PHYSICAL REVIEW VOLUME 126, NUMBER 2 APRIL 15, 1962

Behavior of Scattering Amplitudes at High Energies, Bound States, and Resonances*†

R. BLANKENBECLER AND M. L. GOLDBERGER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received December 4, 1961)

An exact Fourier-Bessel representation of the scattering amplitude is introduced and discussed for potential scattering and field theory. It is shown to contain the Mandelstam representation as a special case. The representation automatically satisfies unitarity exactly in the high-energy limit even in the many-channel situation. The behavior of the scattering amplitude for large momentum transfers is discussed and it is demonstrated that this limit is directly connected with the formation of bound states and resonances. A selection rule governing the ordering of resonances is derived. A variational principle for calculating the asymptotic dependence on momentum transfer in potential scattering is formulated. Some interesting relations between the asymptotic behaviors of π-π, π-N, and N-N are developed, and related to the single-particle poles and low-energy resonances.

I. INTRODUCTION

THE behavior of scattering amplitudes at high energies and at large values of momentum transfer is very important in the formulation of dynamical computational schemes in dispersion theory. In the course of studying this problem we have been led to reconsider an approximation scheme which has been often used in nonrelativistic quantum mechanics. It is variously known as the eikonal method (in which form it is very ancient) or impact parameter method.[1] A rough description is as follows: Consider a very high energy particle moving through a region of interaction with a force center. To the extent that one can construct a sufficiently localized wave packet, it is reasonable to speak of the particle passing at a certain distance b from the center. Assuming the deflection is small, one computes the approximate change in phase of the wave function and from this the scattering amplitude is obtained in a standard fashion. The outstanding virtue of the representation thus obtained is that by its very form, it satisfies unitarity automatically in the high-energy limit.

In Sec. II we review the canonical impact-parameter approximation and introduce a modification of it which is more closely related to dispersion theory. An exact representation modeled after the approximate one is given which is related to the familiar double dispersion relation known to hold in potential scattering for a special class of potentials. In Sec. III it is shown how the problem is, in principle, solved with the aid of the unitarity condition. Furthermore the relation between impact parameter and angular momentum is established.

An important relation between bound states, resonances, and behavior of the scattering amplitude for large momentum transfers is developed in Sec. IV. The relevance to the subtraction problem in dispersion theory is pointed out.

A variational principle for calculating the positions of bound states and resonances (which are closely related to the behavior of the scattering amplitude for large momentum transfer) is formulated in terms of complex angular momenta. Two applications of the principle are made and worked out in detail.

An example is treated in Sec. V which allows one to follow all of these points in detail.

The extension of the formalism to coupled channels is taken up in Sec. VI and finally the much more complicated problems of field theory are treated in Sec. VII. No numerical results are obtained but some interesting relations between the asymptotic behaviors of π-π, N-N, and π-N are developed and related to low-energy resonances.

In an Appendix we prove that the scattering amplitude can be analytically continued to values of the complex angular momentum for which $\mathrm{Re}\,l<-\frac{1}{2}$. There has been widespread misunderstanding on this point.

II. THE HIGH ENERGY APPROXIMATION

The general characteristics of the scattering of particles of high energies is that scattering angles are small and the angular distributions take the form of diffraction patterns. These features suggest the use of an eikonal approximation to describe the phenomenon. Such a description has been discussed within the framework of both nonrelativistic and relativistic quantum mechanics by many authors.[1] The most comprehensive treatment has been given in an excellent article by

---

* Supported in part by the Air Force Office of Scientific Research, Air Research and Development Command.

† Except for the variational principle of Sec. IV and the Appendix, this is a detailed version of the paper presented to the International Conference on Weak and Strong Interactions at La Jolla, California, June, 1961 (unpublished). At that conference we suggested the possibility of associating Regge poles with "elementary" particles as well as with dynamical resonances. Professor Chew and his collaborators had been discussing such ideas for some time independently and their views have appeared in several papers published since the present work was submitted: G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962), and G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1204 (1962). There are a number of points of similarity between our results and those of Chew *et al.* We have learned recently of work by a number of Russian physicists, notably V. Gribov and G. Domokos, on the same subject carried out at about the same time, but we have not seen their results.

[1] G. Molière, Z. Naturforsch. **2A**, 133 (1947); G. Parzen, Phys. Rev. **80**, 261 (1950); D. Saxon and L. Schiff, Nuovo cimento **6**, 614 (1957).

Glauber.[2] The eikonal approximation is based on a computation of the phase change suffered by a particle passing (in a semiclassical sense) through a scattering region at a distance b from the scattering center on an essentially straight line trajectory. The resulting wave function is

$$\psi=\exp\left\{i\mathbf{k}\cdot\mathbf{r}-\frac{i}{2k}\int_{-\infty}^{z}dz' V[(z'^2+b^2)^{\frac{1}{2}}]\right\}, \quad (2.1)$$

where the z axis will be chosen parallel to the average of the initial and final momentum vectors. From this wave function the scattering amplitude is computed in the standard way to be

$$f(k,\theta)=-ik\int_0^\infty b\,db\,J_0\left(2kb\sin\frac{\theta}{2}\right)\times\{\exp[i\chi_e(b^2,k)]-1\}, \quad (2.2)$$

where k is the momentum in the center-of-mass system, θ is the scattering angle, and $J_0(x)$ is the zero-order Bessel function. The eikonal phase function χ_e is given by

$$\chi_e(b^2,k)=-\frac{1}{2k}\int_{-\infty}^{\infty}dz\,V[(z^2+b^2)^{\frac{1}{2}}]. \quad (2.3)$$

We have chosen units such that $\hbar=1$ and $2M=1$, where M is the reduced mass. In a somewhat more accurate approximation,

$$\chi_e(b^2,k)=\int_{-\infty}^{\infty}dz\,(\{k^2-V[(z^2+b^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}-k), \quad (2.4)$$

but we shall confine our attention to the simpler form (2.3), which is valid for large k.

One possible shortcoming of this representation as far as dispersion theory is concerned, is that the simple singularities of the phase function χ as a function of energy have an alarming tendency to show up in the scattering amplitude as essential singularities. In order to avoid this trouble let us attempt to find an approximate wave function which does not contain the potential in an exponential manner and in addition changes both the phase and the amplitude of the wave. One such choice is

$$\psi=\left\{1+\frac{i}{4k}\int_{-\infty}^{z}dz'\,V[(z'^2+b^2)^{\frac{1}{2}}]\right\}^{-2}\exp(i\mathbf{k}\cdot\mathbf{r}). \quad (2.5)$$

This wave function[3] satisfies the Schrödinger equation to the same order as (2.1), namely to terms of order $(1/k^2)$ and leads to the scattering amplitude

$$f(k,\theta)=\int_0^\infty b\,db\,J_0\left(2kb\sin\frac{\theta}{2}\right)n(b)[1-in(b)/2k]^{-1}, \quad (2.6)$$

where

$$n(b)=-\frac{1}{2}\int_{-\infty}^{\infty}dz\,V[(z^2+b^2)^{\frac{1}{2}}]. \quad (2.7)$$

We shall see later that the form (2.5) follows naturally from dispersion theory.

The result (2.6) can, of course, be rewritten in the form of the eikonal approximation, and the resultant phase function turns out to be

$$\chi=2\tan^{-1}[\tfrac{1}{2}\chi_e(b^2,k)]. \quad (2.8)$$

Thus, as a function of k, χ has quite a different character from χ_e.

The condition for validity of the eikonal approximation,[4] and also the approximation (2.7), is roughly the demand that

$$|V(1/k)|\ll k^2.$$

The integration variable b may be identified with the classical impact parameter and this general scheme is sometimes referred to as the impact parameter approximation. It is clear that the regime of small energies is not in general covered by this approximation. Each of the representations (2.2) and (2.6) has advantages and disadvantages, and we will feel free to use the most convenient one whenever possible. As we shall see later, only (2.6) can be easily generalized to field theory.

The most interesting feature of these high-energy approximations for our later purposes is the fact that they automatically satisfy the optical theorem for large k. To study this property for the eikonal approximation, consider the case of a complex phase function $\chi=\chi_R+i\chi_I$ and evaluate the imaginary part of the scattering amplitude in the forward direction:

$$\mathrm{Im}f(k,0)=k\int_0^\infty b\,db\,[1-\exp(-\chi_I)\cos\chi_R].$$

Next we compute a quantity related to the total scattering cross section:

$$I=\frac{1}{4\pi}\int d\Omega\,f^*(k,\theta)f(k,\theta)$$
$$=k\int_0^\infty b\,db\,[1-\exp(-\chi_I)\cos\chi_R]-\frac{k}{2}\int_0^\infty b\,db\,[1-\exp(-2\chi_I)]+O(1/k),$$

[2] R. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1958), p. 315.

[3] It is undoubtedly known to Jost and many others that $\exp(-i\mathbf{k}_0\cdot\mathbf{r})\psi_{k_0}(r)$ regarded as a function of the magnitude of $|\mathbf{k}_0|=k$, is analytic in the upper half k plane for any fixed r, aside from poles on the imaginary axis corresponding to bound states. This is trivially proved from the usual scattering integral equation. Our proposed high-energy wave function satisfies this requirement.

[4] J. Schwinger, Phys. Rev. **94**, 1367 (1954).

where we have used the fact that under an integral,

$$\int_0^{2k} x\,dx\,J_0(xb)J_0(xb')=\frac{1}{b}\delta(b-b')+O(1/k^2). \quad (2.9)$$

If these equations are now combined, we find

$$\frac{4\pi}{k}\,\mathrm{Im}f(k,0) =\int d\Omega\,|f(k,\theta)|^2+2\pi \times\int_0^\infty bdb\,[1-\exp(-2\chi_I)]+O(1/k^2). \quad (2.10)$$

A similar discussion can be given for the second form of the high-energy approximation Eq. (2.6). In order to have the possibility of satisfying unitarity exactly, or to allow for true absorption, we must generalize the representation somewhat. We write, with $s\equiv k^2$,

$$f(k,\theta)=\int_0^\infty bdb\,J_0\left(2kb\sin\frac{\theta}{2}\right)\frac{N(b^2,s)}{D(b^2,s)}, \quad (2.11)$$

where

$$D(b^2,s)=1-\int_0^\infty\frac{ds'}{\pi}\frac{1}{2(s')^{\frac{1}{2}}}\frac{\mathrm{Re}N(b^2,s')}{s'-s-i\epsilon}. \quad (2.11a)$$

If we wish to insure that f satisfies a dispersion relation in s for fixed momentum transfer, $N(b^2,s)$ must be analytic in the s plane cut along the positive real axis. By contour integration one finds

$$D(b^2,s)=1-iN(b^2,s)/2k. \quad (2.11b)$$

The optical theorem becomes

$$\frac{4\pi}{k}\,\mathrm{Im}f(k,0) =\int d\Omega\,|f(k,\theta)|^2+\frac{4\pi}{k} \times\int_0^\infty bdb\,\mathrm{Im}N|D(b^2,s)|^{-2}+O(1/k^2). \quad (2.12)$$

In the limit of large energies, the $(1/k^2)$ terms may be dropped. These results show that the optical theorem is satisfied because the second terms in (2.10) and (2.12) then have a simple physical interpretation. For example, the factor $\exp(-2\chi_I)$ is simply the intensity of a wave of unit amplitude transmitted through the region of interaction at a distance b from the center. Thus $[1-\exp(-2\chi_I)]$ is a measure of the absorption that has taken place. If the functions χ and N are real in the limit of high energies, then the scattering amplitudes automatically satisfy the purely elastic optical theorem.

If there is physical absorption present it is clear that we must have

$$\int_0^\infty bdb\,\{1-\exp[-2\chi_I(b^2,s)]\}>0$$

$$\int_0^\infty bdb\,\mathrm{Im}N(b^2,s)|D(b^2,s)|^{-2}>0$$

for large s. We cannot conclude in general that χ_I or $\mathrm{Im}N$ are definitely positive, but to the extent that one can construct wave packets sufficiently well localized to speak of scattering at a definite impact parameter it is evident that both χ_I and $\mathrm{Im}N$ associated with absorptive processes must be positive. Even in cases where there is no absorption, one can imagine choosing χ_I or $\mathrm{Im}N$ in such a way as to satisfy the elastic unitarity condition and hence the optical theorem, for *all* positive energies. In this circumstance we have no physical basis for requiring these functions to be positive. The possibility of choosing χ_I or $\mathrm{Im}N$ in the above fashion will be discussed in Sec. III.

It is the automatic unitarity aspect of the high-energy approximation which we would like to exploit. Our purpose here is to cast the Mandelstam representation for $f(k,\theta)$, which is valid for Yukawa-type potentials,[5] into the impact parameter form. Although the Mandelstam representation displays the analyticity of the scattering amplitude in the variables k and $\cos\theta$ in an admirable fashion, the unitarity condition is extremely awkward to deal with, especially in the high-energy, large-momentum-transfer regime.

In order to motivate our procedure, let us consider the high-energy approximation for a simple Yukawa potential of unit range which is not strong enough to produce bound states. The phase function according to (2.3) is

$$\chi_e(b^2,k)=-\frac{\lambda}{2k}\int_{-\infty}^\infty dz(z^2+b^2)^{-\frac{1}{2}}\exp[-(z^2+b^2)^{\frac{1}{2}}] =-(\lambda/k)K_0(b), \quad (2.13)$$

where $K_0(b)$ is the modified Bessel function of the second kind with pure imaginary argument as defined by Watson.[6] Some simple properties of this function are

$$K_0(b)\simeq-\ln(\gamma b/2),\quad b\to 0,$$
$$\to(\pi/2b)^{\frac{1}{2}}e^{-b},\quad b\to\infty,$$

where $\ln\gamma$ is the Euler constant, $0.5772\cdots$. An integral representation for $K_0(b)$ which is more useful than that

[5] R. Blankenbecler, M. Goldberger, N. Khuri, and S. Treiman, Ann. Phys. **10**, 62 (1960); T. Regge, Nuovo cimento **14**, 951 (1959); A. Klein, J. Math. Phys. **1**, 41 (1960).

