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Multiple Production of Photons in Quantum Electrodynamics*

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A formalism intended to evaluate the expectation value of physical quantities directly has been applied to the processes involving multiple production of photons in quantum electrodynamics. This has been achieved by constructing the Green's function which is done by expanding it in terms of hard and soft photon parts of the electromagnetic field. A general treatment of infrared divergence is given. The cross section for n-photon production in Coulomb scattering and pair annihilation has been evaluated. The limitation of the expansion of the Green's function in terms of hard and soft photon parts is examined. A plausible generalization regarding the cross section of any primary process in quantum electrodynamics at high energies with n-photon emission in the final state is enunciated.

1. INTRODUCTION

THE production of photons during the interaction of charged particles among themselves or with external electromagnetic field has been a topic of interest to the theoretician. It is a straightforward application of quantum electrodynamics, but the complexity of the equations prevents a completely satisfactory solution of the problem. The techniques developed during one and a half decades have not made the task easier.[1,2]

Recently a formalism, which follows Schwinger's extension[3] of his action principle,[4] has been developed by Bakshi and the author.[5] In this paper the formalism is applied to the problem of multiple production of photons. The details of the formalism will be published elsewhere. Only cursory details will be given here.

Schwinger has extended the action principle to generate the expectation values of physical quantities directly. This is carried out by constructing the transformation function for the temporal development of a physical system in a closed cycle in time (this being a mathematical contrivance) in which the development from the initial to the final time is governed by a dynamics different from that of the other part (the return path). The construction has been achieved using the retarded and advanced Green's functions. In this formulation causality and completeness come out as natural consequences of the fact that the field operators have to satisfy certain physical boundary conditions. Another virtue of this method is that, because the expectation value is evaluated directly, the effect of the presence of all the physically possible complete set of states is automatically taken into account. Thus, the application of this formalism to the quantum electrodynamical problems, as done in subsequent sections, does not lead to any infrared divergence, thus showing that the problem of infrared divergence is due to the computational technique one is using rather than the notion that quantum electrodynamics is formally not a well-defined theory because of infrared divergence.

The required transformation function can also be obtained by the transformation function which describes the development of the system from the initial to the final time by time reflection and a phase transformation.[6] Consider an interacting system of electron and electromagnetic fields in the presence of external sources and external electromagnetic field. The transformation function $\langle 0\sigma_1|0\sigma_2\rangle$ is given by[7]

$$\langle 0\sigma_1|0\sigma_2\rangle=\det(1-e\gamma AG_0{}^+)\times\exp(i\bar{\eta}G^+\eta)\exp(\tfrac{1}{2}iKDK), \quad (1)$$

where $\bar{\eta}$, η are external sources of the electron field, K is the external current distribution, D is the free Green's function for the electromagnetic field, and G^+ is the Green's function for the interacting electron field. Here A stands for $A^e+(1/i)\delta/\delta K$, where A^e is the external electromagnetic field, and $\gamma A\equiv\gamma_\mu A^\mu$. Also $\hbar=c=1$. By applying the transformation,[6]

$$x\longrightarrow -xe^{-\pi i},$$

on (1) we get[5]

$$\langle 0\sigma_2|0\sigma_2\rangle^\pm=C(A_\pm)\exp(i\bar{\eta}G\eta)\exp(\tfrac{1}{2}iKDK), \quad (2)$$

where

$$C(A_\pm)=\det(1-e\gamma A_+G_0{}^+)\det(1-e\gamma A_-G_0{}^-)\times\det(1+I_+{}^+S^-I_-{}^-S^+),$$

with

$$A_\pm=A^e\pm\frac{1}{i}\frac{\delta}{\delta K_\pm};\quad I_\pm{}^\pm=er A_\pm\frac{1}{1-G_0{}^\pm e\gamma A_\pm}$$

$$\eta=\begin{pmatrix}\eta_+\\ \eta_-\end{pmatrix},\quad K=\begin{pmatrix}K_+\\ K_-\end{pmatrix};$$

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[1] S. N. Gupta, Phys. Rev. **98**, 1502 (1955); Phys. Rev. **99**, 1015 (1955).

[2] J. Joseph, Phys. Rev. **103**, 481 (1956).

[3] J. Schwinger, Lectures delivered at the Institute of Theoretical Physics at Brandeis, summer of 1960 (unpublished); Proc. Nat. Acad. Sci. **46**, 1401 (1960); J. Math. Phys. **2**, 407 (1961).

[4] J. Schwinger, Phys. Rev. **91**, 713 (1953); Phys. Rev. **91**, 728 (1953); Phys. Rev. **92**, 1283 (1953); Phys. Rev. **93**, 615 (1954); Phys. Rev. **94**, 1362 (1954).

[5] P. M. Bakshi and the author have applied the formalism to particle production in the presence of external sources and an external electromagnetic field. The details will be published elsewhere.

[6] J. Schwinger, Phys. Rev. **115**, 721 (1959); Ninth Annual International Conference on High-Energy Physics, Kiev, 1959 (Academy of Science, U.S.S.R., 1960).

[7] J. Schwinger, Lectures on the Theory of Coupled Fields, Harvard University, 1954 (unpublished).

with "±" characterizing the external quantities in the forward and the backward developments,

$$G=\begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix},$$

with

$$G_{\pm\pm}=\pm(G_0{}^{\pm}-S^{\mp}I_{\mp}{}^{\mp}S^{\pm})(1+I_{\pm}{}^{\pm}S^{\mp}I_{\mp}{}^{\mp}S^{\pm})^{-1}\times(1-e\gamma A_{\pm}G_0{}^{\pm})^{-1},$$

$$G_{\pm\mp}=\pm i(1-G_0{}^{\pm}e\gamma A_{\pm})S^{\mp}(1+I_{\mp}{}^{\mp}S^{\pm}I_{\pm}{}^{\pm}S^{\mp})^{-1}\times(1-e\gamma A_{\pm}G_0{}^{\pm})^{-1}, \quad (2a)$$

and[4]

$$S^{+}(xx')=\sum_{+p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'),$$

$$S^{-}(xx')=\sum_{-p}\psi_{\lambda p}(x)\bar{\psi}_{\lambda p}(x'),$$

the $\psi_{\lambda p}$'s being the mode functions of the free electron;

$$D=\begin{pmatrix} D_{++} & D_{+-} \\ D_{-+} & D_{--} \end{pmatrix},$$

with

$$D_{\pm\pm}=\pm D_0{}^{\pm},$$

$$D_{+-}=-is^{-}(xx'),$$

$$D_{-+}=-is^{+}(xx'),$$

and[4]

$$s^{-}(xx')=\sum_{\rho k}\bar{A}_{\rho k}(x)A_{\rho k}(x'),$$

$$s^{+}(xx')=S(xx')=\sum_{\rho k}A_{\rho k}(x)\bar{A}_{\rho k}(x'),$$

the $A_{\rho k}$'s being the mode functions of the electromagnetic radiation field.

