Woolly Cusps

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Recent experiments on (K^-,p) scattering led us to the study of the elastic scattering $1+2 \rightarrow 1+2$ in the energy region where the inelastic process $1+2 \rightarrow 3+4$ sets in, for the case that particle 3 is unstable. We call "woolly cusp" the phenomenon which corresponds to the sharp cusp in the stable case. The procedure followed is to consider the inelastic channel to be of the three-body type where the three-body states are parametrized by a Breit-Wigner formula around a mean mass m^* of particle 3. The connection between a woolly and a sharp cusp is made evident. The problem is studied in terms of a two-channel S-wave K matrix. In the two-channel approximation the woolly cusp necessarily shows a decrease in the elastic cross section σ above a characteristic energy. As a function of energy, σ will either show a maximum, or an inflection point. In either case, the energy at which this happens may lie above or below the inelastic threshold for the fictitious case that particle 3 has a sharp mass m^*. The sign and magnitude of the elastic scattering phase shift at this "m^* point" approximately determines which case is actually realized.

I. INTRODUCTION

It was observed by Wigner[1] that scattering and reaction cross sections show an anomaly in the cross section at energies where competing channels open up. Consider for example scattering of particle 1 by particle 2 as well as production of 3 and 4:

$$1+2 \rightarrow 1+2, \quad (1.1)$$

$$\rightarrow 3+4, \quad (1.2)$$

with $m_3+m_4>m_1+m_2$. At the threshold for (3,4) production, the (1,2) scattering cross section has a cusp.[2] The size of the cusp is a matter of detailed dynamics. But the very existence of the effect, regardless of its magnitude, is understandable on essentially kinematical grounds. Thus let T be the transition matrix, denote the (1,2) and (3,4) channels by α and β, respectively, and let $\rho(k)$ be the density of states in channel β. Then, around the threshold

$$T_{\alpha\alpha} \approx T_{\alpha\alpha}{}^t + iT_{\alpha\beta}{}^t\rho(k)T_{\beta\alpha}{}^t, \quad (1.3)$$

where the superscript t refers to values taken at the threshold. By a well-known argument

$$\begin{aligned} \rho(k) &= k, && \text{above threshold,} \\ &= i|k|, && \text{below threshold,} \end{aligned} \quad (1.4)$$

where k is the β-channel momentum. It is again a dynamical consideration which must decide the region of validity of this linear (S-wave) approximation to the influence of the new channel.

In this note we consider the question: What happens to the cusp phenomenon if one of the channel β particles (say 3) is unstable? There is now no longer question of a sharp threshold. Yet it is obvious that, the longer the lifetime of 3 is, the closer we should obtain an anomaly resembling the mathematical cusp for the stable case. Thus, if we consider for example the cusp in $\pi^-+p \rightarrow \Lambda+K$ at ΣK threshold, the instability of all particles can safely be ignored. However, as a matter of principle, the question remains. On the other hand, there are conceivably cases of practical physical interest to which our question applies. For example, consider (K^-,p) scattering in the energy region where $Y^*+\pi$ sets in, Y^* being some hyperon isobar. We now have one "particle," Y^*, whose lifetime is so short that its finite width cannot be neglected and it is interesting to ask what then takes the place of the sharp cusp.

Actually, observed anomalies in the K^--p scattering at $\approx$400 Mev/c lab momentum[3] aroused our interest in this whole question. As has been pointed out,[4] this momentum corresponds in fact to the "threshold" for the reaction $K^-+p \rightarrow Y_1^*+\pi$, where Y_1^* is the mass 1385 Mev and isotopic spin one hyperon isobar. It should be noted that only if the Y_1^* has spin $\frac{3}{2}$ can there possibly be a significant correlation between the onset of $(Y_1^*+\pi)$ production and K^--p scattering at this energy.[5]

It will be our purpose to give an approximate description of the modifications due to a finite width, in such a way that the transition to the zero-width case is clearly brought out. Any mythology as to whether particle 3 is "elementary" or "composite," will be immaterial to the argument. The procedure is the following. Let 3 be un-

[1] E. Wigner, Phys. Rev. **73**, 1002 (1948).

[2] As is often done, we use the term "cusp" to denote what may either be a mathematical cusp or else a rounded step.

