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Indeterminate Character of the Reduction of the Wave Packet in Quantum Theory*

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Von Neumann's proof of the ineffectiveness of the use of hidden variables to account for the reduction of the wave packet is analyzed. A possible weakness of the argument is found in the manner in which the probabilistic interpretation of quantum theory is employed. An alternative proof is presented which avoids this aspect of quantum theory. The conclusion is drawn that there is no theory consistent with quantum mechanics which can account for the occurrence of events in nature.

I. INTRODUCTION

PHYSICAL theories, whether classical or quantum, have in common a twofold aspect. On the one hand, there is the very formal body of mathematical symbols with their well-defined rules of manipulation, while on the other hand, there is the more or less informal prescription for relating the abstract symbols to the world of experience and observation.[1] It is in the manner in which the relationship between the mathematical symbols and the physical world is established, that classical and quantum theories exhibit their striking disparities. For, as a consequence of the finite value of the quantum of action and the resulting inability to discern a dichotomy between the atomic system being measured and the measuring device with which it is in interaction, concepts of classical mechanics which are simultaneously meaningful and necessary for a complete classical description of a physical situation, are found in atomic phenomena, to be complementary. That is, a refined analysis of the process of observation of atomic systems reveals that a complete system of classical concepts in terms of which we understand the physical world cannot be made manifest simultaneously. Thus, whereas the primary symbols of classical physics are presumed to have their direct, simultaneously meaningful, and measurable, deterministic counter-parts in the physical world, the symbols of quantum theories can only be related to the world of physical observations in a statistical fashion.

Since human intuition and modes of understanding are molded by classical experience, many scientists have found disturbing the inability of quantum theory to provide a deterministic picture or model of atomic processes. Thus much effort has been spent devising ingenious deterministic models whose predictions coincide with quantum mechanics.[2] Fock,[3] on the other hand, has attempted to interpret the existing quantum mechanical formalism as a deterministic materialistic theory by assuming that the individual systems of the objective physical world are correctly and completely represented by wave functions of the theory. The principal characteristic peculiar to quantum mechanics, according to Fock, is that the wave function cannot be directly observed, but must be inferred by performing measurements on an (in general infinite) ensemble of identically prepared systems.

These wave functions do in fact propagate deterministicly via the Schrödinger equation provided the atomic system is not disturbed by a measurement. However, when an atomic system does interact with a classical measuring device a characteristic reduction of the wave packet occurs which is not accountable or determined within the mathematical formalism of the theory, but which belongs more to the realm of the interpretation of the meaning of the symbols of the formalism. Quantum mechanics thus predicts with what probability events will occur, but it appears to have no mechanism for accounting for the fact that events do in fact occur.

Although a deterministic physical theory is not an essential requirement for the maintainence of the materialistic philosophical point of view (namely, that there exists an objective reality outside of any mind or will), it is incumbent upon a proponent of a purported *complete* materialist physical theory to demonstrate that, in addition to being in accord with experimental facts when a suitable interpretation is placed upon the mathematical symbols of the theory, the theory can account for the fact that events, not merely probabilities, occur in the material world.

The principal concern of this paper is to investigate whether quantum mechanics can provide a model of a deterministic materialistic physical theory, that is, whether there is a deterministic mechanism consistent with the presently accepted formalism of quantum mechanics which can account for the evident fact that

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[1] That the latter process in classical mechanics is not as trivial as is frequently assumed, may be seen when a problem is solved by means of canonical transformation theory. The solution is generally not regarded as completed until an inverse canonical transformation is performed, returning one to the original variables in phase space, it being tacitly assumed that these variables are the symbols which are to correspond to the concepts of our experience. For it is a consequence of Hamilton-Jacobi theory that all classical systems having the same number of degrees of freedom, can formally be made to appear identical in a local region of phase space. The different classical systems can be distinguished locally only by specifying how the mathematical symbols are to be related to measurements.