Now the generating function $Q(\lambda)$ of projection operators for photon states is given by[5]

$$Q(\lambda)=\exp[(\lambda-1)\sum_{\rho k}a_{\rho k}{}^{(-)};a_{\rho k}{}^{(+)}], \quad (3)$$

the $a_{\rho k}{}^{(\pm)}$'s being creation and annihilation operators of a photon with momentum k and polarization ρ. When $\lambda=0$, $Q(0)$ is the projection operator for the vacuum state; when $\lambda=1$, $Q(1)=1$ representing completeness. The expectation value of $Q(\lambda)$ can be readily generated using Eq. (2). It is given by[5]

$$\langle Q(\lambda)\rangle=C(A_{\pm})\exp(i\bar{\eta}G\eta)\exp(\tfrac{1}{2}iKDK)\times\exp[(\lambda-1)K_{+}SK_{-}]|_{K_{+}=0=K_{-}}. \quad (4)$$

As a specific example of physical phenomena, the expectation value of $Q(\lambda)$ in Coulomb scattering is given by

$$\langle Q(\lambda)\rangle_{\mathrm{cs}}=-C[\bar{\psi}_{+p}\gamma_0G_{--}\gamma_0\psi_{+p'}][\bar{\psi}_{+p'}\gamma_0G_{++}\gamma_0\psi_{+p}]\times\exp[(\lambda-1)K_{+}SK_{-}]\times\exp(\tfrac{1}{2}iKDK)|_{K_{+}=0=K_{-}}, \quad (5)$$

p and p' being the initial and final momenta of the electron. In the case of a pair annihilation,

$$\langle Q(\lambda)\rangle_{\mathrm{pa}}=-C[\bar{\psi}_{+p}\gamma_0G_{--}\gamma_0\psi_{-p'}][\bar{\psi}_{-p'}\gamma_0G_{++}\gamma_0\psi_{+p}]\times\exp[(\lambda-1)K_{+}SK_{-}]\exp(\tfrac{1}{2}iKDK)\times K_{+}=0=K_{-}, \quad (6)$$

p and p' being the momenta of the electron and positron.

In the subsequent sections we shall deal with the detailed evaluation of the right side of Eqs. (5) and (6) which requires explicit construction of G_{++} and G_{--}, which in turn can be done in certain approximations. Sections 2, 3, and 4 are concerned with approximations and explicit construction of the required Green's functions. Section 5 contains the treatment of infrared divergence and Sec. 6 the energy loss in Coulomb scattering due to infrared photons. In Secs. 7 and 8 the multiple bremsstrahlung is considered, and the validity of the approximations in Secs. 3 and 4 is discussed. Section 9 deals with radiative corrections and comparison with earlier results in the literature. In Sec. 10 the formalism is applied to multiple photon production in pair annihilation. In Sec. 11 a plausible generalization is stated and brief concluding remarks are made.

2. HARD AND SOFT PHOTONS

In the foregoing section a complete formulation to evaluate directly the expectation values has been given. In this and subsequent sections, we shall consider the application of the formalism to the problem of infrared divergence (IRD) and the multiple photon processes in quantum electrodynamics. The expressions given in the last section are complete in their description of the physical processes and contain all the virtual phenomena that are involved. Instead of dealing with the above-mentioned problems in all generality, which at the moment does not look feasible, we will neglect the vacuum polarization [which means putting $C(A_{\pm})$ in Eq. (4) equal to unity] and other virtual processes and make simplifying assumptions which do not look far away from physical reality. At the end of Secs. 8 and 9 we shall make comments on these assumptions and try to estimate corrections due to the neglect of some processes and the assumptions.

As is well known, the infrared divergence problem which occurs in radiative corrections in quantum electrodynamics is almost a real process. In other words, the virtual photons which give rise to IRD are almost real. Hence in order to deal with IRD problem alone it is enough to deal with real photons. Our next problem of multiple photon production in any process in quantum electrodynamics involves real photons. Now we shall develop an approximation technique to deal with these real processes. Before going into details it should be mentioned that our gigantic expressions for G_{++} and G_{--} [Eqs. (2a)], which are the ones that are needed for the description of the processes we shall deal with, become simplified because of energy-momentum conservation. This fact becomes clear if one remembers that S^{+} and S^{-} functions are defined on the mass shell. Bearing this in mind and confining ourselves to the emission processes, G_{++} and G_{--} become

$$G_{++}=G_{A+}{}^{+}=G_0{}^{+}\frac{1}{1-e\gamma A_+G_0{}^{+}}, \tag{7a}$$

$$G_{--}=-G_{A-}{}^{-}=-G_0{}^{-}\frac{1}{1-e\gamma A_-G_0{}^{-}}. \tag{7b}$$

When an external field A^C is present, to the first order in A^C we can write G_{++} and G_{--} as

$$G_{++}=G_{A+}{}^{+}e\gamma A^CG_{A+}{}^{+}, \tag{8a}$$

$$G_{--}=-G_{A-}{}^{-}e\gamma A^CG_{A-}{}^{-}. \tag{8b}$$

At this point it should be mentioned that we will treat Coulomb scattering and multiple bremsstrahlung up to first order in the Coulomb field.

Any theory which tries to explain the physical phenomena must also take into account the fact that they are observed and recorded by an apparatus which has a limited accuracy. In view of this fact, whenever there is emission of photons in a physical process, not all the photons are detected but only those which are above a certain energy which is the energy resolution $\Delta\epsilon_0$ of the apparatus involved. We shall denote all the photons below this energy $\Delta\epsilon_0$ as infrared or *infra-photons*. We shall divide the photons above this energy into two parts in comparison with the energy of the particle involved—electron or positron here. The photons which have energy comparable to that of the particle are called *hard photons*; the photons which have energy above $\Delta\epsilon_0$ but not comparable to the energy of the particle as *soft photons*. In what follows, unless there is a confusion in terminology, we shall mean by soft photons both soft and infra-photons.

3. APPROXIMATIONS

For the processes we are dealing with the derivation of an explicit expression for G_{++} and G_{--} has reduced to the finding of G^+ and G^- because of the relations (7). So, consider the differential equation for Green's function for an electron interacting with an electromagnetic field A:

$$(\gamma p+m-e\gamma A)G=1. \tag{9}$$

The A in Eq. (9), on being replaced by $\pm(1/i)\delta/\delta K_\pm$ and made to act on appropriate quantities as in Eq. (4), gives rise to a description of the emission of photons. Because in the physical processes we are dealing with, there are both hard and soft photons, we shall make the distinction between the two kinds at the outset and represent them in the differential equation as

$$[\gamma p+m-e\gamma(A_S+A_H)]G=1. \tag{10}$$

This distinction gives us an opportunity to treat them on different footings. We shall carry out high-energy approximations on the soft-photon part and perturbation in the usual sense on the hard-photon part. To this end we write

$$G=\frac{1}{\gamma p+m-e\gamma(A_S+A_H)}$$
$$=G^S+G^Se\gamma A_HG^S+G^Se\gamma A_HG^Se\gamma A_HG^S+\cdots, \tag{11a}$$

where

$$G^S=\frac{1}{\gamma p+m-e\gamma A_S}. \tag{11b}$$

Looking ahead with a view to how one calculates the cross section making use of the formalism we have developed, one infers that

$$\sigma=\sigma^{(0)}+\sigma^{(1)}+\cdots+\sigma^{(n)}+\cdots, \tag{12}$$

where $\sigma^{(n)}$ is the cross section of the process in question with n hard photons. The possibility of existence of $\sigma^{(n)}$ in σ must be viewed from the standpoint of conservation laws of energy and momentum. For example, Coulomb scattering can occur with no hard photons and hence the series begins with $\sigma^{(0)}$. In the pair annihilation two hard photons are necessary and, hence, the series begins with $\sigma^{(2)}$.