[6] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1922). Hereafter to be referred to as W.

implied by (2.12) is

$$K_0(b)=\frac{1}{2}\int_0^\infty dz\, J_0(z^{\frac{1}{2}})(z+b^2)^{-1}$$

or more generally,

$$K_0(bx)=\frac{1}{2}\int_0^\infty dz\, J_0(xz^{\frac{1}{2}})(z+b^2)^{-1} =\frac{1}{2}\int_0^\infty dz\, J_0(bz^{\frac{1}{2}})(z+x^2)^{-1}. \quad (2.14)$$

In the same approximation, the numerator function in Eq. (2.11) becomes

$$N(b^2,s)=-\lambda K_0(b). \quad (2.15)$$

Returning to the scattering amplitudes, we note that for sufficiently large k the phase function becomes small and D approaches one, so that we may write

$$\lim_{k\to\infty} f(k,\theta)=-ik\int_0^\infty bdb\, J_0\left(2kb\sin\frac{\theta}{2}\right)[i\chi_e(b^2,k)]$$
$$=\int_0^\infty bdb\, J_0\left(2kb\sin\frac{\theta}{2}\right)N(b^2,\infty)$$
$$=-\lambda\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}]K_0(b)$$
$$=-\lambda/(1-t),$$

where the square of the momentum transfer has been introduced as

$$-t=2k^2(1-\cos\theta)=4s\sin^2\theta/2.$$

The representation (2.14) was used together with (2.9) in order to carry out the integration. This is precisely the first Born approximation for a Yukawa potential, and it is elementary to show that one always gets the correct Born term for any potential in the limit of large s. We may remark that Carter[7] and Khuri[8] have shown that this is a rigorous property of the scattering amplitude in potential scattering.

Let us attempt to write an exact representation for the scattering amplitude corresponding to a Yukawa potential by considering phase functions and numerator functions of the form

$$i\chi=-\frac{1}{\pi}\int_0^\infty dz\,\sigma(z,s)(z+b^2)^{-1}, \quad (2.16a)$$

$$N(b^2,s)=\frac{1}{\pi}\int_0^\infty dz\, W(z,s)(z+b^2)^{-1}. \quad (2.16b)$$

[7] D. S. Carter, thesis, Princeton University, 1952 (unpublished).
[8] N. Khuri, Phys. Rev. **107**, 1148 (1957).

In order to get the first Born approximation correctly, we must require that

$$\lim_{s\to\infty}\sigma(z,s)=\pi\lambda J_0(z^{\frac{1}{2}})/2(-s)^{\frac{1}{2}}, \quad (2.17a)$$

$$\lim_{s\to\infty}W(z,s)=-\pi\lambda J_0(z^{\frac{1}{2}})/2. \quad (2.17b)$$

If we denote the Fourier-Bessel transform of the scattering amplitude by $H(b^2,s)$, then we have

$$f(s,t)=\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}]H(b^2,s) \quad (2.18)$$

and

$$H(b^2,s)=\frac{1}{\pi}\int_0^\infty dz\, h(z,s)(z+b^2)^{-1}. \quad (2.19)$$

We can now read off the weight function $h(z,s)$:

$$h(z,s)=-(-s)^{\frac{1}{2}}[\sin\sigma(z,s)] \times\exp\left[-P\int_0^\infty \frac{dz'}{\pi}\sigma(z',s)(z'-z)^{-1}\right], \quad (2.20a)$$

$$h(z,s)=W(z,s)[D(-z-i\epsilon,s)D(-z+i\epsilon,s)]^{-1}, \quad (2.20b)$$

where we recall the relation (2.11b) between N and D.

It has been assumed that $H(b^2,s)$ does not behave too badly for small b. In particular, if $\sigma(0,s)$ is finite, the leading singularity in $H(b^2,s)$ is

$$H(b^2s)\sim C(s)b^{2\sigma(0,s)/\pi}+\cdots.$$

Therefore, we require that

$$\mathrm{Re}\,\sigma(0,s)/\pi>-1. \quad (2.21)$$

This condition is certainly always satisfied for sufficiently large s or sufficiently weak potentials [see Eq. (2.17a)]. One might expect (2.21) to be violated for low energies if the potential becomes too strong. We will return to this point when bound states and resonances are discussed.

From (2.18) and (2.19), it follows that

$$f(s,t)=\int\frac{dt'}{\pi}\frac{A_3(t',s)}{t'-t}, \quad (2.22)$$

where

$$A_3(t,s)=\frac{1}{2}\int_0^\infty dz\, J_0[(zt)^{\frac{1}{2}}]h(z,s). \quad (2.23)$$

If h is analytic in the cut s plane, then the Mandelstam representation immediately follows. This property of $h(z,s)$ implies that both $\sigma(z,s)$ and $w(z,s)$ enjoy analyticity in s also.

In order to complete our discussion of the Fourier-Bessel representation, we will exhibit an exact expression for the function H in terms of the scattering amplitude. This allows us to discuss certain general properties of χ and N which are important to satisfy for large and small energies. Since the Fourier-Bessel transform provides its own inversion formula [see Eq. (2.9)], we immediately find that

$$H(b^2,s)=\int_0^\infty xdx\, J_0(bx)f(s,\,-x^2). \qquad (2.24)$$

In the special case of the Mandelstam representation,

$$f(s,t)=\int\frac{dt'}{\pi}\frac{A_3(t',s)}{t'-t},$$

we find

$$H(b^2,s)=\int\frac{dt}{\pi}K_0(bt^{\frac{1}{2}})A_3(t,s). \qquad (2.25)$$

Since $A_3(t,s)$ is analytic in the cut s plane, H must also possess this property. We note, quite generally, that

$$i\chi(b^2,s)=\ln[1+H(b^2,s)/(-s)^{\frac{1}{2}}] \qquad (2.26a)$$

and

$$N(b^2,s)=\frac{H(b^2,s)}{1+H(b^2,s)/2(-s)^{\frac{1}{2}}}. \qquad (2.26b)$$

The large s limits of χ and N have been discussed above [see (2.17)]. The threshold behavior in s is also simple; χ has a logarithmic singularity as $s\to 0$ and N vanishes as the square root of s since $H(b^2,0)$ is finite and nonzero. These two requirements are very important in assuring the correct high- and low-energy behavior of the scattering amplitude. The high-energy approximations, (2.3) and (2.7), are seen to violate the low energy limits.

The high-energy approximation has an amusing property which serves to clarify its connection to the Mandelstam representation. For illustrative purposes, let us choose the simplest approximation possible, that given by (2.12) and (2.14). Now if the scattering amplitude is expanded in a power series in the potential strength λ, then the Nth term is seen to involve an integral of the form

$$\lambda^N\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}]K_0{}^N(b).$$

Since $K_0(b)$ falls off exponentially for large b, this integral has a singularity for positive t whenever the growth of the J_0 cannot be controlled by the K_0's. The singularity occurs at $t=N^2$. This is exactly the behavior of the Mandelstam representation and the Born series.[5] Thus for large energies, the high-energy approximation gives the asymptotically correct thresholds in the momentum transfer variable.

Let us examine this point in more detail. If the eikonal form of the high-energy approximation is used, then the weight function $h(z,s)$ is found to be

$$h(z,s)=-(-s)^{\frac{1}{2}}\sin\left[\frac{\pi\lambda}{2(-s)^{\frac{1}{2}}}J_0(z^{\frac{1}{2}})\right]\times\exp\left[\frac{\pi\lambda}{2(-s)^{\frac{1}{2}}}Y_0(z^{\frac{1}{2}})\right], \qquad (2.27)$$

where $Y_0(b)$ is a modified Bessel function of the second kind as defined by W. Using the relation

$$J_0(y)Y_0(y)=-\frac{2}{\pi}\int_1^\infty du\, J_0(2uy)/(u^2-1)^{\frac{1}{2}}, \qquad (2.28)$$

which can be obtained from W, page 441, Eq. (5), the first two terms in the expansion of h can be written in the form

$$h(z,s)=-\frac{\pi\lambda}{2}J_0(z^{\frac{1}{2}})+\frac{\pi\lambda^2}{2(-s)^{\frac{1}{2}}}\int_1^\infty du\, J_0(2uz^{\frac{1}{2}})/(u^2-1)^{\frac{1}{2}}.$$

The integral for A_3 can now be directly carried out:

$$A_3(t,s)=-\pi\lambda\delta(t-1)+\tfrac{1}{2}\pi\lambda^2[-st(t-4)]^{-\frac{1}{2}}\theta(t-4). \qquad (2.29)$$

The first term is correct but the second is not. It has the correct threshold in t for large s, however, since the exact second order expansion for A_3 is

$$A_3(t,s)=-\pi\lambda\delta(t-1)+\frac{\lambda^2}{2}\times\int_0^\infty\frac{ds'}{s'-s}[s't(t-4a')]^{-\frac{1}{2}}\theta(t-4a'), \qquad (2.30)$$

where $a'=1+1/4s'$. A phase function that yields this result to second order is

$$i\chi(b^2,s)=-\frac{\lambda}{(-s)^{\frac{1}{2}}}K_0(b)+\frac{\lambda^2K_0{}^2(b)}{2s}+\frac{\lambda^2}{\pi(-s)^{\frac{1}{2}}}\times\int_0^\infty\frac{ds'}{(s'-s)(s')^{\frac{1}{2}}}\int_1^\infty\frac{du\, K_0(2uba'^{\frac{1}{2}})}{(u^2-1)^{\frac{1}{2}}}. \qquad (2.31a)$$

Similarly, a numerator function which does the same job can easily be found:

$$N(b^2,s)=-\lambda K_0(b)-\frac{\lambda^2K_0{}^2(b)}{2(-s)^{\frac{1}{2}}}+\frac{\lambda^2}{\pi(-s)^{\frac{1}{2}}}\times\int_0^\infty\frac{ds'}{(s'-s)(s')^{\frac{1}{2}}}\int_1^\infty du\frac{K_0(2uba'^{\frac{1}{2}})}{(u^2-1)^{\frac{1}{2}}}. \qquad (2.31b)$$

These functions when substituted into $H(b^2,s)$ will yield a scattering amplitude which has the exact t discontinuity up to $t=9$. Above this point, the discontinuity is only approximate, but all the correct t thresholds in $f(s,t)$ are insured. However, it should be noted that the energy dependence of $H(b^2,s)$ is incorrect near zero energy; the approximation therefore breaks down at some finite value of s. We will return to the problem of constructing N to all orders of perturbation theory at a later juncture.

Before leaving this introductory section on the impact parameter representation, it is useful to show the relation between it and other ways of writing scattering amplitudes. We begin always with the Mandelstam form,

$$f(s,t)=\int\frac{dt'}{\pi}\frac{A_3(t',s)}{t'-t}, \tag{2.32}$$

and note that various representations emerge naturally from different ways of writing $(t'-t)^{-1}$. First, there is the standard partial wave expansion:

$$\frac{1}{t'-t}=\frac{1}{t'+2s(1-z)}=\frac{1}{s}\sum_{l=0}(l+\tfrac{1}{2})Q_l\left(1+\frac{t'}{2s}\right)P_l(z). \tag{2.33}$$

The corresponding impact parameter form we have used is

$$\frac{1}{t'-t}=\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}]K_0[b(t')^{\frac{1}{2}}], \tag{2.34}$$

and it is easy to see the connection between these, since for large $l\equiv bk$,

$$P_l(z)\to J_0[b(-t)^{\frac{1}{2}}],\; Q_l\left(1+\frac{t'}{2b^2}\right)\to K_0\left[l\frac{(t')^{\frac{1}{2}}}{k}\right]$$
$$=K_0[b(t')^{\frac{1}{2}}]. \tag{2.35}$$

An alternative representation in terms of complex angular momenta may be obtained using

$$\frac{1}{t'-t}=\pi\int_0^\infty \lambda d\lambda\frac{\tanh\pi\lambda}{\cosh\pi\lambda}P_{i\lambda-\frac{1}{2}}(t')P_{i\lambda-\frac{1}{2}}(t), \tag{2.36}$$

where the $P_{i\lambda-\frac{1}{2}}$ are Legendre functions, not polynomials. The counterpart of this one is simply to interchange t', $-t$ in Eq. (2.34) and write

$$\frac{1}{t'-t}=\int_0^\infty bdb\, K_0[b(-t)^{\frac{1}{2}}]J_0[b(t')^{\frac{1}{2}}]. \tag{2.37}$$

There may be certain formal advantages in using this version for studying the behavior of amplitudes for large t. We note in this connection an alternate form of our representations. We write Eq. (2.32) using (2.37) as

$$f(s,t)=\int_0^\infty bdb\, K_0[b(-t)^{\frac{1}{2}}]\times\int\frac{dt'}{\pi}A_3(t',s)J_0[b(t')^{\frac{1}{2}}]. \tag{2.38}$$

Expressing $A_3(t',s)$ in terms of $h(z,s)$, Eq. (2.23), we obtain

$$f(s,t)=\frac{2}{\pi}\int_0^\infty bdb\, K_0[b(-t)^{\frac{1}{2}}]h(b^2,s). \tag{2.39}$$

This is a complex impact parameter representation analogous to the complex angular momenta introduced by Regge. The explicit exponential decrease of the amplitude for large $(-t)^{\frac{1}{2}}$ is useful in certain circumstances.