[3] P. Nordin, Phys. Rev. **123**, 2168 (1961); R. Tripp, M. Ferro-Luzzi, and M. Watson, Bull. Am. Phys. Soc. **6**, 350 (1961); Phys. Rev. Letters **8**, 28 (1962). We are indebted to all these authors for discussions on these experimental results.

[4] A. Pais, Revs. Modern Phys. **33**, 492 (1961).

[5] The reasons are the following: The (K^-,p) anomaly shows up as a rather sudden onset of a $\cos^2\theta$ term in the angular distribution at $\sim$400 Mev/c, whereas at lower energies the distribution is nearly isotropic. As was noted in reference 3, this indicates the presence of a considerable $D_{\frac{3}{2}}$ partial wave. On the other hand, the system $(Y_1^*+\pi)$ is essentially in an orbital s wave at the energy concerned and therefore can only be coupled to the (K^-,p) $D_{\frac{3}{2}}$ wave for (spin Y_1^*)$=\frac{3}{2}$.

stable for the decay $3 \rightarrow 5+6$. Then the β channel is considered[6] to be the three-particle channel $4+5+6$. We now parametrize the three-body states[7] in terms of a mean mass m^* of 3 and its half-width Δ. Actually we need more than these two numbers, namely the shape of the distribution around m^*. For this we take a Breit-Wigner formula. This is of course not exact in general, but it serves to bring out the nature of the problem. We then show that in this way the cusp for the stable case, described by Eq. (1.4), is replaced by a "woolly cusp" which is described by a function $\rho(\delta,k)$ given by

$$\rho(\delta,k)=\left[\frac{(k^4+\delta^4)^{\frac{1}{2}}+k^2}{2}\right]^{\frac{1}{2}}+i\left[\frac{(k^4+\delta^4)^{\frac{1}{2}}-k^2}{2}\right]^{\frac{1}{2}},$$
$$\delta=\left[\frac{2m^*m_4\Delta}{m^*+m_4}\right]^{\frac{1}{2}}. \tag{1.5}$$

Here k is the c.m. momentum in the β channel corresponding to a sharp mass m^*. The real part of $\rho(\delta,k)$, when weighed by a two-body mass distribution around m^*, gives an approximate description of the three-body phase space. If the mass distribution contracts to a δ function we clearly obtain Eq. (1.4) for $\Delta=0$ as a limiting case, since $\rho(0,k)=\rho(k)$.

In the next section we develop the theory in terms of a 2 by 2 K matrix[8] corresponding to the channels α and β. As is customary for a sharp cusp, we assume that over the woolly cusp region the K-matrix elements are slowly varying compared to the phase space factors referring to the inelastic channel. In the present case this assumption is somewhat more questionable than for the sharp cusp, because the size of the woolly cusp region may be appreciable. In terms of laboratory momentum we have the following. In the rest system of particle 2, the momentum half-width κ over which the woolly cusp should kinematically be relevant is given by

$$c\kappa=\Delta\frac{m^*+m_4}{m_2}\left(w+\frac{1}{w}\right)+O\left(\frac{\Delta^2}{m^*}\right),$$
$$w=\left[\frac{(m^*+m_4)^2-(m_1+m_2)^2}{(m^*+m_4)^2-(m_1-m_2)^2}\right]^{\frac{1}{2}}. \tag{1.6}$$

In practically interesting situations, orders like $c\kappa\sim 5\Delta$ can well be reached.

As a first orientation we consider in this paper a K matrix which refers specifically to S waves for the states (1,2) and (3,4) but the formalism can readily be extended to include higher angular momenta.[9] For a two-channel problem of the kind of Eqs. (1.1) and (1.2) and with stable particles, conservation of probability demands[10] that the elastic cross section σ decrease right above the threshold for reaction (1.2). Thus there are two types of possible discontinuities, depending on whether σ rises or falls just below threshold. Provided we may consider the cusp effect as a perturbation, the ratio of slopes above and below threshold is fixed entirely by the elastic scattering phase shift at that energy.[10] We will show in Sec. II that corresponding, but somewhat more refined, distinctions can be made also for the woolly cusp. The analog of the "rounded step" type of sharp cusp discontinuity is here an inflection point. To the inverted V-type sharp cusp corresponds a maximum. However, while for the stable case the energy value at which the anomaly occurs is of course the threshold for reaction (1.2), the position of the corresponding characteristic point for the woolly cusp needs further specification. Let us call the m^* point the threshold for the reaction (1.2) if the particle 3 would have a sharp mass m^*. The inflection point or the maximum, as the case may be, may lie above, at or below the m^* point. We show that both distinctions: inflection point or maximum above or below the m^* point are approximately conditioned by the value of the elastic scattering phase shift at the m^* point. (See the discussion of Eqs. (2.42) and (2.43) below.) Actually, because of the requirement of slow variation for K-matrix elements, the region of validity of our calculations should at best extend to energy intervals $\sim\Delta$ below and above the m^* point. Therefore we cannot attach much significance to the occurrence of a point of inflection or of maximum if the phase shifts are such as to put this point at distances $\gg\Delta$ from the m^* point.