We shall now examine the series (11a). We shall confine our attention to the third and the fourth terms, and a similar examination can be made for other terms. It should be mentioned that what follows is not a proof with all mathematical rigor; besides we deal with a part of the expression ignoring the effect of the other part on it. Later it will be shown that the conclusions reached here are true in specific physical processes with only certain energy losses. Consider that part of the expression which gives rise to three photons in a pair annihilation. The terms are

$$G^0e\gamma A_SG^0e\gamma A_HG^0e\gamma A_HG^0+G^0e\gamma A_HG^0e\gamma A_SG^0e\gamma A_HG^0$$
$$+G^0e\gamma A_HG^0e\gamma A_HG^0e\gamma A_SG^0$$
$$+G^0e\gamma A_HG^0e\gamma A_HG^0e\gamma A_HG^0. \tag{13}$$

The object is to show that the main contribution to three-photon production in pair annihilation comes from the first and the third terms. Consider the first and second terms:

$$G^0(e\gamma A_SG^0e\gamma A_H+e\gamma A_HG^0e\gamma A_S)G^0e\gamma A_HG^S. \tag{14}$$

Making use of Fourier representation, we write (14) as

$$\int\frac{e^{ip(x-y)}(m-\gamma p)}{m^2+p^2}\Bigg(e\gamma a(k_s)e^{ik_sy}\frac{m-\gamma(p-k_s)}{m^2+(p-k_s)^2}$$
$$\times e\gamma a(k_h)e^{ik_hy}+e\gamma a(k_h)e^{ik_hy}\frac{m-\gamma(p-k_h)}{m^2+(p-k_h)^2}e\gamma a(k_s)e^{ik_sy}\Bigg)$$
$$\times\frac{m-\gamma(p-k_s-k_h)}{m^2+(p-k_s-k_h)^2}e\gamma a(k_3)e^{ik_3y}\frac{e^{ip'(y-x')}(m-\gamma p')}{m^2+p'^2}$$
$$\times dpdp'dk_hdk_sdk_3dy. \tag{15}$$

We are interested only in the order of magnitude of the quantities. Making use of the fact that $m^2+p^2=0=m^2+p'^2$ because we are dealing with free particles at the ends, we have for

$$\sim -\int \frac{e^{ip(x-y)}(m-\gamma p)}{m^2+p^2}\left(\frac{pa(k_s)}{pk_s}e\gamma a(k_h)+\frac{pa(k_h)}{pk_h}e\gamma a(k_s)\right)$$
$$\times e^{i(k_s+k_h+k_3)y}\frac{m-\gamma(p-k_s-k_h)}{m^2+(p-k_s-k_h)^2}$$
$$\times e\gamma a(k_3)\frac{e^{ip'(y-x')}(m-\gamma p)}{m^2+p^2}dk_s dk_h dk_3 dp dp' dy. \quad (16)$$

The operation of variational differentiation gives $a(k_h)\sim 1/k_h^{\frac{1}{2}}$ and $a(k_s)\sim 1/k_s^{\frac{1}{2}}$. Hence, the ratio of the two terms in (16) is

$$\text{first term/second term}=k_h^{\frac{1}{2}}/k_s^{\frac{1}{2}}, \quad (17a)$$

which is large. This means that in the series (14) significant contributions come from terms in which G^S in the middle is replaced by G^0. This is the expression of the well-known fact that soft photons come out before and after the hard interaction takes place. Hence, (11a) becomes

$$G=G^S+G^S e\gamma A_H G^S+G^S e\gamma A_H G^0 e\gamma A_H G^S+\cdots. \quad (18)$$

Now we shall compare first and last terms of the expression (13):

$$G^0 e\gamma A_S G^0 e\gamma A_H G^0 e\gamma A_H G^0$$

and

$$G^0 e\gamma A_H G^0 e\gamma A_H G^0 e\gamma A_H G^0.$$

The matrix ratio of these two terms is given by

$$G^0 e\gamma A_S G^0 \frac{1}{G^0 e\gamma A_H G^0}. \quad (19)$$

Noting that the free-end Green's functions G^0 in the numerator and denominator refer to the same momentum, the magnitude of the ratio is $\sim k_h^{\frac{3}{2}}/k_s^{\frac{3}{2}}$. Now remembering the fact that the cross section is obtained by integrating over the density of the final states which gives an extra factor k for the effective amplitudes, the effective ratio between the first and the last term of (13) becomes

$$\text{first term/fourth term}=k_h^{\frac{1}{2}}/k_s^{\frac{1}{2}}. \quad (17b)$$

Hence, the contribution to the cross section for three-photon production in pair annihilation comes mainly from two hard photons and one soft photon part. Hence we can write for three-photon pair annihilation:

$$\sigma\sim\sigma^{(2)}. \quad (20)$$

4. CONSTRUCTION OF G^{S_r} AND G^{S_l}

Considerations in the latter half of the last section have reduced our problem of constructing G^+ and G^- to that of constructing G^{S+} and G^{S-}. We also observe that G^S appears only at the extremities in each of the terms except the first term of the series (18). The first term is of any consequence, by energy and momentum considerations, only if an external field is present and then we always get $G^S e\gamma A^C G^S$. Appearance of G^S at the ends simplifies their construction. G^S at the left will be referred to as G^{S_l}, and that at the right G^{S_r}. In constructing these functions we will make use of the following assumptions:

(i) Our photons are real; hence $k^2=0$.

(ii) The photon momenta $k_1, k_2, \cdots k_n\cdots$ can be neglected in comparison with the momenta p and p' of the free particles at the ends. We have the relations $m^2+p^2=0=m^2+p'^2$. Due to the assumptions (i) and (ii) we have (with $2p\sum_i k_i>2\sum_{i\neq j}k_ik_j$)

$$m^2+(p\pm k_1\pm k_2\pm\cdots\pm k_n)^2 \approx \pm 2p(k_1+k_2+\cdots+k_n), \quad (21a)$$

$$m-\gamma(p'+k)\gamma a=\gamma a[m+\gamma(p'+k)]+2(p'+k)a \approx \gamma a(m+\gamma p')+2p'a. \quad (21b)$$

Similarly

$$\gamma a[m-\gamma(p-k)]\approx(m+\gamma p)\gamma a+2pa. \quad (21c)$$

In (21a) and (21c) we are neglecting a term like $(\gamma a)(\gamma k)$ which is very important because of its spin dependence. We shall come to the discussion of this term in Sec. 9.

First consider G^{S_r} which has right coordinate index free. Making use of (11b) and (21) we can write it as

$$G^{S_r}=\sum_{n=0}^{\infty}\frac{1}{(2\pi)^4}\frac{1}{n!}\int dp'\frac{e^{ip'(x-x')}(m-\gamma p')}{m^2+p'^2}\times\left(\frac{e}{(2\pi)^4}\int\frac{p'a(k)}{p'k}e^{ikx}dx\right)^n$$
$$=\frac{1}{(2\pi)^4}\int dp'\frac{e^{ip'(x-x')}(m-\gamma p')}{m^2+p'^2}e^{B_{p'}(x)}. \quad (22a)$$

Similarly

$$G^{S_l}=\frac{1}{(2\pi)^4}\int dp\frac{e^{ip(x-x')}(m-\gamma p)}{m^2+p^2}e^{-B_p(x')}, \quad (22b)$$

where

$$B_{p'}(x)=\frac{e}{(2\pi)^4}\int e^{ikx}\frac{p'a(k)}{p'k}dk. \quad (22c)$$

5. INFRARED DIVERGENCE IN COULOMB SCATTERING

As the first application of the foregoing formalism we shall consider the problem of IRD and multiple bremsstrahlung. Even though the IRD problem involves virtual processes, the divergence itself involves almost real photons. Hence we can use the Green's functions which we have constructed in the last section.