III. THE UNITARITY CONDITION AND PARTIAL WAVES

The unitarity condition allows us to complete our set of dynamical equations for the scattering amplitude. This requirement states that

$$\mathrm{Im}f(s,t_{13})=\frac{k}{4\pi}\int d\Omega_2 f^*(s,t_{12})f(s,t_{23}), \tag{3.1}$$

in a transparent notation: $t_{13}=-2s(1-\hat{k}_3\cdot\hat{k}_1)$, $t_{12}=-2s(1-\hat{k}_1\cdot\hat{k}_z)$, $t_{23}=-2s(1-\hat{k}_2\cdot\hat{k}_3)$. In terms of the transform function $H(b^2,s)$, this condition takes the form

$$\int_0^\infty bdb\, J_0[b(-t_{13})^{\frac{1}{2}}]\,\mathrm{Im}H(b^2,s)$$
$$=\int_0^\infty b_1db_1\int_0^\infty b_2db_2\, H^*(b_1^2,s)\times H(b_2^2,s)I(b_1b_2;s,t_{13}), \tag{3.2}$$

where

$$I=\frac{k}{4\pi}\int d\Omega_2 J_0[b_1(-t_{13})^{\frac{1}{2}}]J_0[b_2(-t_{23})^{\frac{1}{2}}]$$
$$=\frac{k}{2}\int_{-1}^{+1}d(\cos\theta')J_0\left(2kb_2\sin\frac{\theta'}{2}\right)J_0\left(2kb_2\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\right)\times J_0[2kb_2\sin(\theta'/2)\cos(\theta/2)]. \tag{3.3}$$

We have used an elementary addition formula to do the ϕ integral and θ is the scattering angle. This is one of the well-known integrals of Sonine (W, page 377),

$$I=\frac{1}{\pi}\int_0^\pi d\phi\, J_1(2kB)/B, \tag{3.4}$$

where

$$B^2=b_1^2+b_2^2-2b_1b_2\cos(\theta/2)\cos\phi.$$

Using the relation

$$\int_0^\infty bdb\, J_0\left(2kb\sin\frac{\theta}{2}\right)J_0\left(2kb\sin\frac{\theta'}{2}\right)=\frac{\delta[\sin(\theta/2)-\sin(\theta'/2)]}{4s\sin(\theta/2)},$$

the kernel I can be written in the form

$$I=\int_0^\infty bdb\, J_0[b(-t_{13})^{\frac{1}{2}}]G(b;b_1b_2;s), \quad (3.5)$$

where

$$G(b;b_1b_2;s)=\frac{2s}{\pi}\int_0^\pi d\phi\int_0^\pi d\theta\,\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times J_0\left(2kb\sin\frac{\theta}{2}\right)\frac{J_1[2kB(\theta,\phi)]}{B(\theta,\phi)}. \quad (3.6)$$

The unitarity condition in terms of H then becomes

$$\mathrm{Im}H(b^2,s)=\int_0^\infty b_1db_1\int_0^\infty b_2db_2\,H^*(b_1^2,s)\times H(b_2^2,s)G(b;b_1b_2;s). \quad (3.7)$$

If this statement is coupled with the equation expressing the analyticity of the scattering amplitude as a function of the energy,

$$H(b^2,s)=-\lambda K_0(b)+\int_0^\infty \frac{ds'}{\pi}\frac{\mathrm{Im}H(b^2,s')}{s'-s}, \quad (3.8)$$

then we have achieved a one-cut mapping problem for the determination of H which involves only physical values of s.[9] This is to be contrasted with the ordinary partial-wave amplitude which even in the case of potential scattering has both a positive and negative cut. For a general potential, $-\lambda K_0(b)$ is replaced by

$$-\frac{1}{2}\int_{-\infty}^\infty dz\, V[(b^2+z)^{\frac{1}{2}}].$$

We have then a dynamical dispersion approach which can be applied to any potential and not just those representable by a superposition of Yukawa potentials.

In the limit of large s the kernel G takes on a particularly simple form. Going back to (3.3), we see that as the energy becomes large, the Bessel functions oscillate to zero unless θ and θ' are small. Thus by keeping only the linear terms in $\sin(\theta/2)$ and $\sin(\theta'/2)$, we find the limit

$$I\sim\frac{\delta(b_1-b_2)}{2kb_1}J_0(2kb_1\sin(\theta/2)),$$

where the error is of order $(1/s)$ times the leading term. Finally, G becomes

$$G(b;b_1b_2;s)\sim(1/2kb_1b_2)\delta(b_1-b_2)\delta(b_1-b). \quad (3.9)$$

For large s, then, the unitarity condition on H simplifies to

$$\mathrm{Im}H(b^2,s)=(1/2(s)^{\frac{1}{2}})H^*(b^2,s)H(b^2,s). \quad (3.10)$$

The solution to the mapping problem implied by (3.8) and (3.10) is obviously

$$H(b^2,s)=\frac{N(b^2)}{1-N(b^2)/2(-s)^{\frac{1}{2}}}, \quad (3.11)$$

where $N=-\lambda K_0(b)$. We see that both unitarity and analyticity are satisfied for sufficiently large s. This is an improvement over our previous discussion of Eq. (2.6) which demonstrated only that the form (3.11) satisfied the optical theorem.

Unfortunately it does not seem possible to carry out in closed form the integrals which define the kernel G. Nevertheless, the exact unitarity statement, Eq. (3.7), proves that the Fourier-Bessel representation of $f(s,t)$ can in principle form a complete dynamics.

There is another form of unitarity which should be discussed at this point since it has some practical importance. The unitarity condition, when applied to $f(s,t)$ written in the Mandelstam form [valid for Yukawa-like potentials, Eqs. (2.24–2.25)], yields

$$\mathrm{Im}A_3(t,s)=\int\frac{dt_1}{\pi}\int\frac{dt_2}{\pi}\times A_3^*(t_1,s)A_3(t_2,s)K(t_1,t_2,t;s), \quad (3.12)$$

where

$$K=\frac{\pi}{2}\frac{\theta[t^{\frac{1}{2}}-t_1^{\frac{1}{2}}(1+t_2/4s)^{\frac{1}{2}}-t_2^{\frac{1}{2}}(1+t_1/4s)^{\frac{1}{2}}]}{[s\{t-(t_1^{\frac{1}{2}}+t_2^{\frac{1}{2}})^2\}\{t-(t_1^{\frac{1}{2}}-t_2^{\frac{1}{2}})^2\}-t_1t_2t]^{\frac{1}{2}}}. \quad (3.13)$$

This condition can be easily rewritten in terms of the weight function $h(z,s)$ defined by (2.19),

$$\mathrm{Im}h(z,s)=\int\frac{dz_1}{\pi}\int\frac{dz_2}{\pi}h^*(z_1,s)h(z_2,s)\mathcal{G}(z_1,z_2,z;s), \quad (3.14)$$

where

$$\mathcal{G}(z_1,z_2,z;s)=-\frac{1}{8}\int dt_1\int dt_2\int dt\, J_0[(t_1z_1)^{\frac{1}{2}}]J_0[(t_2z_2)^{\frac{1}{2}}]\times J_0[(tz)^{\frac{1}{2}}]K(t_1,t_2,t;s). \quad (3.15)$$

This is the form of unitarity appropriate to the complex impact parameter representation, Eq. (2.39).

It is now possible to develop an iteration procedure for systematically calculating the function N by means of the unitarity condition and analyticity. This procedure starts from the exact formula for N in terms of H:

$$N(b^2,s)=H(b^2,s)/[1+H(b^2,s)/2(-s)^{\frac{1}{2}}], \quad (3.16)$$

and we recall that, [see Eq. (2.25)],

$$H(b^2,s)=\int\frac{dt'}{\pi}K_0[b(t')^{\frac{1}{2}}]A_3(t',s).$$

In order to get an interesting construction procedure, one must rearrange the normal perturbation series. This can be done, for example, by expanding the denominator

[9] In this connection see also D. Fivel (to be published).

of (3.16) to a given order in powers of $K_0(b)$, or, equivalently for large b, in powers of $\exp(-b)$ (recall the properties of K_0). This makes sense because if we are interested in small momentum transfer (large impact parameter) the expansion should converge rapidly. The resulting approximation can be quite different from perturbation theory in that one can put into $\mathrm{Im}A_3$, Eq. (3.12), the A_3's calculated in the previous iteration which contain all powers of the potential. This construction has been carried out to second order in the previous section.

The expansion of the denominator in N is valid in two limits, large s and/or large b. If we restrict our attention to values of t less than, say, nine, then we should get an adequate approximation by expanding N up to terms of order $\exp(-3b)$. Thus for sufficiently large s, the expansion yields

$$N=-\lambda K_0(b)+\int_4^9 dt\, K_0[b(t)^{\frac{1}{2}}]A_3(t,s) +\frac{\lambda^2}{2s}[K_0(b)]^2+O[\exp(-3b)]. \quad (3.17)$$

Now in the region $4<t<9$, A_3 is given exactly by the second Born approximation. Since this expansion will yield the exact t discontinuity of $f(s,t)$ up to $t=9$, it should form a reasonable approximation for smaller values of $|t|$.

If the energy is too small, the expansion will fail since it doesn't satisfy the requisite boundary conditions at $s=0$. The lower limit on the range of validity in energy comes from demanding that the denominator be expandable; i.e.,

$$|N(b^2,s)/2(-s)^{\frac{1}{2}}|<1$$

for the important values of b. Now to determine the relevant values of b, we note that the factor $J_0[b(-t)^{\frac{1}{2}}]$ is essentially unity for values of its argument less than one. The structure of the representation is explored only for larger values. The energy limit therefore becomes [in the crudest approximation, $N\sim-\lambda K_0(b)$]

$$|s^{\frac{1}{2}}|>|\lambda K_0[1/(-t)^{\frac{1}{2}}]| \sim|\lambda|[\pi(-t)^{\frac{1}{2}}/2]^{\frac{1}{2}}\exp[-1/(-t)^{\frac{1}{2}}].$$

If the largest t values considered are minus nine, we find

$$|s^{\frac{1}{2}}|>2^{\frac{1}{2}}\lambda.$$

In the physical case of nucleon-nucleon scattering, the energy limit is (with λ some reasonable value $\sim 2.2\,\mu$ where μ is the meson mass)

$$E=s/m>9\,\mu^2/m=1.3\,\mu\sim 180\text{ Mev}.$$

If the maximum t is restricted to be four, the energy limit decreases to 60 Mev. The above estimates are quite conservative since we have neglected the second and third terms in Eq. (3.17).

The freedom in the $O[\exp(-3b)]$ terms can be used in any phenomenological treatment of the small s dependence. This possibility certainly must be used in order to get the correct behavior at zero energy and to insure the presence of the deuteron bound state in this example. We will examine the problem of bound states and resonances in our representation in some detail later.

Another way of proceeding, which starts from the high-energy result, Eq. (3.11), and works to lower energies, is as follows: One simply inserts (3.11), called H_0, into the unitarity condition (3.7) and calculates an improved $\mathrm{Im}H_1$, and hence an improved H_1. This, in turn, leads to new values of N, D, and H:

$$N_2=D_0H_1,$$
$$D_2=1-N_2/2(-s)^{\frac{1}{2}},$$
$$H_2=N_2/D_2,$$

and so on.

The expansion of $f(s,t)$ in partial waves is interesting because the unitarity condition takes on such a simple form for these amplitudes. Define

$$f_l(s)=\frac{1}{2}\int_{-1}^{+1}dz\,P_l(z)f(s,\,-2s(1-z)) =\int_0^\infty\frac{db}{k}J_{2l+1}(2kb)H(b^2,s). \quad (3.18)$$

From Eq. (2.25) it follows that $H(b^2,0)$ is finite. Therefore the correct threshold behavior for every partial wave is assured. When s gets very large, it is easy to see that

$$f_l(s)\sim-\frac{\lambda}{k}\int_0^\infty db\,J_{2l+1}(2kb)K_0(b) =-(\lambda/2s)Q_l(1+1/2s), \quad (3.19)$$

which is the correct Born approximation result.

Let us re-examine $f_l(s)$ for very large l. If the approximation

$$P_l(\cos\theta)\sim J_0[(l+\tfrac{1}{2})\theta]$$

is used in the partial-wave projection, and the upper limit on the angular integration is extended to infinity, which is justified for sufficiently large l, then one finds

$$f_l(s)=(1/2s)H(b^2,s),\quad bk=l+\tfrac{1}{2}, \quad (3.20)$$

which reinforces our physical interpretation of b as the classical impact parameter. Further, we see that

$$|e^{i\delta}\sin\delta|^{-2}=\left(\frac{2k}{|N|^2}\mathrm{Re}N\right)^2+\left(1+\frac{2k}{|N|^2}\mathrm{Im}N\right)^2.$$

In this limit, then, $\mathrm{Im}N>0$ in order that absorption should always damp the elastically scattered wave.

For fixed k, we note from Eq. (3.20) that for large l, b becomes very large also. Hence only the first Born approximation, which has the longest range, survives:

$$f_l(s)\sim-(\lambda/2s)K_0(l/k). \tag{3.21}$$

This is equivalent to the result (3.19) by virtue of the relation

$$Q_l(1+1/2s)\sim K_0(l/k) \tag{3.22}$$

for $l\gg1$.