After this work was completed we found that the same problem has recently been considered by Baz'.[11] We agree with his conclusions. We believe that the present work is nevertheless of some interest because an alternative and somewhat more general method of derivation is followed here (see also footnote 15 below). Another approach which includes some dynamical effects has been given by Ball and Frazer.[12]

II. FORMALISM

For the sake of simplicity we assume here that all particles have zero spin and isotopic spin. Let $T_{\alpha\alpha}$, $T_{\alpha\beta}$,

[6] The threshold region for (4,5,6) production will play no role in what follows. This implies that m^* is well separated from m_5+m_6, a condition which is satisfied in cases of practical interest. For a discussion of true three-body threshold effects see L. Delver, Nuclear Phys. **9**, 391, 1958; L. Fonda and R. Newton, Phys. Rev. **119**, 1394 (1960).

[7] This is similar to the procedure followed by B. Sakita, Nuovo cimento (to be published), and by R. Dalitz and D. Miller, Phys. Rev. Letters **6**, 562 (1961).

[8] An exposition of the K-matrix formalism is given by R. Dalitz and S. Tuan, Ann. Phys. **3**, 307 (1960).

[9] Unlike the situation for the sharp cusp, such an extension may even be necessary for the (3,4) state in the cusp region. In fact, the "threshold" value m^*+m_4 for (3,4) may lie considerably above the sharp threshold for (4,5,6) production and it can therefore in principle not be excluded that the dominant angular momentum of the (3,4) system is larger than zero right from the onset of reaction (2).

[10] R. Newton, Ann. Phys. **4**, 29 (1958), Sec. V.

[11] A. Baz', J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1511 (1961) [translation: Soviet Phys.—JETP **13**, 1058 (1961)].

[12] J. S. Ball and W. Frazer, Phys. Rev. Letters **7**, 204 (1961).

and $T_{\beta\beta}$ denote the respective transition amplitudes for the reactions (1.1), (1.2), and for $3+4\rightarrow 3+4$. As was stated earlier, the reaction (1.2) should really read $1+2\rightarrow 4+5+6$. However, we assume that $T_{\alpha\beta}$ depends on the (5,6) variables only through the (variable) energy m of $5+6$ in its own center-of-mass frame. m is then just the variable mass of the unstable particle or resonance 3. We neglect any dependence of $T_{\alpha\beta}$ on the orientation of the relative (5,6) momentum with respect to the other momenta involved in the reaction. If we now restrict ourselves to S waves, we have simply $T_{\alpha\beta}=T_{\alpha\beta}(E;m)$ where E is the total c.m. energy. Likewise $3+4\rightarrow 3+4$ really stands for the triple collision $4+5+6\rightarrow 4+5+6$ but we put $T_{\beta\beta}=T_{\beta\beta}(E;m,m')$, where m and m' are the respective initial and final mass distributions of 3.

The requirements of unitarity and of space and time reversal invariance of the S matrix imply that the transition amplitudes T_{ij} satisfy the following conditions for $E\geq m_4+m_5+m_6$:

$$\mathrm{Im}T_{\alpha\alpha}(E)=|T_{\alpha\alpha}(E)|^2\rho_\alpha(E)+\int|T_{\alpha\beta}(E;m)|^2\rho_\beta(E;m)dm, \quad (2.1)$$

$$\mathrm{Im}T_{\alpha\beta}(E;m)=T_{\alpha\alpha}(E)T_{\alpha\beta}^*(E;m)\rho_\alpha(E)+\int T_{\alpha\beta}(E;m')T_{\beta\beta}(E;m',m)\times\rho_\beta(E;m')dm', \quad (2.2)$$