We shall first proceed by evaluating the expectation value of the generating function $Q(\lambda)$ given by Eq. (5) for photon emission in Coulomb scattering.

$$\langle Q(\lambda)\rangle = -[\bar{\psi}_{+p}\gamma_0 G_{--}\gamma_0\psi_{+p'}][\bar{\psi}_{+p}\gamma_0 G_{++}\gamma_0\psi_{+p}] \times\exp[(\lambda-1)K_+SK_-]\exp(\tfrac{1}{2}iKDK). \quad (23)$$

We shall assume that all the photons involved are soft photons. Then Eqs. (8a) and (8b) become

$$G_{++}=G_{A+}{}^{Sl+}e\gamma A^C G_{A+}{}^{Sr+}, \quad (24a)$$

$$G_{--}=-G_{A-}{}^{Sl-}e\gamma A^C G_{A-}{}^{Sr-}. \quad (24b)$$

Now substituting (22) and (24)

$$-[\bar{\psi}_{+p}\gamma_0 G_{--}\gamma_0\psi_{+p'}][\bar{\psi}_{+p'}\gamma_0 G_{++}\gamma_0\psi_{+p}]$$
$$=[\bar{\psi}_{+p}\gamma_0 G_{A-}{}^{Sl-}e\gamma A^C G_{A-}{}^{Sr-}\gamma_0\psi_{+p'}] \times[\bar{\psi}_{+p'}\gamma_0 G_{A+}{}^{Sl+}e\gamma A^C G_{A+}{}^{Sr+}\gamma_0\psi_{+p}]$$
$$=(Ze^2)^2\left(\frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\right)\left(\frac{d^3p'}{(2\pi)^3}\frac{m}{p_0'}\right) \times(\bar{u}_{+p}\gamma_0 u_{+p'}\bar{u}_{+p'}\gamma_0 u_{+p})\int e^{-i(p-p')(x'-x)} \times\frac{e^{[B_{-p'}(x')-B_{-p}(x')]}}{|\mathbf{x}'|}\frac{e^{-[B_{+p'}(x)-B_{+p}(x)]}}{|\mathbf{x}|}, \quad (25a)$$

where

$$B_{\pm p}(x)=\frac{e}{(2\pi)^4}\int e^{ikx}\frac{pa_\pm(k)}{pk}dk. \quad (25b)$$

Averaging over the initial spin states and summing over the final ones, we get for (25)

$$\left[\frac{(Ze^2)^2}{2m^2}(p_0p_0'+\mathbf{p}\cdot\mathbf{p}'+m^2)\left(\frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\right)\left(\frac{d^3p'}{(2\pi)^3}\frac{m}{p_0'}\right)\right] \times\int e^{+i(p-p')(x-x')}\frac{e^{(B_{-p'}-B_{-p})(x')}}{|\mathbf{x}'|}\frac{e^{-(B_{+p'}-B_{+p})(x)}}{|\mathbf{x}|}. \quad (26)$$

Hence Eq. (23) becomes

$$\langle Q(\lambda)\rangle=[\ ] \times\int e^{+i(p-p')(x-x')}\frac{e^{(B_{-p'}-B_{-p})(x')}}{|\mathbf{x}'|}\frac{e^{-(B_{+p'}-B_{+p})(x)}}{|\mathbf{x}|} \times\exp[(\lambda-1)K_+SK_-]\exp[\tfrac{1}{2}iKDK]. \quad (27a)$$

In this equation we mean, by a_+ and a_-

$$a_+=\frac{1}{i}\frac{\delta}{\delta K_+(k)};\quad a_-=\frac{-1}{i}\frac{\delta}{\delta K_-(k)}. \quad (27b)$$

In the momentum representation, we have

$$K_+SK_-=\sum\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{2k_0}K_+(-k)K_-(k), \quad (27c)$$

and

$$K_+D_{++}K_+=\frac{1}{(2\pi)^4}\int dk\frac{K_+(-k)K_+(k)}{k^2-i\epsilon}, \quad (27d)$$

$$K_-D_{--}K_-=-\frac{1}{(2\pi)^4}\int dk\frac{K_-(-k)K_-(k)}{k^2+i\epsilon}. \quad (27e)$$

Making use of the identity

$$\exp\left(C\frac{\delta}{\delta K}\right)F(K)=F(K+C), \quad (28a)$$

Eq. (27a) becomes

$$\langle Q(\lambda)\rangle=[\ ]\int\frac{e^{i(p-p')(x-x')}}{|\mathbf{x}||\mathbf{x}'|} \times\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')}\right] \times\exp\left[i\frac{\alpha}{(2\pi)^3}\int dk\left(\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}\right) \times\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\right]. \quad (28b)$$

Using the identity

$$\lim_{\epsilon\to0}\frac{1}{k^2\mp i\epsilon}=P\frac{1}{k^2}\pm i\pi\delta(k^2), \quad (29a)$$

we have

$$\frac{1}{k^2-i\epsilon}-\frac{1}{k^2+i\epsilon}=2\pi i\delta(k^2). \quad (29b)$$

Making use of (29), Eq. (28b) becomes

$$\langle Q(\lambda)\rangle=[\ ]\int e^{i(p-p')(x-x')} \times\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')}\right] \times\exp\left[-\frac{\alpha}{(2\pi)^2}\int\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\right]. \quad (30a)$$

Both integrals in the last two exponents have gauge-invariant structure. Now the term containing λ is due to emission of photons, and the other exponential is due to that part of the radiative correction which gives rise to infrared divergence. Let us say that the photons produced carry away energy and momentum K $(\mathbf{K},K_0)$, which gives the upper limit of integration.

In order to deal with radiative correction integral we appeal to the lowest order calculations of Schwinger[8]

[8] J. Schwinger, Phys. Rev. **76**, 790 (1949).

and Newton[9] and replace the integral by

$$\int \frac{d^3k}{k_0}\left(\frac{2p-k}{2pk-k^2}-\frac{2p'-k}{2p'k-k^2}\right)^2, \tag{30b}$$

which is chosen to be gauge invariant. Besides it does not need an upper limit as it is convergent. Note that this replacement is not at all necessary in order to show the cancellation of IRD because the lower limit is not affected. We assume that in the above expressions we are dealing with renormalized quantities.

The integrals we want can be evaluated attributing a small mass ϵ to the photon. They can be evaluated exactly using Spence functions. In the high-energy limit they are given by

$$\frac{1}{(2\pi)^2}\int_0^K\left(\frac{p}{pk}-\frac{p'}{pk}\right)^2\frac{d^3k}{k_0} =\frac{1}{\pi}\left[\ln\frac{2|pp'|}{m^2}\left(\ln\frac{m^2}{\epsilon^2}+\frac{1}{2}\ln\frac{2|pp'|}{m^2}-\ln\frac{EE'}{K_0^2}\right)-\ln\frac{m^2}{\epsilon^2}+\ln\frac{EE'}{K_0^2}\right] \tag{31a}$$

and

$$\frac{1}{(2\pi)^2}\int\left(\frac{2p-k}{2pk-k^2}-\frac{2p'-k}{2p'k-k^2}\right)^2\frac{d^3k}{k_0} =\frac{1}{\pi}\left[\ln\frac{2|pp'|}{m^2}\left(\ln\frac{m^2}{\epsilon^2}+\frac{1}{2}\ln\frac{2|pp'|}{m^2}-\frac{1}{2}\right)-\ln\frac{m^2}{\epsilon^2}\right]. \tag{31b}$$

Now consider the emission integral,

$$\lambda\frac{\alpha}{(2\pi)^2}\int_0^K\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')} =\lambda\frac{\alpha}{(2\pi)^2}\left(\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}+\int_{\Delta\epsilon}^K\frac{d^3k}{k_0}\right)\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')}, \tag{32}$$

where $\Delta\epsilon$ is the energy momentum lost due to the infrared part. In any emission process the former gets cancelled by the radiative part. Hence, (32) is effectively

$$=\lambda\frac{\alpha}{(2\pi)^2}\int_{\Delta\epsilon}^K\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')} +\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{pk}\right)^2 e^{-ik(x-x')}. \tag{33}$$

$\Delta\epsilon_0$ is nothing but the energy resolution of the experiment from our definitions in Sec. 2.

[9] R. G. Newton, Phys. Rev. **97**, 1162 (1955); Phys. Rev. **98**, 1514 (1955).