[2] For a critique of some of these attempts see W. Heisenberg in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press, New York, 1955).

[3] V. A. Fock, Czech. J. Phys. **7**, 643 (1957).

we do observe events in nature, and not merely probabilities.

In view of the fact that von Neumann[4] has presented an argument to show that the probabilistic character of quantum mechanics cannot be accounted for by hidden variables which are consistent with the theory, the next section of this paper will be devoted to a summary and analysis of his argument. It will be seen that if one grants von Neumann's characterization of a measurement then a deterministic mechanistic viewpoint might in fact become tenable. In Sec. III it will then be proven that it is in fact this characterization of a measurement which is inconsistent with quantum mechanics, and that there can exist no mechanism consistent with the most elementary assumption of quantum theory which can account for the occurrence of events in nature. Section IV will then conclude with a discussion of some possible implications of this result.

II. VON NEUMANN'S ARGUMENT

Von Neumann, using the fact that one represents observables in quantum mechanics by means of Hermitian operators, readily showed that there are no dispersion-free ensembles. Thus we can not hope to prepare a state of a quantum system for which all observables have predetermined values. However, this fact in itself does not exclude the possibility that the particular value of an observable, which one obtains as the result of a measurement, can be understood by means of a deterministic mechanism. For, in order to realize a measurement on a quantum system it must be made to interact with a classical measuring device, that is with a device which has an enormous (essentially infinite) number of degrees of freedom and whose state is only macroscopically defined. One could readily imagine that the lack of predictability of the outcome of measurements stems from our incomplete knowledge of the initial state of the measuring device with which the system interacts. If only we knew the precise initial quantum state of the measuring device and how each of its degrees of freedom interacts with the system being measured, we might hope to follow the interaction and understand the particular outcome of the experiment. The use of probability in quantum theory, from this point of view, is purely a consequence of the fact that we can never know all the degrees of freedom of a classical measuring device and must therefore average over all the inequivalent microstates of the measuring device which give the same macroscopic reading. The dispersion in the readings obtained from identically prepared quantum systems would be understood as a consequence of the fact that the measuring devices with which the quantum systems interact are identically prepared only in the macroscopic sense. The result of von Neumann, that there are no dispersion-free ensembles could be regarded as demonstrating that the usual interpretation of the quantum mechanical formalism tacitly assumes that quantum systems interact with "average," only macroscopically normalized measuring devices.

It is to counter the possibility of re-introducing determinism into quantum physics in a fashion indicated above that von Neumann addresses the following argument.

A classical measuring device, as we have already noted, has an enormous (in general infinite) number of degrees of freedom, the vast majority of which are unknown, unknowable, and irrelevant for the functioning of the apparatus. (Thus, for example, the precise quantum state of a particular iron atom on the arm of an ammeter is immaterial to the reading of the instrument.) Von Neumann therefore describes the initial state of the measuring device (system II) by the mixture:

$$U_{\mathrm{II}}=\sum_n W_n|\mathrm{II};n,i\rangle\,\langle\mathrm{II};n,i|, \tag{1}$$

where $|\mathrm{II};n,i\rangle$ are some complete set of states for the measuring device and W_n is the probability with which we know that the device is in the nth state. The value of W_n is evidently independent of the state of the atomic system being observed. If the atomic system (system I) is initially in the state $|\mathrm{I};i\rangle$, the initial uncorrelated state of the combined system is described by the mixture

$$U_i=\sum_n W_n|\mathrm{I}+\mathrm{II};n,i\rangle\,\langle\mathrm{I}+\mathrm{II};n,i|, \tag{2}$$

where

$$|\mathrm{I}+\mathrm{II};n,i\rangle\equiv|\mathrm{I};i\rangle\,|\mathrm{II};n,i\rangle. \tag{3}$$