IV. BOUND STATES AND RESONANCES

The previous discussion was carried out under the assumption that the potential was not strong enough to produce bound states. Let us now examine the modifications that must be made in our representation as the potential increases in strength. Regge[10] has proven that the scattering amplitude can increase no faster than a (complex) finite power of t for a Yukawa-type potential. The magnitude of the power is related to the highest angular momentum state which is bound or can resonate. To see how this comes about from our point of view, we will first assume that there are no bound states and also that the scattering amplitude is bounded by some power of t. The consistency of the latter assumption will be demonstrated later by actually calculating the power for large energies. It is convenient to express our assumption in terms of the weight function $A_3(t,s)$:

$$\lim_{t\to\infty}A_3(t,s)=\beta(s)(t)^{\alpha(s)}+\cdots. \tag{4.1}$$

Even if there are bound states present, $\beta(s)$ has no poles in s. From this behavior we see that the small b dependence of $H(b^2,s)$ is [see Eq. (2.25)]

$$\lim_{b\to0}H(b^2,s)=2^{1+2\alpha}\Gamma^2(1+\alpha)\beta(s)b^{-2(1+\alpha)}. \tag{4.2}$$

This is the dominant term in H if $\mathrm{Re}\alpha(s)>-1$, which we shall assume to be the case. From the limiting s behavior of H, it is seen immediately that [see Eqs. (2.26a,b)]

$$i\chi\simeq[\sigma(0,s)/\pi]\ln b^2+\ln[\gamma(s)/(-s)^{\frac{1}{2}}]+\cdots \tag{4.3}$$

$$\simeq-(1+\alpha)\ln b^2+\ln[\gamma(s)/(-s)^{\frac{1}{2}}]+\cdots, \tag{4.4a}$$

and

$$\begin{aligned}N(b^2,s)&\simeq-2i(s)^{\frac{1}{2}}[1+O(b^{2(1+\alpha)})],\\ D(b^2,s)&\simeq-2i(s)^{\frac{1}{2}}b^{2(1+\alpha)}/\gamma(s),\end{aligned} \tag{4.4b}$$

where $\gamma(s)$ is a function proportional to $\beta(s)$ and some gamma functions. Now $H(b^2,s)$ must be analytic in the cut s plane; this will be insured if $\sigma(z,s)$ has the same analyticity in s for all z. Therefore $\alpha(s)$ must enjoy this analyticity also. From the limiting condition on σ for large s, Eqs. (2.17) and (2.31), it follows that

$$\lim_{s\to\infty}\alpha(s)=-1+\frac{\lambda^2}{4s}\ln(1+1/4s)-\frac{i\lambda}{2(s)^{\frac{1}{2}}}\times\left[1-\frac{\lambda}{(s)^{\frac{1}{2}}}\tan^{-1}(1/(s)^{\frac{1}{2}})\right]+\cdots. \tag{4.5}$$

Thus $\alpha(s)$ may be expressed as

$$\alpha(s)=-1+\int_0^\infty\frac{ds'}{\pi}\,\mathrm{Im}\alpha(s')(s'-s)^{-1}. \tag{4.6}$$

We are well aware that the analyticity of $\alpha(s)$ has not been proven.[11] In addition there may be some delicacies in taking the limit of large t and large s.

To achieve some further insight into these problems, let us consider the question from the standpoint of the N/D solution, Eq. (3.11), which satisfies analyticity and asymptotic unitarity. We have

$$\begin{aligned}f(s,t)&=-\lambda\int_0^\infty bdb\,J_0[b(-t)^{\frac{1}{2}}]K_0(b)\left[1+\frac{i\lambda}{2k}K_0(b)\right]^{-1}\\&=\frac{4s}{i\lambda}\int_0^\infty bdb\,J_0[b(-t)^{\frac{1}{2}}]\left[\frac{2k}{\lambda}+iK_0(b)\right]^{-1},\end{aligned} \tag{4.7}$$

since $(-t)$ is nonzero. For large values of $(-t)$, only the small-b behavior is important and therefore the small argument expansion of $K_0(b)$ may be used. The identity

$$\left[\frac{2k}{\lambda}-i\ln\frac{\gamma b}{2}\right]^{-1}=\int_0^\infty dx\exp\left[-x\left(\frac{2k}{\lambda}-i\ln\frac{\gamma b}{2}\right)\right],$$

allows the b integral to be carried out with the aid of W, p. 391, and the scattering amplitude becomes

$$f(s,t)=-\frac{4s}{\lambda}\int_0^\infty xdx\,(-t)^{-1-i(x/2)}2^{ix}\Gamma(1+\tfrac{1}{2}ix)\times\Gamma^{-1}(1-\tfrac{1}{2}ix)\exp\left[-x\left(\frac{2k}{\lambda}-i\ln\frac{\gamma}{2}\right)\right]. \tag{4.8}$$

Since the integrand peaks at (approximately)

$$1/x_0=2k/\lambda+i\ln[2(-t)^{\frac{1}{2}}/\gamma],$$

one finds

$$f(s,t)\simeq-\frac{4s}{\lambda}\frac{2^{ix_0}}{(-t)}\frac{\Gamma(1+ix_0/2)}{\Gamma(1-ix_0/2)}\left[\frac{2k}{\lambda}+i\ln\frac{2(-t)^{\frac{1}{2}}}{\gamma}\right]^{-2}. \tag{4.9}$$

We are interested in large (k/λ) and large $(-t)$; therefore x_0 may be set equal to zero.

At fixed (but large) (k/λ), we see that the approximate scattering amplitude does *not* behave as a power of

[10] T. Regge, Nuovo cimento **18**, 947 (1960).

[11] The analyticity of $\alpha(s)$ has recently been rigorously established by C. Bottino, P. Longoni, and T. Regge (to be published).

$(-t)$ asymptotically. However, in the limit in which we expect the approximation for N/D to be most accurate, namely

$$(k/\lambda) \gg \ln[(-t)^{\frac{1}{2}}], \tag{4.10}$$

$f(s,t)$ does behave as $(-t)^{\alpha}$, where

$$\alpha(s) = -1 - i\lambda/2k, \tag{4.11}$$

which agrees with Eq. (4.5) for large s. Note that Eq. (4.9) is much less restrictive than one would naively expect. It requires only that the momentum transfer be small compared to $\exp[(s)^{\frac{1}{2}}/\lambda]$ and *not* $[(s)^{\frac{1}{2}}/\lambda]$.

It may seem surprising that the limiting behavior of $\alpha(s)$ for large s must insure the recovery of the first Born approximation since $\alpha(s)$ was defined essentially in terms of the double spectral function, Eq. (4.1), which is the scattering amplitude with the first Born approximation subtracted out. To see how this comes about, we reconsider the approximate evaluation of Eq. (4.7) but in a slightly different form. If the first Born approximation is explicitly exhibited, the scattering amplitude becomes

$$f(s,t) = -\lambda(1-t)^{-1} + \frac{i\lambda^2}{2k}\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}] \times K_0^2(b)\left[1+\frac{i\lambda}{2k}K_0(b)\right]^{-2}.$$

Evaluating the integral in the same manner as before, we find

$$f(s,t) = -\lambda(1-t)^{-1} + \lambda[(-t)^{-1} - (-t)^{\alpha(s)}] + \cdots, \tag{4.12}$$

where $(-t)$ has been assumed to be large, but the energy is also sufficiently large that Eq. (4.10) obtains. The function $\alpha(s)$ is given by Eq. (4.11). This result has two important properties. For large t the Born term is cancelled by the first term in the square bracket and one achieves the desired form,

$$f(s,t) = -\lambda(-t)^{\alpha(s)} + \cdots. \tag{4.13}$$

If the square bracket is expanded as a power series in the coupling constant, then f becomes

$$f(s,t) = -\lambda(1-t)^{-1} + \frac{i\lambda^2}{2k}\frac{\ln(-t)}{(-t)} + \cdots. \tag{4.14}$$

We thus see how the power law dependence on $(-t)$ comes from summing the entire perturbation series in which each term has a quite different asymptotic dependence on $(-t)$.

Now let us examine the large $(-t)$ limit of the scattering amplitude more generally. For large $(-t)$ only the small-b behavior of $H(b^2,s)$ plays a role. Using Eq. (4.2) in the representation leads to

$$f(s,t) = -\beta(s)(-t)^{\alpha(s)}[\sin\pi\alpha(s)]^{-1}. \tag{4.15}$$

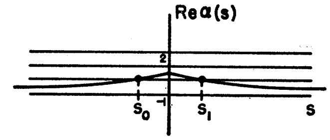

FIG. 1. An S-wave bound state.

This of course behaves as a function of t just as we forced it to, namely $(-t)^{\alpha}$; it also follows directly by substituting (4.1) into a dispersion integral. However, one does not then have any knowledge of the power $\alpha(s)$; in particular, the limiting value given by (4.5) is not easily discernable.

Now $\alpha(s)$ is real for negative s and complex for positive s. If α were to pass through zero for negative s on its way up from the limiting value $\alpha(-\infty) = -1$, then the amplitude f would possess a pole in energy corresponding to an S-wave bound state. It would be quite accidental if $\beta(s)$ were to vanish at the point where $\alpha(s) = 0$. The pole when $\alpha(s)$ is (-1) has been artificially introduced by our approximate treatment of the lower limit in the evaluation of (2.25).

The relation of bound states to the subtraction problem in a dispersion relation in t is now clear. For large t, the scattering amplitude behaves like $(-t)^{\alpha(s)}$, and if there is a bound S state, $\mathrm{Re}\alpha(s)$ looks typically like Fig. 1. The important point is that for s between s_0 and s_1, $\mathrm{Re}\alpha(s)$ lies between zero and one. Thus, one subtraction is required in the t dispersion integral to insure convergence if s is in this region. We may note that independent of the number of bound states, an unsubtracted dispersion relation in t holds if s is large enough.

If the potential is increased still further, $\mathrm{Re}\alpha(s)$ may cross one for negative s and a P-wave bound state has formed. We see that it must have less binding energy than the S wave. Now a P-wave bound state produces a pole in the scattering amplitude of the form

$$(t+2s_2)(s-s_2)^{-1}.$$

The constant term, $2s_2$, has been missed in our discussion because only the most singular part of H at $b=0$ has been studied.

Actually another S-wave bound state might be formed before the P wave. This can happen if the function H has the behavior

$$H(b^2,s) = Ab^{-2(1+\alpha)} + A_1 b^{-2(1+\alpha_1)} + \cdots,$$

where $\alpha > \alpha_1$. The second term has been neglected in our previous discussion, but can be easily included. The second bound state occurs when $\alpha_1(s)$ vanishes. We conjecture that there are in fact an infinite number of such α's, since by increasing the potential strength, one may form an arbitrary number of bound states.

Even if there are no bound states, subtractions still might be formally necessary if $\mathrm{Re}\alpha(s)$ behaves as in Fig. 2. This type of behavior would necessitate a subtraction in the t dispersion relation since for s in the range between s_2 and s_3 the integral would not be abso-

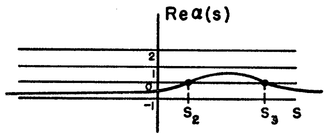

FIG. 2. S-wave resonances.

lutely convergent. However, the integral actually does exist if done with sufficient care by virtue of the oscillations in t due to the imaginary part of $\alpha(s)$. Now if $\mathrm{Im}\alpha(s)$ is small in the neighborhood of the point where $\mathrm{Re}\alpha(s)=0$, then there will be a scattering resonance in $f(s,t)$ due to the fact that the sine almost vanishes. These would be S-state resonances since the t dependence is essentially lost at these points. The more conventional way of stating this fact is that there is a complex value of s on the second sheet such that $\alpha(s)=0$. This then produces the familiar Breit-Wigner pole. At these values of the energy, the resonances have a pure angular momentum character. If the imaginary part of $\alpha(s)$ is large, these points may not show up as obvious resonances in the scattering amplitude, even though the phase shift does pass through $\pi/2$.

It is useful to note that as $\mathrm{Re}\alpha(s)$ rises through integer values with increasing energy, the amplitude takes the following form in the neighborhood of $\mathrm{Re}\alpha(s_0)=l$:

$$\frac{\beta(s)(-t)^{\alpha(s)}}{\sin\pi\alpha(s)}\simeq\frac{\beta(s_0)(-1)^l(-t)^l}{\pi[l-\alpha(s)]}=\frac{\text{const}\times t^l}{s_0-s-i\Gamma/2},$$

where

$$\Gamma/2=\mathrm{Im}\alpha(s_0)[d\,\mathrm{Re}\alpha(s_0)/ds_0]^{-1}.$$

[We have assumed that $\mathrm{Im}\alpha(s_0)$ is very small]. This corresponds to a true Breit-Wigner resonance only if $\mathrm{Im}\alpha(s_0)>0$. It is unreasonable to expect $\mathrm{Im}\alpha$ to vanish in the physical region, consequently when $\mathrm{Re}\alpha(s)$ *decreases* through an integer we do not have a resonance of the usual variety (i.e., $\Gamma>0$).

This type of analysis suggests an interesting classification of resonances and bound states. If $\mathrm{Re}\alpha(s)$ is plotted, it is clear that after one resonance has occurred, the *next* resonance in energy must obey the selection rule

$$\Delta l=1, \tag{4.16}$$

unless a new family of resonances has appeared. Two such families of resonances are plotted in Fig. 3. Since a new family must start with an S-wave resonance, there is no ambiguity in telling them apart experimentally. This rule must be modified if there are symmetry requirements, such as the Pauli principle, which restricts the available angular momentum states, or if exchange potentials are present. (See Sec. VII.)

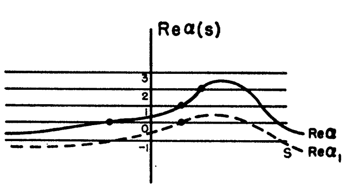

FIG. 3. Two families of resonances.