We shall confine ourselves to the IRD problem. Hence, we are not interested in the soft photon emission. This means that $K=\Delta\epsilon$ and $\lambda=0$. Hence, the probability that an electron of momentum $\mathbf{p}$ gets scattered with momentum $\mathbf{p}'$ with energy and momentum loss $\Delta\epsilon$ due to the infra-photons is given [from (26), (30a), and (33)] by

$$P'(\Delta\epsilon)=[\ ]\int dxdx'\frac{e^{i(p-p')(x-x')}}{|\mathbf{x}|\,|\mathbf{x}'|} \times\exp\left[\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}(\ )^2e^{-ik(x-x')}\right] \times\exp\left[-\frac{\alpha}{(2\pi)^2}\int(\ )^2\frac{d^3k}{k_0}\right]. \tag{34}$$

We have not yet imposed the restriction that the total energy momentum lost by infrared photon emission is $\Delta\epsilon$. This is done by introducing the Fourier representation for $\delta(\Delta\epsilon-\sum k)$. Then (34) becomes

$$P(\Delta\epsilon)=[\ ]\frac{1}{(2\pi)^4}\int dxdx'dy\frac{e^{i(p-p'-\Delta\epsilon)(x-x')}}{|\mathbf{x}'|\,|\mathbf{x}|}e^{i\Delta\epsilon y} \times\exp\left[\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}(\ )^2e^{-iky}\right] \times\exp\left[-\frac{\alpha}{(2\pi)^2}\int(\ )^2\frac{d^3k}{k_0}\right], \tag{35}$$

$$=\frac{1}{(2\pi)^4}[\ ] \times\int dxdx'dy\frac{e^{i(p-p'-\Delta\epsilon)(x-x')}}{|\mathbf{x}'|\,|\mathbf{x}|}e^{i\Delta\epsilon y+f(y)} \times\exp\left[\frac{\alpha}{(2\pi)^2}\left(\int_0^{\Delta\epsilon}+\int\right)\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\frac{d^3k}{k_0}\right], \tag{36a}$$

where

$$f(y)=\exp\left[\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)(e^{-iky}-1)\right]. \tag{36b}$$

We see from (36b) that as $k\to 0$, $(e^{-iky}-1)=-iky$ and hence the integral is not divergent. In (36a) the integrals in the last exponential also cancel out their divergent parts. Hence, (36a) becomes

$$P(\Delta\epsilon)=\frac{1}{(2\pi)^4}[\ ] \times\int dxdx'dy\frac{e^{i(p-p'-\Delta\epsilon)(x-x')}}{|\mathbf{x}|\,|\mathbf{x}'|}e^{i\Delta\epsilon y+f(y)} \times\exp\left[\frac{\alpha}{(2\pi)^2}\int_{\Delta\epsilon}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\frac{d^3k}{k_0}\right]. \tag{36c}$$

Using (31a and b), we get

$$P(\Delta\epsilon)=\frac{1}{(2\pi)^4}[\ ]\int\cdots\exp\left[-\frac{\alpha}{\pi}\left(\ln\frac{2|pp'|}{m^2}-1\right)\times\ln\frac{EE'}{(\Delta\epsilon_0)^2}+\frac{\alpha}{2\pi}\ln\frac{2|pp'|}{m^2}\right].\qquad(36\text{d})$$

Now the cross section for Coulomb scattering with loss of energy and momentum $\Delta\epsilon$ due to infrared radiation is given by

$$\frac{d\sigma^{(0)}}{d\Omega}(\Delta\epsilon)=\frac{1}{(2\pi)^6}\frac{(Ze^2)^2}{2m^2}(p_0p_0'+\mathbf{p}\cdot\mathbf{p}'+m^2)\times\frac{|\mathbf{p}'|}{|\mathbf{p}|}\frac{1}{(\mathbf{p}-\mathbf{p}'-\Delta\boldsymbol{\varepsilon})^4}\left(\int e^{i\Delta\epsilon y+f(y)}dy\right)$$
$$\times\exp\left[-\frac{\alpha}{\pi}\left(\ln\frac{2|pp'|}{m^2}-1\right)\times\ln\frac{EE'}{(\Delta\epsilon_0)^2}+\frac{\alpha}{2\pi}\ln\frac{2|pp'|}{m^2}\right],\qquad(37\text{a})$$

where

$$p_0'=p_0-\Delta\epsilon_0.\qquad(37\text{b})$$

It is very desirable to have no reference to the direction of total momentum $\Delta\boldsymbol{\varepsilon}$ of the photons emitted. This requires the evaluation of the four-dimensional integral $(\int\cdots dy)$ in (37a) which does not seem possible to evaluate analytically. This is the case with all the processes we are treating in subsequent sections. One can do the calculation ignoring the recoil effect. Foldy, Ford, and Yennie[10] have shown that the dynamical effects due to recoil are of order m^2/ME (M being the mass of the nuclear target) as compared to the complete neglect of recoil. These effects become very important in processes like $e+e\rightarrow e+e+\gamma$ when one of the initial electron is at rest. Besides the dynamical effects, one has to take into account the interaction of the target electron with the electromagnetic field due to recoil.

In the next section we shall evaluate the cross section ignoring the total momentum of the photons emitted but taking into account the energy conservation. This effectively means that we are neglecting the dynamical effects of recoil but taking into account the kinematical corrections due to the energy loss.

6. LOSS OF ENERGY IN COULOMB SCATTERING DUE TO INFRARED EMISSION

We shall first do the calculation for energy loss only due to the infra-photons. Instead of introducing a four δ-function restriction on (34), we shall introduce $\delta(\Delta\epsilon_0-\sum k_0)$. Then instead of (37) we have

$$\frac{d\sigma^{(0)}}{d\Omega}(\Delta\epsilon_0)=\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}\left(1-\frac{\Delta\epsilon_0}{E^2}-\frac{v^2\Delta\epsilon_0\cos^2(\theta/2)}{1-v^2\sin^2(\theta/2)}\right)\times\frac{1}{2\pi}\left(\int dy_0\,e^{i\Delta\epsilon_0y_0+f(y_0)}\right)\exp\left(-\frac{\alpha}{\pi}\cdots\right).\qquad(38)$$

Defining

$$C=\frac{2\alpha}{\pi}\left[\ln\frac{2|pp'|}{m^2}-1\right],\qquad(39\text{a})$$

we have[11]

$$\left(\int dy_0\,e^{i\Delta\epsilon_0y_0+f(y_0)}\right)=\int dy_0\,e^{i\Delta\epsilon_0y_0}\times\exp\left[\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon_0}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2(e^{-ik_0y_0}-1)\right]=2\pi\frac{C}{\Delta\epsilon_0}F(C),\qquad(39\text{b})$$

where

$$F(C)=e^{-\gamma C}/\Gamma(1+C),\quad\gamma=\text{Euler's constant}.\qquad(39\text{c})$$

Hence (38) becomes

$$\frac{d\sigma^{(0)}}{d\Omega}(\Delta\epsilon_0)=\frac{C}{\Delta\epsilon_0}F(C)\left(1-\frac{\Delta\epsilon_0}{E^2}-\frac{v^2\Delta\epsilon_0\cos^2(\theta/2)}{1-v^2\sin^2(\theta/2)}\right)\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}\times\exp\left[-\tfrac{1}{2}C\ln\frac{EE'}{(\Delta\epsilon_0)^2}+\tfrac{1}{4}C+\frac{\alpha}{2\pi}\right].\qquad(40)$$

Now suppose the energy resolution of the experiment is $\Delta E>\Delta\epsilon_0$; then one has to integrate (40) over $\Delta\epsilon_0$ from 0 to ΔE. We shall do this assuming that ΔE is very small so that $p_0\approx p_0'$. Then $E\approx E'$, C is independent of $\Delta\epsilon_0$, and

$$\left(1-\frac{\Delta\epsilon_0}{E^2}-\frac{v^2\Delta\epsilon_0\cos^2(\theta/2)}{1-v^2\sin^2(\theta/2)}\right)\approx1.$$

Hence we get

$$\left(\frac{d\sigma^{(0)}}{d\Omega}(\Delta E)\right)=\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}F(C)e^{-C\ln(E/\Delta E)}e^{\frac{1}{4}(C+2\alpha/\pi)}.\qquad(41)$$

This expression was first obtained by Yennie *et al.*[11] The examination of (41) reveals that as $\Delta E\rightarrow0$ the cross section becomes zero, corroborating the con-

[10] L. L. Foldy, K. W. Ford, and D. R. Yennie, Phys. Rev. **113**, 1147 (1959).