It is now assumed that the two systems come into interaction via the Schrödinger equation. Thus the final state of the combined system is represented by the mixture

$$U_f=SU_iS^\dagger. \tag{4}$$

If each of the states $S|\mathrm{I}+\mathrm{II};n,i\rangle$ has the form

$$S|\mathrm{I}+\mathrm{II};n,i\rangle=|\mathrm{I};n\rangle\,|\mathrm{II};n,f\rangle, \tag{5}$$

where $|\mathrm{I};n\rangle$ is an eigenstate of some operator N, and $|\mathrm{II};n,f\rangle$ are some other fixed complete set of states for the apparatus,[5] the interaction would have the character of a measurement, since the system is transformed into a mixture of eigenstates of the operator N. In fact in this case we would have

$$U_f=\sum_n W_n|\mathrm{I};n\rangle\,|\mathrm{II};n,f\rangle\,\langle\mathrm{I};n|\,\langle\mathrm{II};n,f|. \tag{6}$$

[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), pp. 437–439.

[5] The index n evidently refers to the eigenvalue of the quantity being measured, and may *now* be regarded as indicating the reading of the apparatus, (apart from the question of the enormous degeneracy of the states of the apparatus labeled by a macroscopic reading). The particular *value* which the apparatus reads, of course depends on $|\mathrm{II};n,f\rangle$, which in turn depends on both $|\mathrm{II};n,i\rangle$ *as well as* $|\mathrm{I};i\rangle$. Thus, the final result of the measurement depends on the initial state of both the apparatus and the system, as one should expect.

We could thus conclude that the probability that the system I evolves into eigenstate $|\mathrm{I};n\rangle$ is W_n, which value depends exclusively on our knowledge of the state of apparatus (system II), and is totally independent of the initial state of system I, $|\mathrm{I};i\rangle$. However, from the probability interpretation of quantum mechanics, the probability is $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2$. Thus, von Neumann finds a contradiction between the assumption of a known mixture of the measuring apparatus [Eq. (1)], the deterministic reduction of the wave packet of the atomic system [Eq. (5)], and the probabilistic interpretation of the quantum mechanical formalism.

At this point, it is important to question the justification of equating W_n with $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2$, which is the essential step in obtaining the contradiction. Of course both expressions are probabilities, but are they probabilities referring to the same choice of ensembles? In other words, are those physical situations, to which we usually apply the expression $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2$, specified to the same degree of accuracy as is the situation discussed above, for which we concluded that the probability had to be W_n? For it is notorious that *estimates of probability* are drastically altered if we have different information at our disposal, even though *the objective physical* situation remains unaltered. For example, suppose that in conformity with the materialist hypothesis the apparatus is always in some well defined state, rather than in a mixture of the type described by Eq. (1). It would then follow from the hypothesis of the reduction of the wave packet, Eq. (5), that the interaction with the measuring apparatus "grinds" the system I in a deterministic fashion into some unique eigenstate of the quantity being measured. Should we then require, in conformity with von Neumann's argument, $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2=1$ and therefore, $|\mathrm{I};n\rangle=|\mathrm{I};i\rangle$ up to a phase factor?

It is conceivable that the final state might be uniquely determined via the Schrödinger equation, and that the probability of it occurring should still be taken less than one. For, as indicated at the beginning of this section, the probability might be determined by some sort of averaging over variables which specify those quantum states of the apparatus which yield the same macroscopic reading of the instrument. That is, it might be that the usual probability expression is a more anthropomorphic quantity than is customarily assumed in quantum theory, yielding not the probability of an event occurring, but rather the probability that we think an event will occur.