$$\mathrm{Im}T_{\beta\beta}(E;m,m')=T_{\alpha\beta}(E;m)T_{\alpha\beta}^*(E;m')\rho_\alpha(E)+\int T_{\beta\beta}(E;m,m'')T_{\beta\beta}(E;m'',m')\times\rho_\beta(E;m'')dm'', \quad (2.3)$$

where the range of integration over m is given by $(m_5+m_6)\leq m\leq(E-m_4)$. This corresponds to the physically allowed masses of the unstable particle state 3 at a total energy E. Nonrelativistically $\rho_\alpha(E)$ is the center-of-mass momentum in channel α,

$$\rho_\alpha(E)=q_{12}=\left[\frac{2m_1m_2}{m_1+m_2}[E-(m_1+m_2)]\right]^{\frac{1}{2}}. \quad (2.4)$$

Furthermore,

$$\rho_\beta(E;m)=q_4q_{56}, \quad (2.5)$$

where q_4 is the momentum of particle 4 in the over-all center-of-mass system, and q_{56} is the momentum of particles 5 and 6 in their relative center-of-mass system,

$$q_4(m)=\left[\frac{2mm_4}{m+m_4}[E-(m+m_4)]\right]^{\frac{1}{2}}, \quad (2.6)$$

$$q_{56}=\left[\frac{2m_5m_6}{m_5+m_6}[E-(m_5+m_6)]\right]^{\frac{1}{2}}. \quad (2.7)$$

In terms of the symmetric transition matrix T,

$$T=\begin{pmatrix}T_{\alpha\alpha}(E) & T_{\alpha\beta}(E;m)\\ T_{\alpha\beta}(E;m) & T_{\beta\beta}(E;m,m')\end{pmatrix}, \quad (2.8)$$

and the diagonal phase-space matrix,

$$\rho=\begin{pmatrix}\rho_\alpha(E) & 0\\ 0 & \rho_\beta(E;m)\end{pmatrix}, \quad (2.9)$$

we can write the Eqs. (2.1)–(2.3), in a more compact form as

$$\mathrm{Im}T=T\rho\theta T^*, \quad (2.10)$$

where an integration over m is implied, and θ is a diagonal step-function matrix

$$\theta=\begin{pmatrix}\theta(E-m_1-m_2) & 0\\ 0 & \theta(E-m-m_4)\theta(m-m_5-m_6)\end{pmatrix}, \quad (2.11)$$

where $\theta(x)=1$ for $x>0$, 0 for $x<0$.

We now introduce[8] the reaction matrix K:

$$K=\begin{pmatrix}K_{\alpha\alpha}(E) & K_{\alpha\beta}(E;m)\\ K_{\alpha\beta}(E;m) & K_{\beta\beta}(E;m,m')\end{pmatrix}, \quad (2.12)$$

which is related to the transition matrix T by the equation

$$T-iK\rho T=K. \quad (2.13)$$

Since we have left out the step-function matrix θ in Eq. (2.13) the range of integration over m is now from m_5+m_6 to ∞. For a fixed value of E, $\rho_\beta(E;m)$ becomes imaginary for $m>E-m_4$. The proper analytic continuation is to set $\rho_\beta(E;m)=i|\rho_\beta(E;m)|$ in this interval. It can be readily verified from Eq. (2.13) that Eqs. (2.1)–(2.3) are satisfied if the K matrix is real.[13] We now assume[14] that around the m^* point all K matrix elements vary slowly with E as compared to their variation with respect to m. Accordingly, we put

$$K_{\alpha\alpha}(E)=K_{\alpha\alpha}, \quad (2.14)$$

$$K_{\alpha\beta}(E;m)=K_{\alpha\beta}\phi(m), \quad (2.15)$$

$$K_{\beta\beta}(E;m,m')=K_{\beta\beta}\phi(m,m'), \quad (2.16)$$

where $K_{\alpha\alpha}$, $K_{\alpha\beta}$, and $K_{\beta\beta}$ are real constants referring to the fixed value m^*+m_4 for the E variable. The function $\phi^2(m)$ can be interpreted as the mass distribution of 3 and the average mass m^* of this unstable particle is then defined by

$$m^*=\int_{(m_5+m_6)}^{\infty}m\phi^2(m)dm. \quad (2.17)$$

[13] For the corresponding relativistic phase-space terms ρ_α and ρ_β, Eq. (2.13) no longer defines a real K matrix in the physical region. The reason is that the relativistic momentum does not remain imaginary below threshold due to an additional branch point in the energy square variable. Therefore, our treatment is restricted to nonrelativistic kinematics.