[11] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (New York) **13**, 379 (1961). Most of the references regarding infrared divergence are given in this paper.

jecture of Schwinger.[8] The above equation gives the cross section one would get with an apparatus of energy resolution ΔE.

7. MULTIPLE BREMSSTRAHLUNG

Now we shall consider the production of multiple photons in Coulomb scattering. The expectation value of $Q(\lambda)$, using (30a), (31), and (33), becomes

$$\begin{aligned}\langle Q(\lambda)\rangle &= [\;]\int e^{i(p-p')(x-x')} \\ &\times\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int_{\Delta\epsilon_0}^{K_0}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')}\right] \\ &\times\exp\left[\frac{\alpha}{(2\pi)^2}\int_0^{\Delta\epsilon}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2 e^{-ik(x-x')}\right] \\ &\times\exp\left[-\frac{\alpha}{(2\pi)^2}\int\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\right] \\ &= [\;]\int e^{i(p-p')(x-x')}e^{iK_0y_0+f'(y_0)} \\ &\times\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int_{\Delta\epsilon_0}^{K_0}\frac{d^3k}{k_0}(\;)^2\right] \\ &\times\exp\left[-\frac{\alpha}{\pi}\left(\ln\frac{2|pp'|}{m^2}-1\right)\right. \\ &\left.\times\ln\frac{EE'}{(\Delta\epsilon_0)^2}+\frac{\alpha}{2\pi}\ln\frac{2|pp'|}{m^2}\right], \end{aligned}\tag{42}$$

where

$$f'(y_0)=\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int_0^{K_0}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2(e^{-ik_0y_0}-1)\right].\tag{43}$$

Proceeding in the same fashion as before (neglecting the recoil altogether), we get

$$\begin{aligned}\frac{d\sigma^{(0)}}{d\Omega}(\lambda,K_0) &\approx\frac{C'F(C')}{K_0-\Delta E}F(C)\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} \\ &\times\exp\left[\lambda\frac{\alpha}{(2\pi)^2}\int_{\Delta E}^{K_0}\frac{d^3k}{k_0}\left(\frac{p}{pk}-\frac{p'}{p'k}\right)^2\right] \\ &\times\exp\left[-\tfrac{1}{2}C\ln\frac{EE'}{(\Delta E)^2}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right], \\ &=\frac{C'F(C')}{K_0-\Delta E}F(C)\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} \\ &\times\exp\left[C'\ln\frac{K_0}{\Delta E}-\tfrac{1}{2}C\ln\frac{EE'}{(\Delta E)^2}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right],\end{aligned}\tag{44a}$$

where

$$C'=\lambda\frac{2\alpha}{\pi}\left(\ln\frac{2|pp'|}{m^2}-1\right).\tag{44b}$$

When $K_0=\Delta E$, $\lambda=0$ and $C'F(C')/(K_0-\Delta E)$ is unity. In order to get an expression where the energy loss is always less than K_0, we integrate (44) over K_0 from ΔE to K_0. We observe that p_0' appears always inside the logarithm. Because the logarithm is a slowly varying function, we put $p_0'\sim p_0$ and integrate, the error introduced being of the order of $\alpha K_0/E$. Then we get

$$\begin{aligned}\frac{d\sigma^{(0)}}{d\Omega}(\lambda,K_0<) &\approx\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}F(C')F(C) \\ &\times\exp\left[C'\ln\frac{K_0}{\Delta E}-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right],\end{aligned}\tag{45a}$$

where

$$F(C')=e^{-\gamma C'}/\Gamma(1+C'),\quad \gamma=\text{Euler's constant}.\tag{45b}$$

The extraction of the coefficient of λ^n gives the cross section for n-photon production. As an approximation we put $F(C')=F(C')=1$. Then (45b) becomes

$$\begin{aligned}\frac{d\sigma^{(0)}}{d\Omega}(\lambda,K_0<) &=\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}F(C) \\ &\times\exp\left[C'\ln\frac{K_0}{\Delta E}-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right].\end{aligned}\tag{46a}$$

The cross section for n-photon production is given by

$$\begin{aligned}\left(\frac{d\sigma^{(0)}}{d\Omega}\right)_n &=\frac{1}{n!}\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}F(C)\left(C\ln\frac{K_0}{\Delta E}\right)^n \\ &\times\exp\left[-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right].\end{aligned}\tag{46b}$$

Some remarks about the work done before[1] are in order. The exponential factor in (46b) is left out and so also $F(C)$, the former coming from infrared part of the radiative corrections and the latter from the energy conservation. Both of them provide strong convergence at very high energy, remembering that for $C\ll 1$, $F(C)$ behaves like $e^{-C\ln C}$. Besides, in Gupta's result K_0 has been put equal to E when the soft-photon approximation fails (see next section).

In the equation (46a) if we let $\lambda=1$, we get the cross section for Coulomb scattering with energy loss K_0 when the experimental energy resolution is ΔE.

8. CONTRIBUTION DUE TO THE FIRST HARD-PHOTON TERM

So far, we have calculated the relevant physical quantities under the assumption that the photons emitted are soft. The "proof" we gave in the Sec. 2 is inadequate. Hence, let us examine by taking into account one hard-photon term. Then instead of (24a) and (24b) we have to take

$$G_{++}=\{G_{A+}{}^{Sl+}e\gamma A^C G_{A+}{}^{Sr+} +G_{A+}{}^{Sl+}e\gamma A_H G_0{}^+ e\gamma A^C G_{A+}{}^{Sr+} +G_{A+}{}^{Sl+}e\gamma A^C G_0{}^+ e\gamma A_H G_{A+}{}^{Sr+}\}, \quad (47a)$$

$$G_{--}=-\{+\rightarrow - \text{ in the above expression}\}. \quad (47b)$$

The first terms in G_{++} and G_{--} combine to give, as before, the expression (46). The rest of the terms combine to give $d\sigma^{(1)}/d\Omega(\lambda,K_0<)$. In this expression we shall assume that the total energy loss including the hard photon is $\leq K_0$. Then, proceeding as before, we get the analog of (44a):

$$\frac{d\sigma^{(1)}}{d\Omega}(\lambda,K_0)=\left\{\lambda\frac{C'F(C')}{K_0-\Delta E}F(C)\times\exp\left[C'\ln\frac{K_0}{\Delta E}-\tfrac{1}{2}C\ln\frac{EE'}{(\Delta E)^2}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right]\times\int_0^{K_0}dk_0\,R'(k_0)\left(\frac{K_0}{K_0-k_0}\right)^{1-C'}\right\}. \quad (48a)$$

Here $[K_0/(K_0-k_0)]^{1-C'}$ comes from the integration

$$\int dy_0\, e^{i(K_0-k_0)y_0+f'(y_0)}, \quad (48b)$$

where $f'(y_0)$ is given by (43); k_0 is the energy of the hard photon. $R'(k_0)$ is given by

$$R'(k_0)=k_0\int d\Omega\left(\frac{d\sigma}{d\Omega}\right)_1(\mathbf{k}), \quad (48c)$$

where $(d\sigma/d\Omega)_1$ is the cross section for single bremsstrahlung. The integration over the angle of the photon has been achieved by Racah.[12] We shall have his results for two specific cases:

(i) For $k_0\leq E/2$,

$$R_<'(k_0)\approx\frac{8\alpha k_0}{\pi E^2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}\ln\frac{E}{m}, \quad (49a)$$

(ii) For $k_0>E/2$,

$$R_>'(k_0)\approx\frac{2\alpha k_0}{\pi}\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}\ln\frac{E}{m}\frac{1}{(E-k_0)^2}. \quad (49b)$$

[12] G. Racah, Nuovo cimento **11**, 461 (1934).