More specifically, the expression $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2$ can legitimately be interpreted as the probability of observing the system I in the state $|\mathrm{I};n\rangle$ given (a) that it was initially in the state $|\mathrm{I};i\rangle$ and (b) that nothing is known about the initial quantum state of the measuring system II (apart from that fact it is in some normalized *macroscopic* condition appropriate for it to be used as a measuring device). However, if we have at our disposal more information, namely about the initial state of the measuring apparatus—for example, if we know that it is a mixture as described by Eq. (1)—it is plausible that this added information should alter our estimate of the probability, in fact reduce it to W_n. In this event there would be no reason to expect W_n to equal $|\langle\mathrm{I};n|\mathrm{I};i\rangle|^2$, and the contradiction obtained by von Neumann would be vitiated. It thus appears that, to the very interpretation of the quantum mechanical formalism which von Neumann sought to exclude, his proof does not apply.

III. IMPOSSIBILITY OF DETERMINISM

In view of the treacherous and ambiguous considerations one becomes involved with when considering the use of arguments based on probability, we seek to avoid the probabilistic interpretation of the quantum formalism and inquire whether it is possible for an apparatus to exist having the following attributes:

(a) The initial state of the apparatus can be represented by means of some well-defined wave function $|\mathrm{II};j\rangle$.

(b) When the apparatus interacts via the Schrödinger equation with the system I, assumed to be initially in the state $|\mathrm{I};i\rangle$, it "grinds" the system into an eigenstate of the operator N, $|\mathrm{I};n(i,j)\rangle$, where the eigenvalue $n(i,j)$ is some unique function of the arbitrarily chosen initial states of the system and the apparatus, i and j, respectively.

(c) The apparatus should be a measuring device for the quantity N in the sense that if system I is initially in an eigenstate of N then, independent of the initial state of the apparatus, the apparatus will leave the state of the system unaltered. Or to state this condition more formally, the function $n(i,j)$ should have the property

$$n(n',j)=n', \tag{7}$$

where it is understood that this is intended as a symbolic equation, the symbols standing not for the numerical values of eigenvalues, but rather for the states which they specify.[6]

If it were possible to find an apparatus with the above properties there would be no difficulty in accepting the deterministic position. Such a model conforms, in fact, with the intuitive picture of a measuring apparatus held by many workers who have given only cursory inspection to the problems of measurement, in their fields.

Attribute (a) is a reflection of the materialist position that physical systems in objective reality are correctly represented by wave functions. It is our specialization of Eq. (1) of von Neumann's analysis.

[6] In a more cumbersome, but perhaps more precise notation, Eq. (7) would read

$$|\mathrm{I};n(|\mathrm{I};n'\rangle,\ |\mathrm{II};j\rangle)\rangle=|\mathrm{I};n'\rangle. \tag{7'}$$

Attribute (b) is a statement of the intuitive idea that if we could follow in detail the quantum mechanical interaction between the system being measured and the measuring device, we would understand the precise result of the experiment. It is an expression of the determinist viewpoint, and corresponds to Eq. (5) of von Neumann's analysis. It is possible to broaden attribute (b) in a straight forward fashion to allow for the possibility that system I is destroyed in the course of the measurement. However, in view of the fact that such a generalization would complicate but not otherwise materially alter the following discussion and conclusion, we shall not consider it.

Attribute (c) is the statement of the essential property a system must have if it is to be adequate for performing measurements in quantum theory. One can consider a slight generalization of attribute (c) to make system II more realistic; namely, that there exists a nonempty class of initial states of system II which have attribute (c), thereby leaving open the question of whether system II can be put into states in which it is incapable of functioning as a measuring device. However such a generalization will not materially affect the subsequent argument, and will therefore be ignored. In our argument, attribute (c) will replace von Neumann's recourse to the probabilistic interpretation of quantum mechanics. This attribute is evidently closely related to the probabilistic interpretation, but it is equally evident that we are demanding a simpler and more fundamental requirement of a measurement, which is not subject to possible ambiguous interpretations that we found to be the case with notions of probability.

We now proceed to show that, as a consequence of the principle of linear superposition in quantum theory, the existence of an apparatus having the three attributes enumerated above is impossible.