[14] For the case of two-body channels only, the K-matrix elements are regular functions of E in the physical region, see R. Oehme, Nuovo cimento **20**, 334 (1960). In this case the assumption of slow variation of $K(E)$ is justified. It appears that K is not regular when three-particle channels are involved. Nevertheless, the condition of slow variation used here may still be justified for not too large Δ.

Likewise, the function $\phi^2(m,m')$ can be interpreted as a joint mass distribution for the initial and final states of 3. In our approximation, we consider the mass distribution of the initial and final states of 3 to be independent. Hence

$$\phi(m,m')=\phi(m)\phi(m'). \tag{2.18}$$

If we now substitute Eqs. (2.14)–(2.18) in Eq. (2.13), we obtain

$$[1-iK_{\alpha\alpha}\rho_\alpha(E)]T_{\alpha\alpha}(E)-iK_{\alpha\beta}\int_{(m_5+m_6)}^{\infty}\phi(m')\times\rho_\beta(E;m')T_{\alpha\beta}(E;m')dm'=K_{\alpha\alpha} \tag{2.19}$$

$$-iK_{\alpha\beta}\phi(m)\rho_\alpha(E)T_{\alpha\alpha}(E)+T_{\alpha\beta}(E;m)-iK_{\beta\beta}\phi(m)\int_{(m_5+m_6)}^{\infty}\phi(m')\rho_\beta(E;m')\times T_{\alpha\beta}(E;m')dm'=K_{\alpha\beta}\phi(m), \tag{2.20}$$

$$T_{\beta\beta}(E;m,m')-iK_{\beta\beta}\phi(m)\int_{(m_5+m_6)}^{\infty}\phi(m'')\times\rho_\beta(E;m'')T_{\beta\beta}(E;m'',m')dm''-iK_{\alpha\beta}\phi(m)\rho_\alpha(E)T_{\alpha\beta}(E;m')=K_{\beta\beta}\phi(m)\phi(m'). \tag{2.21}$$

Setting

$$T_{\alpha\beta}(E;m)=T_{\alpha\beta}(E)\phi(m), \tag{2.22}$$

$$T_{\beta\beta}(E;m,m')=T_{\beta\beta}(E)\phi(m)\phi(m'), \tag{2.23}$$

Eqs. (2.10)–(2.21) can be solved readily for T in terms of K. We obtain

$$XT_{\alpha\alpha}=K_{\alpha\alpha}-i(K_{\alpha\alpha}K_{\beta\beta}-K_{\alpha\beta}^2)\rho_\beta(E), \tag{2.24}$$

$$XT_{\alpha\beta}=K_{\alpha\beta}, \tag{2.25}$$

$$XT_{\beta\beta}=K_{\beta\beta}-i(K_{\alpha\alpha}K_{\beta\beta}-K_{\alpha\beta}^2)\rho_\alpha, \tag{2.26}$$

where

$$X=(1-iK_{\alpha\alpha}\rho_\alpha)[1-iK_{\beta\beta}\rho_\beta(E)]+\rho_\beta(E)\rho_\alpha K_{\alpha\beta}^2, \tag{2.27}$$

and

$$\rho_\beta(E)=\int_{m_5+m_6}^{\infty}\rho_\beta(E;m)\phi^2(m)dm, \tag{2.28}$$

while in Eqs. (2.24)–(2.27) ρ_α is to be taken at the fixed argument $E=m^*+m_4$. Note that the entire energy dependence of the transition amplitudes T_{ij} is therefore given by $\rho_\beta(E)$. This is the counterpart of the woolly cusp case of the k dependence in Eq. (1.3) through $\rho(k)$ only.[15]

If we substitute for $\phi^2(m)$ a Breit-Wigner distribution

$$\phi^2(m)=\frac{\Delta/\pi}{(m-m^*)^2+\Delta^2}, \tag{2.29}$$

we obtain for $\Delta/m^*\ll 1$ [extend the lower limit in the integral (2.28) to $-\infty$]

$$\rho_\beta(E)=q\left[\frac{2m^*m_4}{m^*+m_4}[E-m^*-m_4+i\Delta]\right]^{\frac{1}{2}}, \tag{2.30}$$

where $q=q_{56}(m^*)$, the decay momentum of 3 evaluated at $m=m^*$, see Eq. (2.7). In Eq. (2.6) put $q_4(m^*)\equiv k$. Then we have

$$\rho_\beta(E)=q[k^2+i\delta]^{\frac{1}{2}}=q\rho(\delta,k), \tag{2.31}$$

where $\rho(\delta,k)$ was defined in Eq. (1.5).

