x+s-s')+(m^2-s)(\mu^2+x),\\
w_3&=(s'+t/2-\mu^2-m^2)(x-s'+m^2)-(t/2-2\mu^2)(s+m^2),\\
w_4&=(t/2)(s'-s+m^2)+(x-\mu^2)(s'-\mu^2-m^2)+2\mu^2(s-m^2).
\end{aligned}\tag{A13}$$

Then

$$\begin{aligned}
\mathcal{G}_0(s,t)&=\frac{4\pi}{[w_1(4\mu^2-t)]^{\frac{1}{2}}}\left\{\arctan\frac{w_2}{[w_1(4\mu^2-t)]^{\frac{1}{2}}}-\arctan\frac{w_3}{[w_1(4\mu^2-t)]^{\frac{1}{2}}}\right\} &&\text{for}\quad (4\mu^2-t)w_1>0,\\
&=\frac{4\pi(w_2-w_3)}{w_2w_3} &&\text{for}\quad (4\mu^2-t)w_1=0,\\
&=\frac{2\pi}{[(t-4\mu^2)w_1]^{\frac{1}{2}}}\ln\left[\frac{w_4+[w_1(t-4\mu^2)]^{\frac{1}{2}}}{w_4-[w_1(t-4\mu^2)]^{\frac{1}{2}}}\right] &&\text{for}\quad (4\mu^2-t)w_1<0,
\end{aligned}\tag{A14}$$

with all arctangents defined in the range $-\pi/2\leqslant\arctan\leqslant\pi/2$.