In parallel with von Neumann's argument and in conformity with attribute (a), we assume that the initial uncorrelated state of the combined system of the object being measured and measuring apparatus can be represented by the state vector[7]

$$|\mathrm{I+II}; i,j\rangle \equiv |\mathrm{I}; i\rangle |\mathrm{II}; j\rangle, \tag{8}$$

where $|\mathrm{I}; i\rangle$ and $|\mathrm{II}; j\rangle$ represent the initial states of the atomic system and the apparatus respectively.

The systems I and II are now brought into interaction and due to this interaction the combined system evolves into a final state

$$|\mathrm{I+II}; f\rangle = S|\mathrm{I+II}; i,j\rangle, \tag{9}$$

where the only property of the Schrödinger equation we require is that it is linear.

If we are considering an apparatus which is to measure the property N of system I it will be convenient to expand $|\mathrm{I+II}; f\rangle$ in terms of a complete set of eigenstates, $|\mathrm{I}; n\rangle$ of the operator N. Thus,

$$|\mathrm{I+II}; f\rangle = \sum_{n,k} |\mathrm{I+II}; n,k\rangle \times \langle \mathrm{I+II}; n,k|S|\mathrm{I+II}; i,j\rangle, \tag{10}$$

where

$$|\mathrm{I+II}; n,k\rangle \equiv |\mathrm{I}; n\rangle\, |\mathrm{II}; n,k\rangle. \tag{11}$$

The states of the apparatus, $|\mathrm{II}; n,k\rangle$, which appear in the expansion [Eq. (10)] are denoted by two letters to indicate that there are many (in general, an infinite number of) states of the appartus II which correspond to a given reading of the instrument. The index n labels the class of states of the apparatus II which are correlated with system I in such a manner that one can infer from the instrument reading that system I is in the eigenstate $|\mathrm{I}; n\rangle$. The index k is to emphasize that such states of the apparatus II are degenerate, and to indicate the "hidden variables" which, hopefully, when averaged over in some suitable fashion, were to give the usual probability interpretation for the state vectors of system I.[8]

Our investigation now focuses on the question of whether an operator exists whose properties are in conformity with the remaining attributes (b) and (c). From attribute (b), which is the hypothesis of the reduction of the wave packet, we conclude that there exists a *unique* $n(i,j)$ such that, for some k

$$\langle \mathrm{I+II}; n(i,j),k|S|\mathrm{I+II}; i,j\rangle \neq 0, \tag{12}$$

and for all k

$$\langle \mathrm{I+II}; n',k|S|\mathrm{I+II}; i,j\rangle = 0, \quad n' \neq n(i,j). \tag{13}$$

Attribute (c), the statement that system II is a measuring device, requires that for each j there is some k such that

$$\langle \mathrm{I+II}; n,k|S|\mathrm{I+II}; n,j\rangle \neq 0, \tag{14}$$

and that for all j and k

$$\langle \mathrm{I+II}; n',k|S|\mathrm{I+II}; n,j\rangle = 0, \quad n' \neq n. \tag{15}$$

Let us now consider the particular initial state of system I:

$$|\mathrm{I}; i\rangle = \alpha|\mathrm{I}; n_1\rangle + \beta|\mathrm{I}; n_2\rangle, \tag{16}$$

where $\alpha\beta(n_2 - n_1) \neq 0$ and $|\mathrm{I}; n_1\rangle$ and $|\mathrm{I}; n_2\rangle$ are understood to be eigenstates of the operator N, which apparatus II is purportedly measuring. From Eq. (8) we

[7] I am indebted to Professor Y. Aharonov for the observation that some proponents of determinism may take issue with Eq. (8) on the grounds that all elements of the universe may in fact be correlated due to the past history of universe. However, the explicit form of Eq. (8) enters in no essential way in the subsequent argument. It is only required that we make a slight, rather evident, modification of the notation in Eq. (16).