The total S-wave cross sections for processes (1.1) and (1.2) are given by

$$\sigma_{\alpha\alpha}(E)=4\pi|T_{\alpha\alpha}(E)|^2 \tag{2.32}$$

$$\sigma_{\beta\alpha}(E)=(4\pi/\rho_\alpha)|T_{\beta\alpha}(E)|^2\,\mathrm{Re}\rho_\beta(E), \tag{2.33}$$

respectively. Substituting Eqs. (2.24) and (2.25) in Eqs. (2.32) and (2.33) we obtain

$$\sigma_{\alpha\alpha}(E)=\frac{4\pi}{\rho_\alpha^2}\frac{x^2+2\rho_2x[xz-y^2]+(\rho_1^2+\rho_2^2)(xz-y^2)^2}{1+x^2+2\rho_1y^2+2\rho_2[z+x(xz-y^2)]+(\rho_1^2+\rho_2^2)[z^2+(xz-y^2)^2]}, \tag{2.34}$$

$$\sigma_{\alpha\beta}(E)=\frac{4\pi}{\rho_\alpha^2}\frac{y^2\rho_1}{1+x^2+2\rho_1y^2+2\rho_2[z+x(xz-y^2)]+(\rho_1^2+\rho_2^2)[z^2+(xz-y^2)^2]}, \tag{2.35}$$

where $x=K_{\alpha\alpha}\rho_\alpha$, $y^2=K_{\alpha\beta}^2\rho_\alpha$, and $z=K_{\beta\beta}$ are real constants and

$$\begin{aligned}\rho_1&=\mathrm{Re}\rho_\beta=2^{-\frac{1}{2}}[(k^4+\delta^4)^{\frac{1}{2}}+k^2]^{\frac{1}{2}}q,\\ \rho_2&=\mathrm{Im}\rho_\beta=2^{-\frac{1}{2}}[(k^4+\delta^4)^{\frac{1}{2}}-k^2]^{\frac{1}{2}}q.\end{aligned} \tag{2.36}$$

Note that

$$\rho_1\rho_2=q^2\delta^2/2,\quad \rho_1^2+\rho_2^2=q^2(k^4+\delta^4)^{\frac{1}{2}}. \tag{2.37}$$

For $|k^2|\lesssim\delta^2$ we have ρ_1 and $\rho_2\lesssim\delta^2$, and for sufficiently small δ we need to keep only linear terms in ρ_1 and ρ_2 in Eqs. (2.34) and (2.35). Hence

$$\sigma_{\alpha\alpha}\simeq\frac{4\pi}{\rho_\alpha^2}\frac{x^2}{(1+x^2)}\left[1-\frac{2y^2}{(1+x^2)}\left(\rho_1+\frac{\rho_2}{x}\right)\right], \tag{2.38}$$

$$\sigma_{\beta\alpha}\simeq\frac{4\pi}{\rho_\alpha^2}\frac{y^2\rho_1}{(1+x^2)}. \tag{2.39}$$

[15] Our K-matrix treatment implies that $\phi(m)$ is real. This leads us to believe that the present method is not general enough because in equations like (2.22) one expects ϕ in general to be complex cf. reference 7. However, for $\Delta^*/m\ll 1$ the restriction to real $\phi(m)$ should not affect the elastic scattering cross section.