We shall consider (i) first. With (49a) the integral in (48a) gives rise to

$$\frac{d\sigma^{(1)}}{d\Omega}(\lambda,K_0)=[\quad]\frac{8\alpha}{\pi E^2}\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}\times\ln\frac{E}{m}\left(K_0{}^2-\frac{C'}{C'+1}K_0{}^2\right), \quad (50)$$

C' is very small; for 1-Bev electrons and an energy loss of 20 Mev, $C'\leq 0.06$. Hence the second term can be neglected. Hence (50) combined with (44a) gives for $K_0<E/2$

$$\frac{d\sigma}{d\Omega}(\lambda,K_0)=\frac{d\sigma^{(0)}}{d\Omega}(\lambda,K_0)+\frac{d\sigma^{(1)}}{d\Omega}(\lambda,K_0)=\frac{C'F(C')}{K_0-\Delta E}F(C')\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}\times\exp\left[C'\ln\frac{K_0}{\Delta E}-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right]\times\left(1+\lambda\frac{K_0}{C'}R_<(K_0)\right), \quad (51a)$$

where

$$R_<(K_0)=\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}R_<'(K_0). \quad (51b)$$

As before, in order to get an expression where the energy loss is less than or equal to $K_0<E/2$, we have to integrate over K_0 from ΔE to K_0. This operation yields [after approximations similar to (46a)]

$$\frac{d\sigma}{d\Omega}(\lambda,K_0<)=\left(\frac{d\sigma}{d\Omega}\right)_{\rm el}F(C)F(C')\times\exp\left[-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)+C'\ln\frac{K_0}{\Delta E}\right]\times\left(1+\lambda\frac{4\alpha}{\pi}\frac{K_0{}^2}{E^2}\ln\frac{E}{m}\right). \quad (52)$$

For 1-Bev electrons and $K_0=E/2$ we get that the second term is smaller than the first by a factor of thirty. Hence, we have about 3% error by neglecting the second term. Hence, for $K_0<E/2$, the soft-photon term gives the main contribution. With this term, the

production cross section for n-photon production is

$$\left(\frac{d\sigma}{d\Omega}(\lambda,K_0<)\right)_n=\frac{1}{n!}F(C)\left(C\ln\frac{K_0}{\Delta E}\right)^{n-1}$$
$$\times\exp\left[-C\ln\frac{E}{\Delta E}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right]$$
$$\times\left\{C\ln\frac{K}{\Delta E}+n\frac{4\alpha}{\pi}\frac{K_0{}^2}{E^2}\ln\frac{E}{m}\right\}. \quad (53)$$

We shall now consider the case where $K_0>E/2$. From the expression (48a) we infer, as $0\leq C'<1$, that the integral has a major contribution coming from $K_0=k_0$. Hence, the expansion about the point K_0 will provide an approximate evaluation of (48a) for $K_0>E/2$. Up to the second term in Taylor series we have

$$R_>'(k_0)=\frac{2\alpha}{\pi}\ln\frac{E}{m}\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}\left\{\frac{K_0}{(E-K_0)^2}+(k_0-K_0)\right.$$
$$\left.\times\left[\frac{1}{(E-k_0)^2}+\frac{2K_0}{(E-K_0)^3}\right]\right\}, \quad (54)$$
$$=R_>(K_0)\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}\left[1-\frac{(K_0-k_0)}{K_0}\frac{E+K_0}{E-K_0}\right].$$

Now

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma^{(0)}}{d\Omega}+\left(\frac{d\sigma^{(1)}}{d\Omega}\right)_{E/2}+\left(\frac{d\sigma^{(1)}}{d\Omega}\right)_{>E/2}. \quad (55)$$

Using the above expression, integration limits of k_0 in 2nd term being from 0 to $E/2$ and integration limits in the last term being from $E/2$ to K_0, we get

$$\frac{d\sigma}{d\Omega}(\lambda,K_0)=\frac{C'F(C')}{K_0-\Delta E}F(C)\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}$$
$$\times\exp\left[C'\ln\frac{K_0}{\Delta E}-\tfrac{1}{2}C\ln\frac{EE'}{(\Delta E)^2}+\frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right]$$
$$\times\left\{1+R_>(K_0)\left(\frac{K_0}{C'}-\left(K_0-\frac{E}{2}\right)\frac{E+K_0}{E-K_0}\right)\right.$$
$$\left.+\frac{(K_0-E/2)}{C'}R_<(K_0)\right\}, \quad (56)$$

$d\sigma/d\Omega(\lambda,<K_0)$ is given by integration of K_0 from ΔE to K_0. Upon doing the integrations approximately, one finds that the last two terms give rise to a contribution between $2C$ and $3.5C$ times the first term when $K_0=0.75E$ and, hence, the error in neglecting the hard-photon term is between 20 and 35%, with $C\approx0.1$ which is the case for 1-Bev electrons. Taking higher terms in the Taylor series does not help the situation. Hence for $K_0>E/2$ the soft-photon part alone does not account for the loss or the production; higher hard-photon terms come into play.

In the above treatment we have extended the range of integration of the soft-photon energy from 0 to K_0 where $K_0>E/2$. This is not valid because in our construction of G^{S_l} and G^{S_r} we have assumed that the energy and momentum of soft photons are negligible as compared to that of the electron. One could remove this restriction provided one takes into account all the neglected terms in the series. The contribution due to these neglected terms is very small when $K_0<E/2$. When $K_0>E/2$, there is no distinction between hard and soft photons as one has to take into account these terms. Hence, two things happen when $K_0>E/2$: (i) The hard-photon term makes significant contribution; (ii) there is no distinction between hard and soft photons. This makes the expansion in terms of hard and soft photons lose its meaning for $K_0>E/2$.

9. RADIATIVE CORRECTIONS AND DISCUSSION

In this section we shall briefly discuss (i) the construction of G^{S_l} and G^{S_r}, (ii) the estimation of radiative corrections, and (iii) the comparison of the results derived before in the literature with ours.

In our construction of G^{S_l} and G^{S_r} we have ignored the spin-dependent terms entirely. Though these terms do not contribute to the energy loss or emission process, their contribution to the radiative corrections is significant. Schwinger[8] has calculated the radiative correction to first order. From that we infer that the contribution of the $(\gamma a)(\gamma k)$ term is logarithmic. By looking at the second-order radiative correction calculations by Newton,[9] one feels heuristically that in general this logarithmic term should appear in the exponent along with the infrared part of the radiative correction; the same might be true of the logarithmic contribution due to vacuum polarization which is entirely due to the determinant in Eq. (3.36) and which has been put equal to unity in our approximation. Because of the logarithmic nature of these terms, the major part of the radiative corrections to the particular process of Coulomb scattering comes from this exponential structure. Hence $e^{-\delta}$ in Schwinger's work[8] would give most of the radiative corrections to Coulomb scattering. A systematic method of evaluating the radiative correction using its infrared part has been given by Yennie *et al.*[11]

Some remarks about the work done before are in order. Yennie *et al.*[11] have obtained an expression similar to Eq. (12) by using classical arguments combined with Feynman diagrams. They seem to have superimposed infrared photons not on the scattering amplitude but on the probability, itself, thus getting an expression which is the sum of different hard-photon cross sections corrected for infrared photon superposition. Even from their most general treatment one cannot get (37a) or (40); one could get (41). Because of the reasons mentioned above, their treatment of infrared divergence is deficient, in addition to their nonrigorous treatment of overlapping divergence.

10. MULTIPLE PHOTON PRODUCTION IN PAIR ANNIHILATION

This problem can be given as general a treatment as the Coulomb scattering. Here, also, using the result for three-photon production in pair annihilation calculated by Joseph,[2] one can show that if the difference between the initial energy and the energy of the two primary photons is less than half the total energy, the "loss" is primarily due to the soft photons. But we shall not go into details. We shall just give the derivation for the cross section of multiple photon production when the energy loss is less than half the total energy and hence is mainly due to soft photons.