In terms of partial waves, the resonances show up in the following way. Using the result (3.18), the small-b behavior of H leads to

$$\begin{aligned}f_l(s)&=\int_0^\infty\frac{db}{k}J_{2l+1}(2kb)H(b^2,s)\\&=\beta(s)\Gamma^2(1+\alpha)(2k)^{2\alpha}\\&\qquad\times\Gamma(l-\alpha)\Gamma^{-1}(l+2-\alpha)+\cdots.\end{aligned} \tag{4.17}$$

Thus the resonances and bound states occur when

$$\mathrm{Re}\alpha(s)=l. \tag{4.18}$$

This shows incidentally that the resonant behavior cannot be associated with $\beta(s)$ since it occurs multiplicatively for all angular momenta.

We have argued that any absorption that is present should manifest itself in a positive imaginary contribution to χ or to N. Therefore, the same must be true of α. For the imaginary part of α coming from absorption, we would then have

$$\mathrm{Im}\alpha^a(s)>0.$$

This means that the absorption *increases* $\alpha(s)$ for s below the rise of $\mathrm{Im}\alpha^a(s)$. Absorption will therefore tend to drive $\alpha(s)$ up through a resonance,[12] which may occur well below the inelastic threshold.

A powerful approach has been developed by Regge[10] to discuss the large momentum transfer behavior of the scattering amplitude. He extends the S-matrix elements $\exp(2i\delta_l)$ to complex values of the angular momentum and shows that there are singularities in the l plane. The poles correspond precisely to the singularities in Eq. (4.15) when $l=\alpha(s)$. He shows that there is a term in the scattering amplitude of the form

$$\beta(s)P_\alpha(-z)/\sin\pi\alpha(s),$$

which is the same as (4.15) for large t.

Up to this point we have been able to give a quantitative discussion of $\alpha(s)$ for large s only. In order to improve this situation, let us consider directly the radial Schrödinger equation for an arbitrary complex value of the angular momentum. We write the identity

$$\alpha(\alpha+1)\equiv I(s)=\frac{-\displaystyle\int_0^\infty dr\,\psi\left[-\frac{\partial^2}{\partial r^2}+V(r)-s\right]\psi(r)}{\displaystyle\int_0^\infty dr\,\psi\psi/r^2}. \tag{4.19}$$

This is easily seen to be a variational principle for $I(s)$

[12] J. Ball and W. Frazer, Phys. Rev. Letters 7, 204 (1961), also L. Cook and B. Lee (to be published), have discussed this effect in the partial wave case for the higher pion-nucleon resonances.

if the wave functions are defined to vanish at the origin sufficiently fast and to behave at infinity like $\exp[-(-s)^{\frac{1}{2}}r]$.

To clarify the significance of the complex angular momentum, consider the problem of the direct integration of the Schrödinger equation. If we consider an arbitrary negative energy and integrate outward from the origin, where the wave function vanishes like r^{l+1}, we must choose l so that the solution vanishes exponentially at infinity. The true bound-state eigenvalues follow from the requirement that $l=\alpha(s)=$positive integer. For positive energies the solution regular at the origin has the general form for large r:

$$\psi\simeq(-1)^l \exp[(-s)^{\frac{1}{2}}r]-S_l(s)\exp[-(-s)^{\frac{1}{2}}r]; \quad (4.20)$$

the critical value of l for a given s is that for which we have only an outgoing wave, which means $S_l(s)$ becomes infinitely large. These are the singularities in the l plane referred to above, $l=\alpha(s)$. This is in contrast to the usual way of proceeding in which one fixes the angular momentum and then chooses the energy so that the wave function satisfies both boundary conditions. Here the energy is fixed and the angular momentum is chosen to make the wave function satisfy the requisite boundary conditions.

To illustrate the use of the variational principle let us consider negative energies and a Yukawa potential of unit range. We choose a trial function of the form

$$\psi=r^\sigma \exp(-\gamma r), \quad (4.21)$$

where $\gamma^2=-s$ and σ is the variational parameter. In the exact solution, the behavior of the wave function at the origin should be r^{l+1}. This trial function in Eq. (4.19) leads to

$$I=-\sigma^2-(2\sigma-1)(\lambda/2\gamma)(2\gamma/2\gamma+1)^{2\sigma}. \quad (4.22)$$

In performing the integrals in I we have assumed that σ is greater than one-half. We will use Eq. (4.22) to define I for *all* σ.

Our next task is to find the extremum of this expression by varying σ. This is in general very tedious to do analytically but there are two limits in which it is quite simple.

The first simplification occurs for large γ. Then I may be expanded in powers of $(1/\gamma)$ and one finds that the appropriate values of σ and I are

$$\sigma=-(\lambda/2\gamma)(1+1/2\gamma+\cdots)$$

and

$$I=(\lambda/2\gamma)(1+\lambda/2\gamma)+\cdots. \quad (4.23)$$

This leads to

$$\alpha(s)=-\tfrac{1}{2}\pm\tfrac{1}{2}(1+4I)^{\frac{1}{2}}$$
$$=-\tfrac{1}{2}\pm\tfrac{1}{2}(1+2I-2I^2+\cdots).$$

The appropriate choice of the sign of the square root turns out to be the lower one, as we shall demonstrate, hence

$$\alpha(s)=-1-\lambda/2\gamma+O(1/\gamma^3), \quad (4.24)$$

which agrees precisely with our previous result, Eq. (4.5). One might worry that the square root in $\alpha(s)$ will introduce a branch point at the value of s for which I is $(-\frac{1}{4})$. We would like to argue that this branch point does not in fact occur. The point is, that from the definition of I, we see that it has an absolute minimum of $(-\frac{1}{4})$ for real $\alpha(s)$. If this minimum occurs for a finite value of γ, say γ_0, then we can write

$$I(\gamma)=-\tfrac{1}{4}+(\gamma-\gamma_0)^2\tfrac{1}{2}I''(\gamma_0)+\cdots, \quad (4.25)$$

and in the neighborhood of γ_0 we find

$$\alpha(s)=-\tfrac{1}{2}+(\gamma-\gamma_0)[\tfrac{1}{2}I''(\gamma_0)]^{\frac{1}{2}}+\cdots. \quad (4.26)$$

Thus the point γ_0 is *not* a branch point of the function $\alpha(s)$.

In order to demonstrate this in detail let us return to our example, Eq. (4.13). The absolute minimum of I occurs when $\sigma=\frac{1}{2}$. The value of γ which makes

$$\left.\frac{\partial I(\gamma,\sigma)}{\partial\sigma}\right|_{\sigma=\frac{1}{2}}=0,$$

can be determined exactly for our model to be

$$\gamma_0=-\lambda-\tfrac{1}{2}.$$

The potential strength λ is negative for an attractive potential and we will assume that γ_0 is positive, i.e., $\lambda<-\frac{1}{2}$.

If we now expand I to second order about the minimum point by writing

$$\gamma=\gamma_0+x,$$
$$\sigma=\tfrac{1}{2}+\delta,$$

then the extreme value of I is found by varying δ and we find

$$I=-\tfrac{1}{4}+\frac{x^2}{4\lambda^2}\left[2\ln\left(\frac{2\lambda}{2\lambda+1}\right)+1\right]^{-1}+O(x^3).$$

Therefore,

$$\alpha(s)=-\tfrac{1}{2}+\frac{(\gamma+\lambda+\frac{1}{2})}{2\lambda}\times\left[2\ln\left(\frac{2\lambda}{2\lambda+1}\right)+1\right]^{-\frac{1}{2}}+\cdots. \quad (4.27)$$

This example bears out our general argument that $\alpha=-\frac{1}{2}$ is not a branch point. In an appendix we prove, using the methods of Regge, that for a Yukawa potential, α may be continued analytically below $\mathrm{Re}\,\alpha=-\frac{1}{2}$.

As another example of the variational principle, let us try to calculate the value of λ required to produce a bound S state at zero energy, again for a Yukawa potential. The wave function (4.21) is not very appropriate in this case and therefore we choose

$$\psi=1-e^{-ar}. \quad (4.28)$$

This leads to

$$I(\gamma=0,\,a)=\left[\frac{a}{2}+\lambda\ln\frac{(1+a)^2}{(1+2a)}\right]\Big/2a\ln 2. \quad (4.29)$$

The value of a required to extremize this expression is independent of λ and is found numerically to be approximately $a=1.40$. Therefore

$$I=(1+0.60\lambda)/4\ln 2. \quad (4.30)$$

The condition for $\alpha(0)$ to vanish is that I vanish, and this requires

$$\lambda=-1.67. \quad (4.31)$$

The exact value of λ required to produce a zero-energy bound state is $\lambda=-1.6798$. The variational principle therefore seems to be quite accurate.

Let us return to the problem of subtractions in the Mandelstam representation. It is well known that the scattering amplitude is an analytic function of the potential strength λ. From our previous discussion we expect that as the potential is made more attractive, a bound state is formed when $\alpha(s)$ vanishes for some real, negative value of s. In order to follow the formation of the bound state in detail, we will write our representation in such a way that the continuation in λ (or in α) can be carried out in a transparent way.

If $\mathrm{Re}\alpha(s)$ exceeds zero, then the Fourier-Bessel representation [Eq. (2.18)] breaks down at the lower limit of the b integration. Therefore let us consider the identity

$$H(b^2,s)=[H(b^2,s)-\bar{\beta}(s)b^{-2(1+\alpha)}]+\bar{\beta}(s)b^{-2(1+\alpha)}, \quad (4.32)$$

where

$$\bar{\beta}(s)=(1/\pi)2^{1+2\alpha}\Gamma^2(1+\alpha)\beta(s).$$

We have assumed that $\mathrm{Re}\alpha$ is negative. As $\mathrm{Re}\alpha$ increases through zero, the term in the square bracket still behaves in a reasonable manner at $b=0$. As long as $\mathrm{Re}\alpha<0$, we may write Eq. (4.32) as

$$H(b^2,s)=\int_0^\infty\frac{dz}{\pi}(z+b^2)^{-1}[h(z,s)-\bar{\beta}(s)z^{-(1+\alpha)}\sin\pi(1+\alpha)]+\bar{\beta}(s)b^{-2(1+\alpha)}. \quad (4.33)$$

The scattering amplitude then becomes

$$f(s,t)=\int_0^\infty\frac{dz}{\pi}K_0(z^{\frac{1}{2}}(-t)^{\frac{1}{2}})\times[h(z,s)-\bar{\beta}(s)z^{-(1+\alpha)}\sin\pi(1+\alpha)]-(-t)^{\alpha}\beta(s)[\sin\pi\alpha]^{-1}. \quad (4.34)$$

This equation can be continued in $\alpha(s)$ without difficulty. The first term remains well defined for values of α less than unity. The second term can be continued by hand and one sees that there is a pole at the point $\alpha(s)=0$, as expected.

If an explicitly subtracted Mandelstam representation is desired, one can easily achieve this goal by using the dispersion form for $K_0(b)$, Eq. (2.14), and also the formula

$$\beta(s)\int_0^\infty\frac{dt'}{\pi}t'^{\alpha(s)}(t'-t)^{-1}=-\beta(s)(-t)^{\alpha(s)}[\sin\pi\alpha]^{-1}.$$

The absorptive part, $A_3(t,s)$, turns out to be

$$A_3(t,s)=\frac{1}{2}\int_0^\infty dz\,J_0(z^{\frac{1}{2}}t^{\frac{1}{2}})\times[h(z,s)-\bar{\beta}(s)z^{-(1+\alpha)}\sin\pi(1+\alpha)]+\beta(s)t^{\alpha(s)}. \quad (4.35a)$$

This formula is valid if $\mathrm{Re}\alpha<1$. The last term with its explicit factor of t^α shows the need for a subtracted dispersion relation. This form for A_3 is to be compared with its analytic continuation to values of $\mathrm{Re}\alpha<0$, Eq. (2.23):

$$A_3(t,s)=\frac{1}{2}\int_0^\infty dz\,J_0(z^{\frac{1}{2}}t^{\frac{1}{2}})h(z,s). \quad (4.35b)$$

One can also show, by dispersing $b^2H(b^2,s)$ instead of just $H(b^2,s)$ in b^2, that for $\mathrm{Re}\alpha>0$,

$$A_3(t,s)=\frac{1}{2}\int_0^\infty dz\,[J_0(z^{\frac{1}{2}}t^{\frac{1}{2}})-1]h(z,s). \quad (4.35c)$$

These three forms are equivalent and are analytic continuations of each other. They can be rewritten in a universally valid fashion by using the fact that for any value of t less than one, A_3 vanishes. In particular, we have

$$A_3(0,s)=0.$$

Introducing the function

$$I(z,s)=\frac{1}{2}\int_z^\infty dz'\,h(z',s),$$

and integrating by parts, one gets the same result for each of the three forms for $A_3(t,s)$:

$$A_3(t,s)=-\int_0^\infty dz\,I(z,s)\frac{\partial}{\partial z}J_0(z^{\frac{1}{2}}t^{\frac{1}{2}}). \quad (4.36)$$

This form holds for any value of α less than unity. We see that the bound-state pole position and residue, and the single dispersion integrals are all completely specified by an analytic continuation in the coupling constant.

V. THE COULOMB SCATTERING AMPLITUDE

It will prove instructive to illustrate our previous general discussion by means of an example. Even though there are some difficulties with the long-range character of the Coulomb potential, we will consider this case because a cutoff can be introduced and the problem can be solved exactly. In a nonforward direction the scattering amplitude takes the form[2]

$$f(s,t)=(-s)^{\frac{1}{2}}\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}](4a^2/b^2)^{1+\alpha(s)}, \quad (5.1)$$

where

$$1+\alpha(s)=-(Ze^2M/\hbar^2)[(-s)^{\frac{1}{2}}+1/a]^{-1}, \quad (5.2)$$

and "a" is a parameter which has been introduced to make the integrals well defined and will be allowed to go to infinity at the end of the calculation. A positive (negative) Z corresponds to a repulsive (attractive) potential. The integral over b is well defined if $\alpha(s)$ is negative. Thus, we require that

$$-(Ze^2M/\hbar^2)a<1.$$

This restriction will be relaxed later.