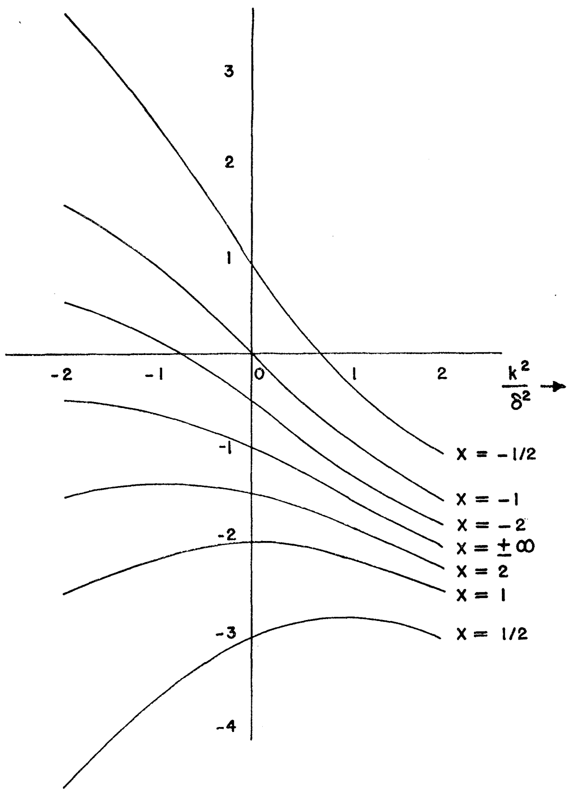

FIG. 1. Plot of $-(\rho_1+x^{-1}\rho_2)$ as a function of $k^2\delta^{-2}$, for various values of x.

We now examine the behavior of the elastic scattering cross section $\sigma_{\alpha\alpha}$ near $k^2=0$. Evaluating the first and second derivatives with respect to k^2, we find

$$\frac{d\sigma_{\alpha\alpha}}{dk^2}=-\frac{4\pi}{\rho_\alpha^2}\frac{x^2y^2}{(1+x^2)^2}\frac{1}{[k^4+\delta^4]^{\frac{1}{2}}}\left[\rho_1-\frac{\rho_2}{x}\right], \tag{2.40}$$

$$\frac{d^2\sigma_{\alpha\alpha}}{(dk^2)^2}=-\frac{4\pi}{\rho_\alpha^2}\frac{x^2y^2}{(1+x^2)^2}\frac{1}{2[k^4+\delta^4]^{\frac{3}{2}}}\{([k^4+\delta^4]^{\frac{1}{2}}-2k^2)\rho_1 + ([k^4+\delta^4]^{\frac{1}{2}}+2k^2)\rho_2/x\}. \tag{2.41}$$

The extremum for $\sigma_{\alpha\alpha}(k^2)$ is therefore given by

$$\rho_1=(1/x)\rho_2, \tag{2.42}$$

which can be satisfied provided $x>0$. At this extremal point,

$$\frac{d^2\sigma_{\alpha\alpha}}{(dk^2)^2}=-\frac{4\pi x^2y^2}{\rho_\alpha^2(1+x^2)^2}\frac{\rho_1}{(k^4+\delta^4)}<0, \tag{2.43}$$

and therefore the extremum is always a *maximum in* $\sigma_{\alpha\alpha}$. For $0<x\leq 1$ the maximum occurs for $k^2\geq 0$, while for $1<x$ it occurs for $k^2<0$. In order that this maximum appear within the range of validity of our approximation $|k^2|<\delta^2$, it is clear that $x\sim 1$. For $x<0$ we can only have an inflection point.[16]

These results are illustrated in Fig. 1 where we have plotted $-(\rho_1+\rho_2/x)$, the energy-dependent part of the elastic cross section $\sigma_{\alpha\alpha}$, Eq. (2.38) as a function of k^2/δ^2 for various values of the parameter x. It can be readily verified from Eq. (2.22) that in the limit $\delta\rightarrow 0$, $x=\tan\delta_0$ where δ_0 is the S-wave scattering phase shift at the m^* point for the inelastic process (1.2). This remains approximately valid for finite but narrow width δ.

Note added in proof. The transition amplitude $T_{\beta\beta}$ for which the unitarity condition [Eqs. (2.1)–(2.3)] applies contains an unconnected process in which particle 3 is produced and decays without interaction with 1 or 4. Such a contribution can be eliminated by considering the discontinuity of the transition amplitudes T_{ij} in E across the physical cut instead of the unitarity condition [see R. Blankenbecler, Phys. Rev. **122**, 983 (1961)]. It is then possible to factor out a complex propagator $\phi(m)$ and get rid of the difficulty that we encountered in factoring the K matrix (see reference 14). For small widths Δ, however, the results of this paper remain essentially unchanged.

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[16] The case of a maximum at $k^2=0$ is mentioned in reference 11. It should be noted that some intermediate steps in the cited paper are not strictly correct. The quantity ϵ_{cd} in reference 11 is actually the energy of the decay particles in their own rest frame, and not (as was stated) in the rest frame of the reaction.