The expectation value of the generating function $Q(\lambda)$ in the case of pair annihilation is given by

$$\langle Q(\lambda)\rangle = -[\bar{\psi}_{+p}\gamma_0 G_{--}\gamma_0\psi_{-p}'][\bar{\psi}_{-p}\gamma_0 G_{++}\gamma_0\psi_{+p}] \times e^{(\lambda-1)K+SK-}e^{\frac{1}{2}iKDK}. \quad (57)$$

By the considerations of Secs. 2 and 3 we will calculate the contributions to $Q(\lambda)$ with two hard photons. Then G_{++} and G_{--} are given by

$$G_{++} = G_{A+}{}^{Sl+}e\gamma A_{H+}G_0{}^{+}e\gamma A_{H+}G_{A+}{}^{Sr+}, \quad (58a)$$

$$G_{--} = -G_{A-}{}^{Sl-}e\gamma A_{H-}G_0{}^{-}e\gamma A_{H-}G_{A-}{}^{Sr-}. \quad (58b)$$

Using these, we have

$$\begin{aligned} -[\bar{\psi}_{+p}\gamma_0 G_{--}\gamma_0\psi_{-p'}][\bar{\psi}_{-p'}\gamma_0 G_{++}\gamma_0\psi_{+p}] &= [\bar{\psi}_{+p}\gamma_0(G_{A+}{}^{Sl+}e\gamma A_{H+}G_0{}^{+}e\gamma A_{H+}G_{A+}{}^{Sr+})\gamma_0\psi_{-p'}] \\ &\quad + [\bar{\psi}_{-p'}\gamma_0(G_{A-}{}^{Sl-}e\gamma A_{H-}G_0{}^{-}e\gamma A_{H-}G_{A-}{}^{Sr-})\gamma_0\psi_{+p}] \quad (59a) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{e}{(2\pi)^4}\right)^4 \left(\frac{d\mathbf{p}}{(2\pi)^3}\frac{m}{p_0}\right)\left(\frac{d\mathbf{p}'}{(2\pi)^3}\frac{m}{p_0'}\right) \\ &\quad \times \int e^{-i(x'-x)(p+p')}e^{i(k_1'+k_2')x'}e^{i(k_1+k_2)x}dxdx' \\ &\quad \times \Big\{ \bar{u}_{+p}\gamma a_-(k_1')\frac{m-\gamma(p'-k_1')}{m^2+(p'-k_1')^2}\gamma a_-(k_2') \\ &\quad \times u_{-p'}\bar{u}_{-p'}\gamma a_+(k_1)\frac{m-\gamma(p+k_2)}{m^2+(p+k_2)^2}\gamma a_+(k_2)u_{+p} \\ &\quad \times e^{(B_{-p'}-B_{-p})(x')-(B_{+p'}-B_{+p})(x)}dk_1dk_2dk_1'dk_2' \Big\}. \quad (59b) \end{aligned}$$

In Eq. (59b) it is understood that the a's on the line correspond to the hard photons and the operation of variational differentiation has to be carried out on the hard part of the exponential, e^{K+SK-} and $e^{\frac{1}{2}KDK}$. Besides, it should be noticed that this differentiation gives rise to radiative corrections when operating on $e^{\frac{1}{2}iKDK}$ [without vacuum polarization because we have put the determinant equal to unity in Eq. (4) in our approximation]. In our treatment of the problem here, we shall neglect these and carry out the cross-section calculation and include the major part of this effect in the exponential as we did in the case of Coulomb scattering [see after Eq. (30a) and also Sec. 9].

We shall work in the center-of-mass system of the electron and positron. During the calculation we assume that the total momentum of the soft photons, $\mathbf{K}$, is equal to zero. This is a physically good assumption for high-energy electrons and positrons. Proceeding as before, the generator for the cross section when the energy loss is K_0, which is less than half the total energy of the initial system, is given by

$$\sigma(\lambda,K_0) = \lambda^2 \frac{C'F(C')}{K_0-\Delta E}F(C)\sigma_2 \times \exp\left[C'\ln\frac{K}{\Delta E} - C\ln\frac{E}{\Delta E} + \frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right], \quad (60a)$$

where

$$\sigma_2 = \frac{4\pi\alpha^2}{(2E-K_0)^2}\ln\left(\frac{2E-K_0}{m}\right). \quad (60b)$$

Here σ_2 is just the cross section for the production of two photons in pair annihilation when the total available energy is $2E-K_0$. Integrating over K_0 from ΔE to K_0 to get the generator with the loss of energy less than K_0, we have

$$\sigma(\lambda,K_0<) = \lambda^2\frac{\pi\alpha^2}{E^2}\ln\frac{2E}{m}F(C)F(C') \times \exp\left[C'\ln\frac{K}{\Delta E} - C\ln\frac{E}{\Delta E} + \frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right], \quad (61)$$

after having neglected K_0 inside the logarithmic term and ignoring 0.75 as compared to $1/C$, C being approximately 0.06 for 1-Bev electrons. Hence the cross section for the production of n photons out of which $(n-2)$ are soft photons with total energy $K_0<E$ is given by

$$\sigma_n(K_0<) = \alpha^2\frac{\pi}{E^2}\ln\frac{2E}{m}\frac{F(C)}{(n-2)!}\left(C\ln\frac{K}{\Delta E}\right)^{n-2} \times \exp\left[-C\ln\frac{E}{\Delta E} + \frac{1}{4}\left(C+\frac{2\alpha}{\pi}\right)\right]. \quad (62)$$

The remarks made at the end of Sec. 7 regarding the results derived in the literature (Gupta[1] and Joseph[2]) hold in the case of pair annihilation also.

11. GENERALIZATION AND CONCLUSION

In the foregoing treatment of multiple photon production, we have calculated the cross section for Coulomb scattering and pair annihilation, in the course of which some approximations have been made. We infer from the examination of these approximations (see last paragraph of Sec. 8) that the treatment is valid only when the energy loss is less than half the total initial energy of the system, all the correction terms being small. When the energy loss is greater than half the total initial energy, the corrections are no longer small and exact evaluation of them is essential. With these limitations in mind, the following generalization appears plausible.

The contribution to the multiple-photon-production cross section in any primary process in quantum electrodynamics in the range of high energy is predominantly due to the soft photons if the energy lost by the primary process is less than half the initial energy. More specifically, if there are m photons in the final state of the primary process and s electron-positron momenta involved (both final and initial), the generator for the production cross section is given by

$$\sigma(\lambda)=\lambda^m\sigma\frac{1}{(2\pi)^4}\int dy\, e^{iKy+f'(y)}\exp\left(\lambda\int_{\Delta E}^{K}I\right)\exp\left(-\int_0 I\right),$$

where

$$I=\frac{\alpha}{(2\pi)^2}\frac{d^3k}{k_0}\left(\sum_f\frac{p_f}{p_fk}\epsilon(\lambda_f)-\sum_i\frac{q_i}{q_ik}\epsilon(\lambda_i)\right)^2$$

and

$$f'(y)=\exp\left[\lambda\int_0^K I(e^{-iky}-1)\right];\quad \begin{aligned}\epsilon(\lambda)&=+1,\quad \lambda>0\\ &=-1,\quad \lambda<0.\end{aligned}$$

σ is the modified cross section of the primary process due to the loss of energy momentum; $K=(\mathbf{K},K_0)$ is the total soft-photon energy momentum or the energy momentum lost by the primary process and is less than half the initial total energy.

If $\lambda=1$, $\sigma(1)$ gives the cross section for the loss of energy due to the emission of soft photons which is predominant when K_0 is less than half the total initial energy.

Though we have a method of treating multiple production with greater ease, we have had to make simplifying assumptions, the effect of some of which is hard to estimate when the energy loss is larger than half the total initial energy of the system. An improved treatment requires exact or better-approximated expressions for G_{++} and G_{--} than have been used before, Besides, the effect of vacuum polarization has to be included, which means the evaluation of the expressions taking $C(A_\pm)$ into account. Because of these reasons the foregoing treatment is not as complete as one would desire it to be.

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