The integral over the impact parameter can be carried out with the aid of W and the result is

$$f(s,t)=-\frac{2(-s)^{\frac{1}{2}}(1+\alpha)}{t}\frac{\Gamma(-\alpha)}{\Gamma(2+\alpha)}(-a^2t)^{1+\alpha}, \quad (5.3)$$

which gives the correct cross section in the limit of infinite "a" since all dependence on "a" is isolated in a phase factor.

If we make use of the representation

$$(4a^2/b^2)^{1+\alpha}=-\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{dz}{z+b^2}(4a^2/z)^{1+\alpha}, \quad (5.4)$$

then we find

$$h(z,s)=-(-s)^{\frac{1}{2}}[\sin\pi\alpha](4a^2/z)^{1+\alpha}. \quad (5.5)$$

This leads immediately to

$$A_3(t,s)=[2\pi(-s)^{\frac{1}{2}}/t](a^2t)^{1+\alpha}\Gamma^{-2}(1+\alpha). \quad (5.6)$$

It is now possible to rewrite $f(s,t)$ in double dispersion form. The result is

$$f(s,t)=\frac{2Ze^2M}{\hbar^2 t}+\int_0^\infty\frac{ds'}{\pi}\int_0^\infty\frac{dt'}{\pi}\frac{\rho(s',t')}{(s'-s)(t'-t)}, \quad (5.7)$$

where

$$\rho(s,t)=-(\pi s^{\frac{1}{2}}/t)\times[(a^2t)^{1+\alpha}\Gamma^{-2}(1+\alpha)+(a^2t)^{1+\alpha^*}\Gamma^{-2}(1+\alpha^*)]. \quad (5.8)$$

The infinite range of the Coulomb potential manifests itself in the fact that the spectral function $\rho(s,t)$ is nonzero in the entire positive quadrant of the $s-t$ plane.

The formation of bound states can be studied with this model either by performing a continuation from the second sheet[13] or by the method discussed in the previous section. The interesting factors in the scattering amplitude are in the same form that occurred before:

$$\Gamma(-\alpha)(-t)^\alpha=-\pi(-t)^\alpha\Gamma^{-1}(1+\alpha)[\sin\pi\alpha]^{-1}.$$

For example, the lowest bound S state yields a pole in the scattering amplitude of the form

$$4a^2\frac{s_0(s_0^{\frac{1}{2}}+1/a)}{s+s_0}, \quad (5.9)$$

where

$$s_0^{\frac{1}{2}}=[-(Ze^2M/\hbar^2)-(1/a)].$$

The potential has been assumed attractive and a must be large enough so that s_0 is real. The bound states are seen to occur with the correct energy (as a goes to infinity) and with the correct degeneracy. That is, the pole that occurs at the point $\alpha(s_L)=L$ has a residue that varies as t^L. This form contains bound states at s_L with all angular momentum up to the value L.

It is also instructive to consider the variational principle for $\alpha(s)$ in the Coulomb case. Using the same trial function as before, Eq. (4.21), we find

$$I=-\sigma^2-(2\sigma-1)Ze^2M/\hbar^2\gamma.$$

The extreme value of I is

$$I=(Z/a_0\gamma)(1+Z/a_0\gamma),$$

where $a_0=\hbar^2/Me^2$. The function $\alpha(s)$ turns out to be

$$\alpha(s)=-(1+Z/a_0\gamma)$$

and therefore the bound energy levels are given by[14]

$$\gamma^2=(Z/a_0)^2(1+\alpha)^{-2},$$

where we require α to be 0, 1, 2, $\cdots$.

VI. COUPLED CHANNEL CASE

It has been demonstrated[15] that the Mandelstam representation holds for the coupled channel case if the potentials are of the Yukawa form. The binding energies of the channels must also obey certain inequalities to avoid the problem of anomalous thresholds. In order to treat the many-channel problem in terms of the high-energy approximation, we are naturally led to a matrix notation. Thus we would generalize the phase function to a symmetric matrix. Instead of discussing this representation, we will confine ourselves to the N/D solution since it can be easily generalized to field theory.

In order to satisfy the matrix unitarity condition

$$\mathrm{Im}f(s,t_{13})=\int\frac{d\Omega_2}{4\pi}f(s-i\epsilon,\,t_{12})\rho(s)f(s+i\epsilon,\,t_{23}), \quad (6.1)$$

where ρ is a diagonal phase space matrix, we consider

[13] R. Blankenbecler, M. Goldberger, S. MacDowell, and S. Treiman, Phys. Rev. **123**, 692 (1961).

[14] This result was derived independently by N. Bohr, Phil. Mag. **26**, 1 (1913).

[15] L. Fonda, L. Radicati, and T. Regge, Ann. Phys. **12**, 68 (1961).

the matrix form[16]

$$f(s,t)=\int_0^\infty bdb\, J_0[b(-t)^{\frac{1}{2}}]N(b^2,s)D^{-1}(b^2,s), \quad (6.2)$$

where

$$D(b^2,s)=1-\int_0^\infty \frac{ds'}{\pi}(s'-s)^{-1}r(s')\,\mathrm{Re}N(b^2,s') \quad (6.3)$$

and

$$\rho_i(s)=2k_i^2(s)r_i(s),$$

where $k_i^2(s)$ is the square of the wave number in the appropriate channel. Performing the standard manipulations, it is easily demonstrated that this scattering amplitude satisfies

$$\mathrm{Im}f(s,0)=\int\frac{d\Omega}{4\pi}f(s-i\epsilon)\rho f(s+i\epsilon)+\int_0^\infty bdb\,[D^T(s-i\epsilon)]^{-1}\times\mathrm{Im}ND^{-1}+O(1/s). \quad (6.4)$$

Thus we see that for s values large compared to the excitation energies of the channels, unitarity is satisfied. Again, if we choose ImN appropriately, unitarity can be satisfied exactly.

Let us examine the problem of bound states and resonances in the many-channel case. Again only the small-b limit of N and D^{-1} need be considered. As in the single-channel problem, we expect that

$$N(0,s)=\text{constant matrix},$$
$$\det[D(b^2,s)]=C(s)b^{2[1+\alpha(s)]}+\cdots.$$

Therefore, the scattering amplitude becomes in the large t limit:

$$f(s,t)=B(s)(-t)^{\alpha(s)}[\sin\pi\alpha(s)]^{-1}+\cdots, \quad (6.5)$$

where $B(s)$ is a matrix. This demonstrates the fact that if a bound state occurs in one channel, it must occur in all channels in order to satisfy unitarity. This statement is also seen to be true for resonances. Finally, we remark that the large-t behavior is the same for *all* channels, a most remarkable result.

VII. FIELD THEORY[17]

As an example of the application of the Fourier-Bessel representation to field theory, we will consider first the case of nucleon-nucleon elastic scattering. This restriction is in no way required or essential. The inelastic effects do not seem important even in the medium energy range for this process and therefore the elastic unitarity approximation would seem reasonable. These effects could be included by using the matrix representation discussed in Sec. VI. The complications due to spin and isotopic spin will be neglected in our discussion, but the effects of both a direct and exchange force will be included.

The amplitude will be considered as a function of the energy squared, s, and the two momentum transfers, t and u. Some useful kinematical relations in the center-of-mass system are

$$\begin{aligned} s+t+u&=4m^2,\\ t&=-2p^2(1-z),\\ u&=-2p^2(1+z),\\ 4p^2&=s-4m^2, \end{aligned} \quad (7.1)$$

where m is the nucleon mass.

The Mandelstam representation is written in the form[18]

$$M(s,t,u)=\int_1^\infty\frac{dt'}{\pi}\frac{A_3(t',s)}{t'-t}+\int_1^\infty\frac{du'}{\pi}\frac{A_2(u',s)}{u'-u}, \quad (7.2)$$

where the absorptive parts take the general form

$$A_i(x,s)=\Gamma_i\delta(x-1)+\theta(x-4)\sigma_i(x)+\int_{4m^2}^\infty\frac{ds'}{\pi}\rho_i(x,s')(s'-s)^{-1}+\int_{4m^2}^\infty\frac{dx'}{\pi}\rho(x',x)(x'+x+s-4m^2)^{-1}, \quad (7.3)$$

and all energies are measured in units of the pion mass.

The unitarity relation in the elastic approximation is

$$\mathrm{Im}M(s,z_{13})=p^2\int\frac{d\Omega_{12}}{4\pi}M^*(s,z_{12})r(s)M(s,z_{23}), \quad (7.4)$$

where

$$r(s)=m/p(s)s^{\frac{1}{2}}$$

and

$$z_{23}=z_{12}z_{13}+(1-z_{12}^2)^{\frac{1}{2}}(1-z_{13}^2)^{\frac{1}{2}}\cos\phi_{12}.$$

The differential cross section in terms of this amplitude is

$$d\sigma/d\Omega=|M|^2(4m^2/s). \quad (7.5)$$

The impact parameter representation of M is introduced as

$$M=\int_0^\infty bdb\,\{J_0[b(-t)^{\frac{1}{2}}]H_3(b^2,s)+J_0[b(-u)^{\frac{1}{2}}]H_2(b^2,s)\}, \quad (7.6)$$

[16] J. Bjorken, Phys. Rev. Letters **4**, 473 (1960); J. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961); R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

[17] We use the term "field theory" in the sense usually employed by dispersion theorists; there is no implication that the theory has necessarily anything to do with Lagrangians or "axiomatics." It is sometimes referred to as the "S-matrix theory."

[18] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

where

$$H_i(b^2,s)=\int_1^\infty \frac{dx}{\pi}K_0(bx^{\frac{1}{2}})A_i(x,s). \quad (7.7)$$

Our next step is to find a form for H_i which automatically guarantees that elastic unitarity is satisfied for large energies. As we shall see, the energy only needs to be large compared to the range of the interaction, and this can still be in the elastic unitarity region. Since both direct and exchange forces will be considered, we are free to attempt to satisfy unitarity in both the forward and backward directions. Carrying out the same manipulations that led to Eq. (3.10), we find two conditions in the field theory case:

$$\text{forward,} \quad \mathrm{Im}H_3(b^2,s)=r(s)[H_3{}^*H_3+H_2{}^*H_2], \quad (7.8)$$

$$\text{backward,} \quad \mathrm{Im}H_2(b^2,s)=r(s)[H_3{}^*H_2+H_2{}^*H_3]. \quad (7.9)$$

In order to solve these coupled equations, it is convenient to introduce amplitudes which are even and odd under exchange:

$$H_\pm(b^2,s)=H_3(b^2,s)\pm H_2(b^2,s). \quad (7.10)$$

In terms of these functions, asymptotic unitarity takes the simple form

$$\mathrm{Im}H_\pm(b^2,s)=r(s)|H_\pm(b^2,s)|^2. \quad (7.11)$$

Let us consider writing $H_\pm$ in the form

$$H_\pm(b^2,s)=N_\pm(b^2,s)D_\pm{}^{-1}(b^2,s), \quad (7.12)$$

where

$$D_\pm(b^2,s)=1-\int_{4m^2}^\infty \frac{ds'}{\pi}r(s')[\mathrm{Re}N_\pm(b^2,s')](s'-s)^{-1}, \quad (7.13)$$

and the quantities $N_\pm(b^2,s)$ are complex analytic functions of s. If they become real for large energies, then Eq. (7.11) is automatically satisfied. We entertain the possibility of complex $N_\pm$ in order to satisfy elastic unitarity exactly for all energies. Even in the case where inelastic processes are important, the form (7.12) is still appropriate, since it sums approximately those graphs which contribute to elastic unitarity. It is seen from Eqs. (7.7) and (7.3) that the H's and N's have both a positive and negative cut in s. In any approximation scheme, we can always make sure that the negative cuts in M combine in such a way that only positive cuts in s, t, and u are present in the final Mandelstam form.

Before discussing the general problem of bound states and resonances in field theory, let us briefly outline a practical application of this representation. We will try to construct a reasonable approximation to the scattering amplitude which is valid for small momentum transfers and large energies. This regime suggests using the type of approach discussed in connection with potential scattering at the end of Sec. II. That is, the t and u discontinuity will be calculated exactly up to some chosen momentum transfer and the representation will be used to automatically insure the satisfaction of unitarity in the s channel at high energies. This is similar to the peripheral collision model,[19] but with the limitations of unitarity enforced, at least approximately. Only the one- and two-pion exchange terms will be considered exactly, since they should yield the most important contribution for large impact parameters. Therefore, we will choose

$$\begin{aligned}N_\pm(b^2,s)=&(\Gamma_3\pm\Gamma_2)K_0(b)-(\Gamma_3\pm\Gamma_2)^2K_0{}^2(b)I(s)\\&+\int_4^9\frac{dx}{\pi}K_0(bx^{\frac{1}{2}})[A_3(x,s)\pm A_2(x,s)]\\&+\int_9^\infty\frac{dx}{\pi}K_0(bx^{\frac{1}{2}})[A_3(x,s)\pm A_2(x,s)], \quad (7.14)\end{aligned}$$

where

$$I(s)=\int_{4m^2}^\infty\frac{ds'}{\pi}r(s')(s'-s)^{-1}.$$

If the functions $H_\pm=N_\pm D_\pm{}^{-1}$ are expanded, then it is easily seen that the t and u discontinuities are given exactly up to the value nine by the first three terms alone. In addition, we see that the absorptive parts A_3 and A_2 are given exactly in the region $4<t<9$ by the elastic or two-pion contribution. We will assume that these quantities are known in terms of nucleon-antinucleon annihilation into two pions. For the region $9<t<\infty$, there are several approximations which seem reasonable. The simplest is to choose the elastic contribution plus an arbitrary function which would insure the correct low energy behavior of $H_\pm$ and would also produce the S-state deuteron pole from H_+. It is also possible to choose these absorptive parts so that the two- and three-pion resonances are included and perhaps even some sort of absorptive potential.