[8] The discussion of the choice of notation given here is intended only for intuitive clarity and is not pertinent to the subsequent argument. The only relevant assumption is that the operator N has a complete set of eigenstates of system I, $|\mathrm{I}; n\rangle$ in terms of which the final state of the combined system, $|\mathrm{I+II}; f\rangle$ can be expanded.

can now conclude that for all n

$$\langle \mathrm{I+II}; n,k|S|\mathrm{I+II}; i,j\rangle = \alpha\langle \mathrm{I+II}; n,k|S|\mathrm{I+II}; n_1,j\rangle + \beta\langle \mathrm{I+II}; n,k|S|\mathrm{I+II}; n_2,j\rangle. \quad (17)$$

If we set n equal to n_1 in Eq. (17), we have as a consequence of Eqs. (14), and (15), that for some k

$$\langle \mathrm{I+II}; n_1,k|S|\mathrm{I+II}; i,j\rangle = \alpha\langle \mathrm{I+II}; n_1,k|S|\mathrm{I+II}; n_1,j\rangle \neq 0. \quad (18)$$

Therefore, from Eqs. (12) and (13), we must conclude that for this particular choice of i and j

$$n(i,j) = n_1. \quad (19)$$

However, if, in Eq. (17), we set $n = n_2$ we can conclude in the identical fashion that, for the same choice of i and j,

$$n(i,j) = n_2. \quad (20)$$

We, thereby, contradict the fundamental determinist requirement of attribute (b), namely, the uniqueness of $n(i,j)$.

IV. CONCLUSION

The import of the argument just presented is that although quantum mechanics can deal properly with the relative probability of events occurring, there is no mechanism or physical theory consistent with the formalism of quantum mechanics which can account for the fact that events do in fact occur. The fact that events occur is a tacit assumption made in the language which we use to interpret the symbols which occur in the quantum mechanical formalism.

Depending upon one's tastes there are several possible positions one can have with regard to this state of affairs:

(1) One can demand no more from a physical theory than that the rules of manipulation of its formalism be precise, and that there exists a precise interpretation of its symbols so that its predictions can be investigated and verified. Anything more is a vacuous play upon words and an appeal to man's notoriously poor intuition for realms where one has no experience; or—

(2) The behavior of man and of inanimate matter are precisely the same not from the nineteenth century mechanistic point of view, but rather from the point of view that the behavior of both are governed by laws which are deterministic for statistical aggregates, but which, in detail, allow and, in fact, require individual fluctuations. These fluctuations may be either (a) random or (b) governed by volition. A proponent of position (2a) could assert that there are no laws of nature, but only laws of probability, for ensembles which are determined by criteria or concepts which the observer may find appropriate or convenient. A proponent of position (2b) might thereby be led to the consideration of teleological forces in nature—a concept which, in fact, is rather closely related to the existence of action principles and symmetry principles; or—

(3) Since we do observe events and not merely correlations, quantum mechanics cannot be an accurate or complete physical theory. It must be modified, and the argument of Sec. III can indicate in which direction. At some point, the linearity of the theory has to break down. Perhaps the representation space for the physical states is not a linear vector space, but has a slight curvature, so that locally (with respect to some suitably chosen topology)[9] one still can add vectors, however, over large distances this becomes meaningless. The "curvature" of this space would give a measure of the degree to which a system may be regarded as quantum mechanical versus classical.

Purely apart from the attempt to reintroduce determinism into microphysics, an alteration of the formalism of quantum mechanics along the lines here indicated is probably desirable. This is due to the fact that there exists an approximate superselection rule between states of a quantum system, which becomes more and more exact as the states become macroscopically distinguishable.[9] It would therefore appear preferable to have a formalism which delimits the possibility of forming linear superpositions of states.

There are undoubtedly many other points of view which we may have slighted. However, we have listed the three possible alternatives which we find most appealing.

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[9] An indication of an appropriate topology is implicit in H. Wakita, Progr. Theoret. Phys. (Kyoto) **23**, 32 (1960).