Just as in the potential scattering case, the bound states and resonances are associated with the small-b behavior of N and D. In fact, if $N_\pm$ approaches a constant, and

$$D_\pm(b^2,s)\simeq b^{2(1+\alpha_\pm(s))}C_\pm(s), \quad (7.15)$$

then we find

$$\begin{aligned}M(s,t,u)=&-\beta_+(s)[\sin\pi\alpha_+(s)]^{-1}[(-t)^{\alpha+(s)}+(-u)^{\alpha+(s)}]\\&-\beta_-(s)[\sin\pi\alpha_-(s)]^{-1}\\&\times[(-t)^{\alpha-(s)}-(-u)^{\alpha-(s)}]+\cdots. \quad (7.16)\end{aligned}$$

The possibility of a more general small-b behavior of D will be considered later. The even (odd) angular momentum bound states and resonances come from the first (second) term in (7.16). For example, the bound states and resonances arise when

$$\mathrm{Re}\alpha_+(s)=2n,$$

$$\mathrm{Re}\alpha_-(s)=2n+1,$$

[19] See, for example, S. D. Drell, Revs. Modern Phys. **33**, 458 (1961); F. Salzman and G. Salzman, Phys. Rev. **121**, 1541 (1961).

where n is an integer. The singularities or resonances which seem to arise when $\mathrm{Re}\alpha_+$ is an odd integer and $\mathrm{Re}\alpha_-$ is an even integer are not generally present. This arises from the fact that (7.16) contains only the leading terms in an asymptotic development in t and u. These leading order terms cancel by virtue of the relation (7.1) between s, t, and u. For example, if $\alpha_+\sim 1$, then the first term in (7.16) becomes

$$-(t+u)\beta_+[\sin\pi\alpha_+]^{-1}=(s-4m^2)\beta_+[\sin\pi\alpha_+]^{-1};$$

it no longer grows like t. The singularity of this term is in general cancelled by the first order terms in the asymptotic development which we have not discussed. Our experience with potential scattering tells us that in place of (7.16) we should have the form

$$M(s,z)=-\gamma_+(s)[P_{\alpha_+}(-z)+P_{\alpha_+}(z)]/\sin\pi\alpha_+(s) -\gamma_-(s)[P_{\alpha_-}(-z)-P_{\alpha_-}(z)]/\sin[\pi\alpha_-(s).^{19a} \quad (7.17)$$

It is plausible from (7.15) and the definition of D that in the case of nucleon-nucleon scattering, we have

$$\alpha_\pm(s)=\alpha_\pm(-\infty)+\int_{4m^2}^{\infty}\frac{ds'}{\pi}\,\mathrm{Im}\alpha_\pm(s')(s'-s)^{-1}. \quad (7.18)$$

There is always the possibility that the integral in (7.18) will not converge and that a subtraction must be performed at some finite point. This would correspond to a type of CDD[20] ambiguity since the arbitrary value of $\alpha_\pm$ which is thereby introduced can be used to produce a "bound state" pole in M no matter how weak the coupling or "potential" becomes. The introduction of elementary particles into a theory can be accomplished by just such a procedure. For example, if the deuteron had to be introduced in this manner, one would write

$$\alpha_+(s)=(s-M_D^2)\int_{4m^2}^{\infty}\frac{ds'}{\pi} \times\mathrm{Im}\alpha_+(s')(s'-M_D^2)^{-1}(s'-s)^{-1}. \quad (7.19)$$

The subtraction has introduced only one arbitrary constant, M_D^2. It may be that the residue of the deuteron pole is still determined dynamically.

It behooves us to remind the reader that even if our assumptions concerning the $\alpha_\pm(s)$ behavior of the scattering amplitude are correct, this does *not* solve the bound-state problem in field theory. One must also show that the bound state shows up in the inelastic continuum, for example the contribution of $\pi+D$ in nucleon-nucleon scattering. This state manifests itself by extending the inelastic cut to a lower threshold, an effect we have not yet demonstrated. A possible mechanism for this extension was given in reference (13) and the poles discussed there are exactly the same as those arising from the $\alpha(s)$ behavior. We therefore expect that the correct behavior of the inelastic thresholds as a bound state is formed is assured by analytic continuation in $\alpha(s)$.

To see how this comes about, we have to examine the structure of the inelastic matrix element M_{23} describing: nucleon + nucleon → pion + nucleon + nucleon, illustrated in Fig. 4. There are many things we don't know about the matrix element but it is certainly true that it must have a term of the form

$$M_{23}=n_{23}(s,w^2,\cdots)/\sin\pi\alpha_+(w^2)+\cdots,$$

where w is the center-of-mass energy of the two nucleons. This insures that there will be a pole as a function of w^2 if a bound state occurs in the nucleon-nucleon system. The other variables have been suppressed. We now suppose that the "potential" is sufficiently weak so that $\alpha_+(w^2)$ does not pass through zero.

The contribution to nucleon-nucleon elastic scattering will have the structure

$$\int_{(2m+1)^2}^{\infty}\frac{ds'}{\pi}(s'-s)^{-1}\int_{4m^2}^{(s'^{\frac{1}{2}}-1)^2}dw^2\frac{\rho_3(s',w^2)\int d\Omega\, M_{23}{}^*M_{32}}{\sin\pi\alpha_+(w^2)\,\sin\pi\alpha_+{}^*(w^2)},$$

where ρ_3 is a three-particle phase space factor. Now as the "potential" strength is increased, $\alpha_+(w^2)$ will vanish at the upper limit and the w^2 integration must be deformed to avoid the end-point singularity of the integrand. This, in turn, forces a deformation of the s' integration. The end result is an "anomalous" threshold at the point $s=(M_D+1)^2$. For details of the type of continuations needed the reader is referred to reference (13).

The contribution to $\mathrm{Im}\alpha_\pm(s)$ coming from each of the inelastic channels should be positive to insure that the incident wave is attenuated by the absorption. [See discussion following Eq. (4.18).] Thus the suggestion that one might need a subtraction in $\alpha_\pm$ is quite reasonable because there are an infinite number of channels. We also see that absorption tends to build up $\alpha_\pm(s)$ and hence may lead to elastic scattering resonances just as in the potential case. We would like to emphasize again that these resonances tend to occur below the inelastic threshold in contrast to the particular situation discussed by Ball and Frazer in reference 12.

We will now prove that at most one subtraction is necessary for $\alpha_\pm(s)$. Froissart[21] and Greenberg and Low[22] have given limits on the maximum angular-momentum

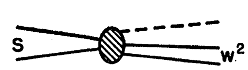

FIG. 4. Pion production in nucleon-nucleon scattering.

[19a] This point is treated in detail in reference 25. We wish to thank Dr. Frautschi for a discussion on his approach to the question.

[20] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956).

[21] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[22] O. Greenberg and F. Low (to be published).

state that can resonate at a given energy. Greenberg and Low, who used only analyticity in the Lehmann ellipse, showed that

$$\mathrm{Re}\alpha(s) < s \times \ln s \times \mathrm{const.}$$

Froissart, using the Mandelstam representation, showed that

$$\mathrm{Re}\alpha(s) < s^{\frac{1}{2}} \times \ln s \times \mathrm{const.}$$

Therefore, if the Mandelstam representation holds, at most one subtraction is needed in the α dispersion relation.

A selection rule concerning the ordering of resonances and bound states holds in field theory just as in potential scattering. The only new point to be considered is that the presence of an exchange potential means that the even and odd angular momentum states must be considered separately.

If an $\alpha_+(s)$ curve is followed below the point $s=0$, then one is in the physical region for the crossed process where t is the energy and s the momentum transfer. Thus the asymptotic behavior in the energy of this process is governed by $\alpha_+(s)$. If we apply Froissart's upper bound on the scattering amplitude, then we must require for large t and negative s that

$$|[(-t)^{\alpha_+}+(-u)^{\alpha_+}-s+4m^2][\sin\pi\alpha_+]| < ct\ln^2 t.$$

The constant terms have been added to produce a pure P-wave term in the limit as discussed in Eq. (7.17). This implies that α_+ must be less than or equal to one since, as α_+ approaches one, l'Hospital's rule yields

$$|i\pi t| < ct\ln^2 t,$$

which is certainly satisfied. Any value of α_+ larger than unity (for negative s) would violate this inequality. In fact, if $\alpha_+(0)=1$, then one gets a constant total cross section in agreement with Pomeranchuk's theorem.[23] It should be emphasized that if for any s, $\mathrm{Re}\alpha_+(s)=1$, there is no corresponding resonance or bound state.

One can produce a scattering amplitude which is as large as that allowed by the upper bound by assuming $H_+(b^2,s)$ to behave as

$$H_+(b^2,s) \sim b^{-2(1+\alpha)}\ln^2 b,$$

for small b. One can carry out the impact parameter integration by differentiating the previous result with respect to α twice. Then we find

$$M \sim -\beta_+[(-t)^{\alpha_+}\ln^2(-t)+(-u)^{\alpha_+}\ln^2(-u)]/\sin\pi\alpha_+ + \cdots.$$

However, if H_+ were to behave this way at the origin, then any bound states and resonances which appear when α passes through an even integer would have a very peculiar angular dependence which doesn't remotely resemble a pure angular momentum state. If this α does not take on even integer values, we cannot exclude such an asymptotic behavior. Resonances and bound states, if any, would then arise from terms which behaved like a pure power of b at the origin.

It is appropriate at this time to recall the relation between $\alpha(s)$ and the complex angular momentum discussed in Sec. IV. We have tacitly assumed that the only singularities in the angular momentum plane are poles, a point which can be proved in potential theory. It is far from clear that life is so simple in field theory. This is a rather involved problem to which we hope to return elsewhere. It seems that due to resonating particle pairs in the inelastic contributions to unitarity that there appear in the elastic amplitude terms of the form

$$M \sim \int dw^2(-t)^{\gamma(w^2,s)}B(w^2,s).$$

Thus, in general, cuts in the l plane may be expected. It would be interesting to study the extension of Regge's formalism to the multi-channel problem in potential theory.

Another physical scattering process to which the family theorem may be applied is pion-nucleon scattering. The resonant families for $T=\frac{3}{2}$, $J=l+\frac{1}{2}$ are presumably as shown in Fig. 5.[24] The $D_{\frac{5}{2}}$ and $F_{7/2}$ resonances occur at laboratory energies of 1.2 and 1.4 Bev, respectively. The familiar $P_{\frac{3}{2}}$ resonance is a member of the same "odd" family as the $F_{7/2}$. It is very surprising that there is no $S_{\frac{1}{2}}$ member of the "even" family to which the $D_{\frac{5}{2}}$ belongs. If the $S_{\frac{1}{2}}$ resonance is absent, there is a strong suggestion of a need for a subtraction in the even amplitudes. There is some experimental indication of a $J=\frac{3}{2}$ resonance at 0.85 Bev which could be one of a second "odd" family with $J=l+\frac{1}{2}(P_{\frac{3}{2}})$, or a member of a completely different "even" family with $T=\frac{3}{2}$, $J=l-\frac{1}{2}(D_{\frac{3}{2}})$.

In Fig. 6 we show the analogous curves for the states with $T=\frac{1}{2}$, $J=l-\frac{1}{2}$. The $D_{\frac{3}{2}}$ and $F_{\frac{5}{2}}$ resonances are at a laboratory energy of 0.6 and 0.9 Bev. The $P_{\frac{1}{2}}$ "resonance" is what is commonly called the nucleon and is a member of the same "odd" family as the 0.9-Bev $F_{\frac{5}{2}}$ resonance.

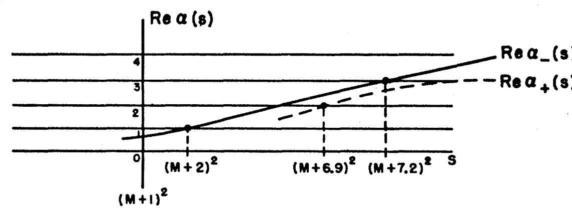

FIG. 5. Pion-nucleon scattering families—$T=\frac{3}{2}$, $J=L+\frac{1}{2}$.

---

[23] The possible connection between the Pomeranchuk theorem and the trajectory of a Regge pole has been discussed in detail by G. Chew and S. Frautschi, Phys. Rev. Letters 7, 394 (1961).

[24] W. M. Layson (to be published). R. Omnes and G. Valladas (to be published).

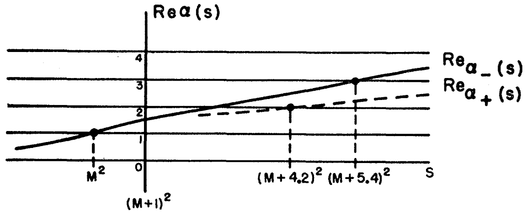

FIG. 6. Pion-nucleon scattering families—$T=\frac{1}{2}$, $J=L-\frac{1}{2}$.

It is quite striking that there are no resonances which can be definitely associated with the quantum numbers $T=\frac{3}{2}$, $J=l-\frac{1}{2}$ or $T=\frac{1}{2}$, $J=l+\frac{1}{2}$. We have been unable to think of any symmetry principle to exclude these. It is amusing to recall that the observed classification is just that found in strong-coupling theories. From the present standpoint it is reasonable to regard the $T=\frac{1}{2}$ states as the more strongly interacting ones since their associated α-curves lie higher than those for $T=\frac{3}{2}$.

As remarked above, the pole in the $T=\frac{1}{2}$ amplitude which appears when $\alpha_-(M^2)=1$ and the high-energy resonances all have the same character. In the language of potential theory, the nucleon is to be regarded as a bound state and is no more fundamental than any of the other resonances. Even if a subtraction were necessary for $\alpha_-(s)$, one is free to subtract at the nucleon mass (in which case the nucleon would be "elementary") or at the 0.9-Bev resonance (which would then be "elementary"). In any case, one may experimentally determine whether the nucleon pole corresponds to α_- going through unity at $s=m^2$ by following the curve into the physical region of the crossed processes. A specific experimental test involving the crossed nucleon pole in pion-nucleon scattering has been suggested by Frautschi, Gell-Mann, and Zachariasen.[25]

The concept of a "dynamical" nucleon opens the door for some interesting speculations. Suppose we regard the t channel (namely $\pi\pi\rightarrow N\bar{N}$) in pion-nucleon scattering as a potential. Imagine further that the annihilation process is characterized by the parameters g, which measures the pion-nucleon interaction, and λ, which measures in some sense the π-π interaction. We admit therefore that there are nucleons in the world of the appropriate mass. What we have not introduced yet is the concept of the single nucleon pole. If we now require that $\alpha_-(M^2)=1$, we get one relation between g^2, λ and we get another from the fact that the residue of the pole thus created must be proportional to g^2. If we had some physical basis (such as the Pomeranchuk theorem) for fixing the value of α_- at some point we would have the exciting possibility of actually computing the nucleon mass in terms of the pion mass. This procedure should work for any and every particle since the single-particle poles will occur in some reaction. Remarks of much the same religious nature have been frequently made by Chew.[26]

As an explanation of the fact that some of the α_+ curves do *not* seem to pass through zero, we offer the following model which may be totally unrelated to reality. Assume that the field theoretic $\alpha_+(s)$ behaves like potential scattering in the case of very small coupling constants. Then the $\alpha_+(s)$ should be below zero for all s; that is, there are no bound states. As the coupling parameters are increased, there should be a point at which an S-wave bound state is formed. If the couplings are increased still further, the mass of the bound state decreases and eventually becomes negative, heralding the appearance of a ghost. The condition that this ghost pole be unobservable is that it occur at infinity and this should determine the coupling constants. It would be amusing and economical if such a condition also yielded Pomeranchuk's theorem.

The next point to be considered here is the high-energy behavior of M. Following our previous development, especially the discussion of the coupled-channel case, we expect in the t channel terms of the form

$$\lim_{s\rightarrow\infty} M(s,t,u)=\beta(t)(-s)^{\alpha(t)}[\sin\pi\alpha(t)]^{-1}+\cdots. \quad (7.20)$$

The bound states in this channel occur when $\alpha(t)$ is an even (odd) integer. Since these bound states must also occur in the pion-pion system, the $\alpha(t)$, and hence the large-s behavior, must be the same [see Eq. (6.5)] for the two reactions. Thus we see in the actual physical case that certain combinations of the amplitudes in nucleon-nucleon scattering, for example, have exactly the same asymptotic behavior for large energies as do corresponding combinations of pion-nucleon and also pion-pion scattering amplitudes. The general statement here is that the amplitudes for all sets of targets and projectiles that interact by the exchange of the same physical states have the same asymptotic dependence on the energy.

The last point which should be mentioned concerns the application of the Fourier-Bessel representation to problems like pion-pion scattering where the crossed process is the same as the one under study. It is clear that one can handle such a problem with the techniques described here in some iteration sense. The input would be the momentum transfer discontinuity which is calculated from the results of the previous iteration. This scheme shares the difficulty of all calculational procedures suggested so far for such problems—it is probably very difficult to carry out in practice.

VIII. CONCLUSIONS

The impact parameter representation of the scattering amplitude has considerable intuitive appeal as well as

[25] S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. (to be published).

[26] G. Chew, Report to the International Conference on Weak and Strong Interactions, La Jolla, California, June, 1961 (unpublished).

many computational advantages. The ease with which the requirements of unitarity may be satisfied at high energies even in the many-channel situation allows a systematic improvement of the discussion of peripheral collisions. Work along these lines is in progress. The neglect of unitarity in previous treatments is a serious deficiency. The diffraction character of scattering amplitudes at high energies is almost guaranteed by the impact parameter representation if $H(b^2,s)$ is sufficiently smooth for small b^2.

The use of this representation in connection with the "strip" approximation is best illustrated by Eq. (7.14). It is clear that we automatically sum the higher particle exchanges in an approximate manner in order to satisfy unitarity.

We wish to emphasize that much of our discussion concerning bound states and resonances can be carried out without the use of the Fourier-Bessel representation. In the case of potential scattering, for example, one may rely directly upon Regge's work. We feel, however, that our representation provides a rather natural basis for a study of the asymptotic behavior for large momentum transfer and for large energies. Some support for this position is to be found in the fact that we were able to establish the high-energy behavior of $\alpha(s)$ almost trivially for potential scattering. For field theoretic problems, the analog of the Regge approach in potential theory has not yet been given. That is, the assumption that the S-matrix elements are meromorphic in the angular momentum plane has not been deduced from the Mandelstam representation and unitarity. The fact that our representation enables us to at least approximately impose the requirements of unitarity along with the Mandelstam analyticity gives us somewhat greater confidence in our conclusions which are otherwise reached by simply bodily copying the results of the nonrelativistic Regge theory.

APPENDIX

In this Appendix we wish to discuss the analyticity of the S matrix in the complex angular momentum plane. In particular we shall show that for a restricted class of potentials, the domain of analyticity found by Regge may be enlarged. We follow the notation of reference 11. Similar and more general work on this point has been carried out by Froissart.[27]

Introducing $\lambda=l+\frac{1}{2}$ in place of the angular momentum l, the radial Schrödinger equation for r times the wave function takes the form

$$\left[\frac{d^2}{dr^2}+k^2-V-\frac{\lambda^2-\frac{1}{4}}{r^2}\right]\psi(\lambda,k,r)=0. \qquad \text{(A1)}$$

[27] M. Froissart (private communication). We are indebted to Dr. Froissart for helpful conversations on the subject matter of the Appendix.

The particular solution which behaves like $r^{\lambda+\frac{1}{2}}$ near the origin, call it $\phi(\lambda,k,r)$, is defined by the integral equation

$$\phi(\lambda,k,r)=r^{\lambda+\frac{1}{2}}-\frac{1}{2\lambda}\int_0^r dr'\left[\frac{r'^{\lambda+\frac{1}{2}}}{r^{\lambda-\frac{1}{2}}}-\frac{r^{\lambda+\frac{1}{2}}}{r'^{\lambda-\frac{1}{2}}}\right] \times[V(r')-k^2]\phi(\lambda,k,r'). \qquad \text{(A2)}$$

It is conventional to introduce another solution $f(\lambda,k,r)$ which has the asymptotic form

$$f(\lambda,k,r)\rightarrow e^{-ikr}. \qquad \text{(A3)}$$

It was shown in reference 11 that $\phi(\lambda,k,r)$ is an entire function of k and is analytic in λ provided $\text{Re}\lambda>0$; $f(\lambda,k,r)$ is an entire function of λ for $\text{Im}k<0$. From these results, it follows that the Jost function $f(\lambda,k)$ defined by

$$f(\lambda,k)\equiv\left[f(\lambda,k,r)\frac{d\phi(\lambda,k,r)}{dr}-\phi(\lambda,k,r)\frac{df(\lambda,k,r)}{dr}\right]\Big/2\lambda, \qquad \text{(A4)}$$

is analytic in λ, k in the product of the half-planes $\text{Re}\lambda>0$, $\text{Im}k<0$ and is continuous on the boundaries.

It is the analyticity domain of ϕ in the λ plane that limits that of $f(\lambda,k)$ to $\text{Re}\lambda>0$. To see how the limitation arises, let us rewrite (A2) as

$$u(\lambda,k,r)=1+\int_0^r dr'\,K_1(r,r')u(\lambda,k,r'), \qquad \text{(A5)}$$

where

$$u=\phi/r^{\lambda+\frac{1}{2}},\quad K_1\equiv-\frac{1}{2\lambda}\left[\left(\frac{r'}{r}\right)^{2\lambda}-1\right]r'[V(r')-k^2]. \qquad \text{(A6)}$$

The iteration solution of (A5) converges and yields a solution analytic in λ provided that

$$\int_0^r dr'\,|K_1(r,r')|<\infty. \qquad \text{(A7)}$$

If one is given only that $r^2V(r)\rightarrow 0$ as $r\rightarrow 0$, the condition (A7) requires $\text{Re}\lambda>0$. We shall make the explicit assumption that

$$V(r)=A/r^{1+\epsilon}+O(r^{-\delta}), \qquad \text{(A8)}$$

for small r, where both ϵ and δ are less than unity. The second term in the potential, when substituted into (A7) leads to the restriction $\text{Re}\lambda>-(2-\delta)/2$. In the text, we were primarily concerned with possibility of λ reaching the value $-\frac{1}{2}$ (i.e., $l=-1$) so we needn't concern ourselves with the $r^{-\delta}$ term. If we wished to commit ourselves on the precise form of this term, as we shall see, it is possible to get to even smaller values of $\text{Re}\lambda$. We shall concentrate our attention on the first term in (A8), which is the dangerous one.

Our purpose is to rewrite Eq. (A5) in such a way that an explicit continuation from $\mathrm{Re}\lambda>0$ down to $\mathrm{Re}\lambda>-(2-\delta)/2$ may be carried out. To this end, define $v_1=u-1$ and find that v_1 satisfies the integral equation

$$v_1(\lambda,k,r)=B_1(\lambda,k,r)+\int_0^r dr'\, K_1(r,r')v_1(\lambda,k,r'), \quad \text{(A9)}$$

where

$$B_1(\lambda,k,r)=\int_0^r dr'\, K_1(r,r').$$

$$\approx\frac{Ar^{1-\epsilon}}{[2\lambda+1-\epsilon][1-\epsilon]}+\cdots. \quad \text{(A10)}$$

As a function λ we see that B_1 has a single pole at $\lambda=-(1-\epsilon)/2$. Next, introduce $v_1=B_1(1+v_2)$; we obtain for v_2 the equation

$$v_2(\lambda,k,r)=B_2(\lambda,k,r)+\int_0^r dr'\, K_2(r,r')v_2(\lambda,k,r'), \quad \text{(A11)}$$

where

$$K_2(r,r')=K_1(r,r')B_1(\lambda,k,r')/B_1(\lambda,k,r) \quad \text{(A12)}$$

and

$$B_2(\lambda,k,r)=\int_0^r dr'\, K_2(r,r')$$

$$\approx\frac{Ar^{1-\epsilon}}{[2\lambda+2(1-\epsilon)][2(1-\epsilon)]}. \quad \text{(A13)}$$

Proceeding in this fashion, we eventually reach a v_N and B_N which are analytic in λ for

$$\mathrm{Re}\lambda>-N(1-\epsilon)/2. \quad \text{(A14)}$$

If N is chosen so that the above value is less than $-(2-\delta)/2$, we see that u is analytic in λ for $\mathrm{Re}\lambda>-(2-\delta)/2$ except for simple poles at $\lambda=-n(1-\epsilon)/2$, $n=1,\cdots,N$.

Evidently we may define a new function ϕ_R which has the poles just found taken away by writing

$$\phi(\lambda,k,r)=\Gamma\left(\frac{2\lambda}{1-\epsilon}+1\right)\phi_R(\lambda,k,r). \quad \text{(A15)}$$

Similarly we may define from (A4) a regular Jost function $f_R(\lambda,k)$ by

$$f(\lambda,k)=\Gamma\left(\frac{2\lambda}{1-\epsilon}+1\right)f_R(\lambda,k).$$

Since the poles in λ are independent of k, by an appropriate analytic continuation one finds

$$f(\lambda,-k)=\Gamma\left(\frac{2\lambda}{1-\epsilon}+1\right)f_R(\lambda,-k).$$

The S matrix, since it is essentially a ratio of the f's, can be expressed as a ratio of the f_R's. Therefore S is meromorphic for $\mathrm{Re}\lambda>-(2-\delta)/2$. The poles arise from the possible vanishing of $f_R(\lambda,-k)$.

The only relevant soluble example which we know of is the Coulomb potential, $V=A/r$, which was discussed in Sec. VI. In this rather peculiar case, the S matrix is

$$S(\lambda,k)=\Gamma(\lambda+\tfrac{1}{2}+iA/2k)\Gamma^{-1}(\lambda+\tfrac{1}{2}-iA/2k), \quad \text{(A16)}$$

which is meromorphic in the *entire* λ plane. The first singularity in λ is $\lambda=-\frac{1}{2}-iA/2k$ or

$$\alpha(s)=-1-i(A/2k).$$

We note that our high-energy approximation for the Yukawa potential led to precisely the same result [see Eq. (4.5)]. It is reasonable to conjecture that the same would be true for any potential with a $1/r$ singularity at the origin. For a fixed angular momentum, as the energy increases, the inner region, where the potential is strongest, dominates. This also follows from the well-known fact that in the limit of large k, the correct expansion parameters are (A/k) and (r/k), where r is the inverse range of the potential. If the factor (r/k) is to appear, it must occur with (A/k), and therefore it cannot show up in $\alpha(s)$ to order $(1/k)$.